2A03 Sample Midterm 2

1. (a) $V(h,x) = 2hx^2$

$A(h,x) = 2x^2 + 2(hx + 2hx)$

$= 2x^2 + 6hx$

(b) $\nabla V = (2x^2, 4hx)$

$\nabla A = (6x, 4x + 6h)$

Lagrange: A min $\Rightarrow$

$\nabla A = \lambda \nabla V$ some λ

$6x = \lambda 2x^2 \quad \Rightarrow \lambda = 3/x$

$4x + 6h = \lambda 4hx$

$= 12h$

$\Rightarrow x = 3h$

$10 = V(h,x) = 2hx^2 \Rightarrow x = \sqrt{5/h}$

so $3h = \sqrt{5/h}$

$\Rightarrow 9h^2 = 5/h$

$\Rightarrow h^3 = 5/9$

so $\underline{h = \sqrt[3]{5/9}}$

is only crit point, so must be min

2. (a) $\|\underline{c}'(t)\| = \sqrt{(\frac{3}{2}t^{1/2})^2 + 1^2 + 1^2}$

$= \sqrt{\frac{9}{4}t + 2}$

so $\ell(t) = \int_0^t \sqrt{\frac{9}{4}\alpha + 2}\, d\alpha = \int_0^t \frac{d}{dt}\left(\frac{8}{27}(\frac{9}{4}\alpha + 2)^{3/2}\right) d\alpha$

$= \frac{8}{27}\left((\frac{9}{4}t + 2)^{3/2} - 2^{3/2}\right)$

$s = \ell(t) = \frac{8}{27}\left((\frac{9}{4}t + 2)^{3/2} - 2^{3/2}\right)$

$\Leftrightarrow \frac{27}{8}s + 2^{3/2} = (\frac{9}{4}t + 2)^{3/2}$

$\Leftrightarrow \frac{4}{9}\left((\frac{27}{8}s + 2^{3/2})^{2/3} - 2\right) = t$

$\ell^{-1}(s) =$

so $k(s) := \underline{c}(\ell^{-1}(s))$

$= \left(\left(\frac{4}{9}\left((\frac{27}{8}s + 2^{3/2})^{2/3} - 2\right)\right)^{3/2}, \frac{4}{9}(\frac{27}{8}s + 2^{3/2})^{2/3} - 2, \frac{4}{9}(\frac{27}{8}s + 2^{3/2})^{2/3} - 2\right)$

is an arc-length parametrisation

(b) $\cancel{k(1) = \left(\left(\frac{4}{9}\left((\frac{27}{8} + 2^{3/2})^{2/3} - 2\right)\right)^{3/2}, \frac{4}{9}(\frac{27}{8} + 2^{3/2})^{2/3} - 2, \frac{4}{9}(\frac{27}{8} + 2^{3/2})^{2/3} - 2\right)}$

$\underline{\ell^{-1}(1) = \frac{4}{9}\left((\frac{27}{8} + 2^{3/2})^{2/3} - 2\right) \text{ seconds}}$

3. $(t^2)^3 = ((t^3+1) - 1)^2 = t^6$

so $\underline{c}(t) = (t^2, t^3+1)$ satisfies $x^3 = (y-1)^2$

$\underline{c}(1) = (1,2)$

$\underline{c}(2) = (4,9)$

so if we take $\underline{c}$ to have domain $[1,2]$, it parametrises the curve with the correct orientation.

So $\int_C \underline{F} \cdot d\underline{s} = \int_{\underline{c}} \underline{F} \cdot d\underline{s} = \int_1^2 \underline{F}(\underline{c}(t)) \cdot \underline{c}'(t)\, dt$

$= \int_1^2 (t^3, t^2) \cdot (2t, 3t^2)\, dt$

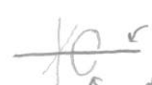
$= \int_1^2 2t^4 + 3t^4\, dt$

$= \int_1^2 5t^4\, dt = 2^5 - 1^5$

$\underline{= 31}$

4. (a) $\text{dom}\, \underline{F} = \mathbb{R}^3 \setminus \{(x,y,z) \mid y = 0 \text{ and } z = 0\}$

$= \mathbb{R}^3 \setminus [x\text{-axis}]$

(b) No (e.g. 

)

(c) Yes: $\underline{F} = \nabla f$ where $f(x,y,z) = \frac{1}{2}\log(y^2+z^2)$

(Another way to see it: $\text{curl}\, \underline{F} = 0$, and the circulation around a loop as in (b) is 0, so all circulations are 0, so $\underline{F}$ is a grad. vector field.)