1. Consider the function

$$f(x,y) = \frac{xy(x^2+y^4)}{x^2+y^2}, \quad (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}.$$

[2] (a) Show that f is continuous at $(0,0)$ and find the limit

$$f(0,0) := \lim_{(x,y)\to(0,0)} f(x,y).$$

[2] (b) Compute the x-partial derivative at $(0,0)$ by using the definition

$$\frac{\partial f}{\partial x}(0,0) := \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}.$$

[2] (c) Compute $\frac{\partial f}{\partial x}(x,y)$ by using chain rule and prove that it is continuous at $(0,0)$.

(a) Check behavior as $(x,y) \to (0,0)$ in polar coordinates

$$\left.\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}\right\} \quad f(x,y) = \frac{r^2\cos(\theta)\sin(\theta)\left(r^2\cos^2(\theta) + r^4\sin^4(\theta)\right)}{r^2\left(\cos^2\theta + \sin^2\theta\right)}$$

The limit corresponds to $r \to 0$

$$= r^2\cos(\theta)\sin(\theta)\left[\cos^2(\theta) + r^2\sin^4(\theta)\right]$$

$f(x,y) \longrightarrow 0$ as $r \to 0$. (because $\cos(\theta)$, $\sin(\theta)$ are bounded)

Let $f(0,0) := \lim_{(x,y)\to(0,0)} f(x,y) = 0$. Then, $f(x,y) = \frac{xy(x^2+y^4)}{x^2+y^2}$ is defined everywhere on $\mathbb{R}^2$.

(b) $f(h,0)=0$ $\Rightarrow$ $\dfrac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = 0.$

$f(0,0)=0$

(c) $$\frac{\partial f}{\partial x} = \frac{y(3x^2+y^4)}{(x^2+y^2)} - \frac{2x^2y(x^2+y^4)}{(x^2+y^2)^2} = \frac{y[(x^2+y^2)(3x^2+y^4)-2x^2(x^2+y^4)]}{(x^2+y^2)^2}$$

$$= \frac{y[x^4+3x^2y^2-x^2y^4+y^6]}{(x^2+y^2)^2}$$

Consider again the limit $(x,y)\to(0,0)$ in polar coordinates.

$$\frac{\partial f}{\partial x} = \frac{r\sin(\theta)\,[r^4\cos^4(\theta) + 3r^4\cos^2(\theta)\sin^2(\theta) - r^6\cos^2(\theta)\sin^4(\theta) + r^6\sin^6(\theta)]}{r^4(\cos^2\theta+\sin^2\theta)^2}$$

$$= r\sin(\theta)\,[\cos^4(\theta) + 3\cos^2\sin^2\theta - r^2\cos^2(\theta)\sin^4(\theta) + r^2\sin^6(\theta)]$$

As $r\to 0$, $\dfrac{\partial f}{\partial x} \to 0 = \dfrac{\partial f}{\partial x}(0,0)$

Therefore, $\dfrac{\partial f}{\partial x}$ is continuous at $(0,0)$.

2. Consider the function

$$f(x, y) = x^4 - 2x^2 + y^2 + 3,$$

in the disk $D = \{(x, y) \in \mathbb{R}^2 : \quad x^2 + y^2 \leq 1\}$.

[2] (a) Find critical points by using the first derivative test.

[2] (b) Identify local minima and maxima in D by using the second derivative test.

[2] (c) Find the global minimum and global maximum in D.

(a) f is differentiable everywhere (as a polynomial)

$$\frac{\partial f}{\partial x} = 4x^3 - 4x = 0 \qquad\qquad x(x^2-1) = 0$$

$$\Rightarrow \qquad y = 0$$

$$\frac{\partial f}{\partial y} = +2y = 0$$

Critical points are: $(0,0)$; $(1,0)$; $(-1,0)$

All three points belong to the unit closed disk D.

(b) $\frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$(0,0)$: $Hf = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$ - saddle point

$\frac{\partial^2 f}{\partial y^2} = 2$

$(\pm 1, 0)$: $Hf = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$ - local minimum

$\frac{\partial^2 f}{\partial x \partial y} = 0$

(c) $f(\pm 1, 0) = 2$

$f(0,0) = 3$

On the boundary of D: $x^2 + y^2 = 1$

$$f = x^4 - 2x^2 + (1 - x^2) + 3 = x^4 - 3x^2 + 4 \equiv g(x)$$

Check critical point of $g(x)$ for $x \in [-1, 1]$:

$$g'(x) = 4x^3 - 6x = 0 \qquad 2x(2x^2 - 3) = 0$$

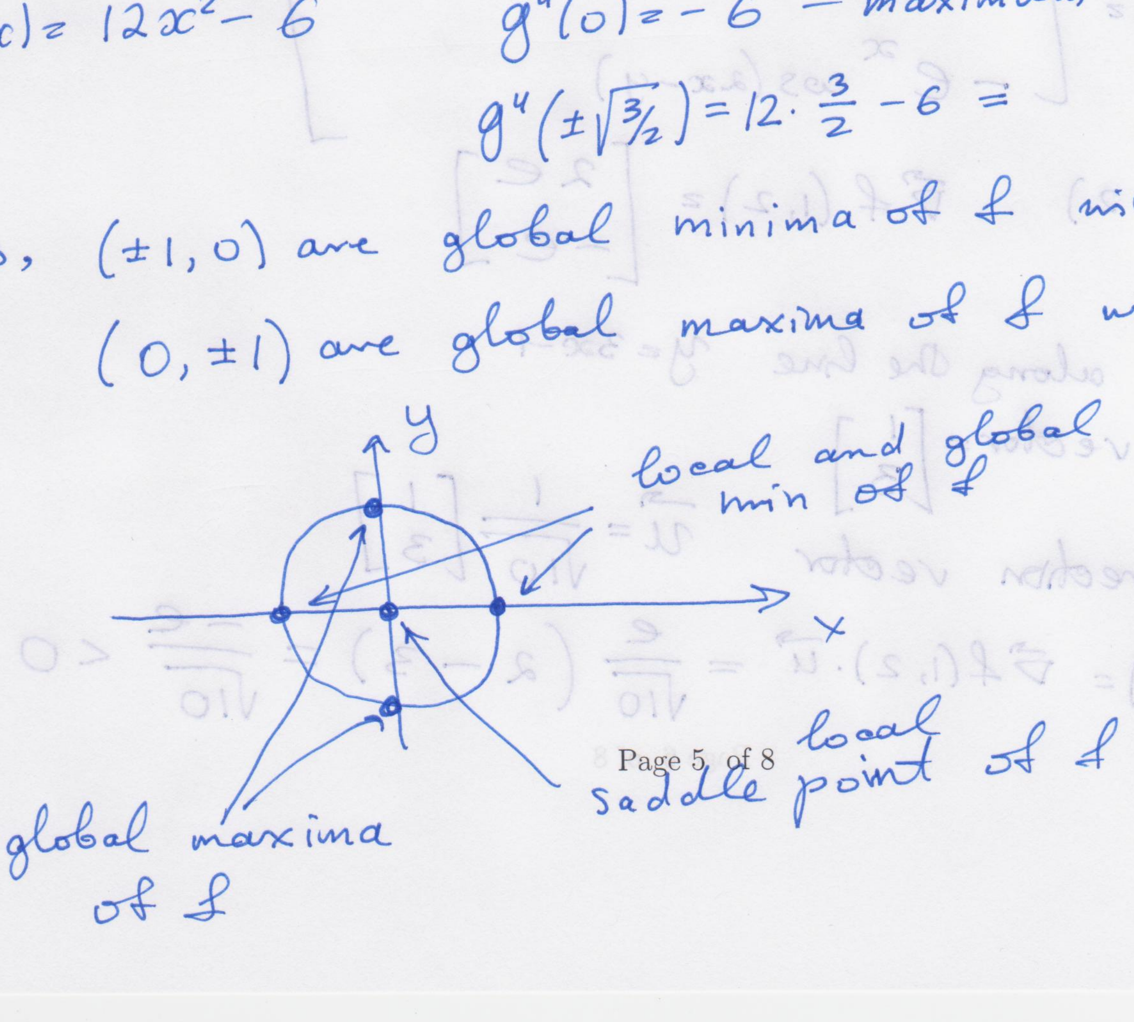
$x = 0$ or $x = \pm\sqrt{3/2}$ (these points do not belong the boundary of D)

$g''(x) = 12x^2 - 6$ $\qquad g''(0) = -6$ — local maximum

$$g''(\pm\sqrt{3/2}) = 12 \cdot \frac{3}{2} - 6 =$$

Thus, $(\pm 1, 0)$ are global minima of f with $f = 2$

$(0, \pm 1)$ are global maxima of f with $f = 4$.



3. Consider the function

$$f(x, y) = e^x \sin(2x - y), \quad (x, y) \in \mathbb{R}^2.$$

[2] (a) Find the gradient vector of f at any point (x, y).

[2] (b) Find the directional derivative of f at the point $(1, 2)$ in the direction of the line $y = 3x - 1$, for increasing values of x.

[2] (c) Find the unit direction vector $\mathbf{u}$ at the point $(1, 2)$, along which the function has the maximal increase.

(a) $\dfrac{\partial f}{\partial x} = e^x \sin(2x - y) + 2e^x \cos(2x - y)$

$\dfrac{\partial f}{\partial y} = -e^x \cos(2x - y)$

$$\vec{\nabla} f(x,y) = \begin{bmatrix} e^x[\sin(2x-y) + 2\cos(2x-y)] \\ -e^x \cos(2x-y) \end{bmatrix}$$

(b) point $(1, 2)$ $\quad \vec{\nabla} f(1,2) = \begin{bmatrix} 2e \\ -e \end{bmatrix}$

direction along the line $y = 3x - 1$

Tangent vector $= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

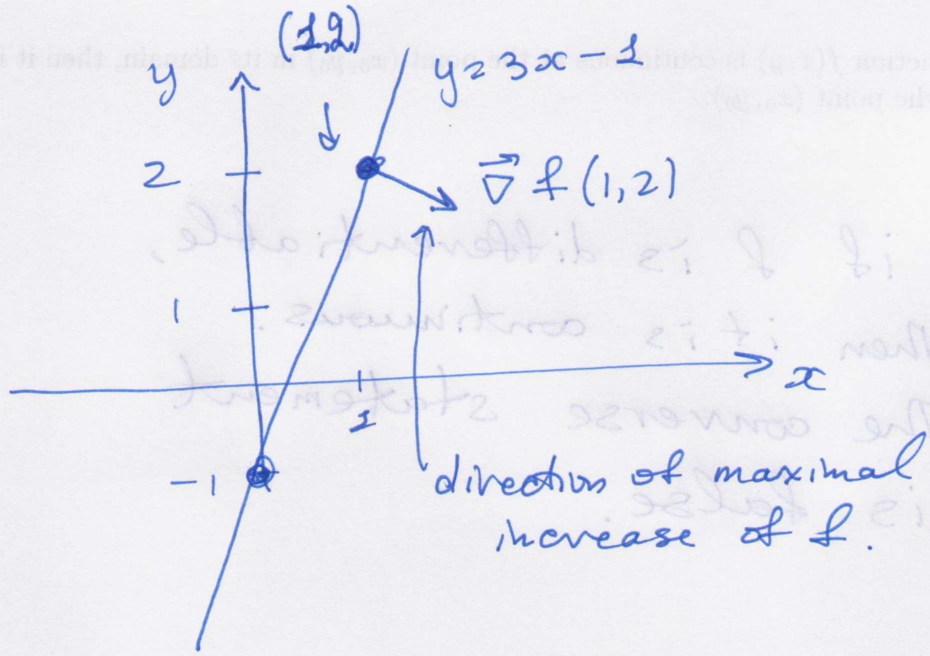
Unit direction vector $\quad \vec{u} = \dfrac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u} = \frac{e}{\sqrt{10}}(2 - 3) = \frac{-e}{\sqrt{10}} < 0$$

(c) $\vec{\nabla} f(1,2)$ shows the direction of the maximal increase of f.

$$\|\vec{\nabla} f(1,2)\| = \sqrt{4e^2 + e^2} = \sqrt{5}\, e$$

$$\vec{u} = \frac{\vec{\nabla} f(1,2)}{\|\vec{\nabla} f(1,2)\|} = \frac{1}{\sqrt{5}e}\begin{bmatrix} 2e \\ -e \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$



4. TRUE or FALSE:

[1] (a) The function $Q(x,y) = 3xy + x$ is the quadratic approximation of the function $f(x,y) = x(1+y)^3$ at the point $(0,0)$.

$f(x,y) = x(1+3y+3y^2+y^3) = x + 3xy + 3xy^2 + xy^3$

$Q(x,y) = 3xy + x$

TRUE: Q is a quadratic approximation of f at $(0,0)$

[1] (b) If a function $f(x,y)$ is continuous at the point (x_0, y_0) in its domain, then it is differentiable at the point (x_0, y_0).

FALSE: if f is differentiable, then it is continuous. The converse statement is false.