2A03 Midterm 1 Sample Answers

1. See exam sheet (⤼)

2. (a) Yes and yes
(since the components are cont's at this point)

(b) ~~By continuity~~ $= f(F(0,1))$

By (a), $f\circ F$ is continuous at $(0,1)$, since $F(0,1) = (1,0)$.

So $\lim_{(x,y)\to(0,1)} f(F(x,y)) = f(F(0,1))$
$= f(1,0)$
$= 0+0$
$= 0$

3. $\nabla f = (3x^2-3, 3y^2-3)$

$\nabla f(2,1) = (9,0)$

$\nabla f(x,y) = (9,0) \Leftrightarrow 3x^2-3=9$, $3y^2-3=0$

$\Leftrightarrow y=\pm 1, x = \pm 2$

so $(x,y) = (2,1), (-2,1), (2,-1)$, or $(-2,-1)$

4 (a) $T_p(r,\theta) = T_c(P(r,\theta))$
$= (r\cos\theta)^2 + (r\cos\theta)(r\sin\theta) - (r\sin\theta)^2$
$= r^2(\cos^2\theta - \sin^2\theta + \cos\theta\sin\theta)$

$DT_p(r,\theta) = DT_c(P(r,\theta))\, DP(r,\theta)$ (chain rule)
$= (2x+y, x-2y) \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

$= (2r\cos\theta + r\sin\theta, r\cos\theta - 2r\sin\theta) \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

$= \left(2r\cos^2\theta + 2r\sin\theta\cos\theta - 2r\sin^2\theta, 4r^2\cos\theta\sin\theta\right)$ $\to r^2(\cos^2\theta - \sin^2\theta - 4\sin\theta\cos\theta)$

5. all derivatives exist and are continuous so order of derivation doesn't matter, so

$\frac{\partial^6}{\partial x\partial y\partial x\partial x\partial x\partial x\partial x}(yx^5 + e^{\cos x})$
$= \frac{\partial^5}{(\partial x)^5}\frac{\partial}{\partial y}(yx^5 + e^{\cos x})$
$= \frac{\partial^5}{(\partial x)^5} x^5$
$= 5! = 120$

6. (a) $f(a+x, b+y) = f(a,b) + \nabla f(a,b)\cdot(x,y) + \frac{1}{2}(x,y)\, Hf(a,b) \binom{x}{y}$ + remainder

$\nabla f(x,y) = (y - 3(x+y)^2, x - 3(x+y)^2)$

$Hf(x,y) = \begin{pmatrix} -6(x+y) & 1-6(x+y) \\ 1-6(x+y) & -6(x+y) \end{pmatrix}$

so Taylor:

$f(a+x, b+y) = f(a,b) + (b-3(a+b)^2)x + (a-3(a+b)^2)y$
$+ \frac{1}{2}\left(-6(a+b)x^2 + 2(1-6(a+b))xy - 6(a+b)y^2\right)$
+ remainder.

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6 (b) $\nabla f(x,y) = 0 \iff y = x = 3(x+y)^2$

$x = 3(2x)^2 \iff x = 0, \text{ or } x = \frac{1}{12}$

so crit points: $(0,0)$ and $(\frac{1}{12}, \frac{1}{12})$

$\text{Det}(Hf(0,0)) = 0 - 1^2 < 0$ so saddle point

$\text{Det}(Hf(\frac{1}{12}, \frac{1}{12})) = (-1)^2 - 0 > 0$ so min/max

$f_{xx}(\frac{1}{12}, \frac{1}{12}) = -1 < 0$, so local max at $(\frac{1}{12}, \frac{1}{12})$

is only local extremum (since all points are interior)

(c) $f(-1,-1) = 1 - (-2)^3 = 9$

$f(-1,-1) > f(\frac{1}{12}, \frac{1}{12}) = (\frac{1}{12})^2 - (\frac{1}{6})^3$

so $(\frac{1}{12}, \frac{1}{12})$ is not a global max

but any global max is a local max,

so f has no global max

sim since no local min,

f has no global min

(d)

on the boundary

$\partial X = \{(x,y) \mid x+y = 0\}$

$= \{(x,y) \mid y = -x\}$

$f(x,y) = -x^2 - 0$

≤ 0 for all x

f clearly has no global min on ∂X

so certainly has no global min on X

To look for a global max, consider f restricted to a line $L_c := \{(x,y) \mid x+y = c\}$

for $c \geq 0$.

Note X is covered by such lines.

Now on L_c,

$f(x,y) = x(c-x) - c^3$

$= -x^2 + cx - c^3$

which has max at $c/2$

of $f(c/2, c/2) = -c^2/4 + c^2/2 - c^3$

$= c^2/4 - c^3 = g(c)$

$g'(c) = c/2 - 3c^2$

$g''(c) = 1/2 - 6c$

$g'(c) = 0 \iff c = 0, c = \frac{1}{6}$

Now considering this cubic,

it looks like

so on $c \geq 0$

it has a global max at $c = 1/6$

with value $(\frac{1}{6})^2/4 - (\frac{1}{6})^3 = (\frac{1}{6})^2(\frac{1}{4} - \frac{1}{6})$

$= \frac{1}{72}\frac{1}{6} = \boxed{\frac{1}{(6)(72)}}$

so since any point of X is on some L_c,

with value therefore $\leq$ the max on that L_c,

this is also the global max of f on X.