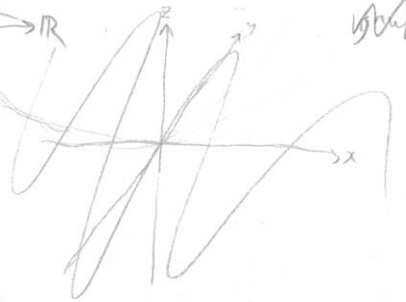
2A03 Lec 1

$f : \mathbb{R}^2 \to \mathbb{R}$

e.g. $f(\underline{x}) = |\underline{x}|$

i.e. $f(x,y) = \sqrt{x^2+y^2}$



$U \subseteq \mathbb{R}^2$

$f : U \to \mathbb{R}$

e.g. temp at a point in this room

$U \subseteq \mathbb{R}^3$

$f : U \to \mathbb{R}^2$

e.g. temp, humidity

$f(x,y,z) = (x+y+z, xyz)$

$f(x,y,z) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x+2y+3z, 2x+3y+4z)$

$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^3 \to \mathbb{R}^3$

vector fields

$f(x,y) = (-y, x)$ Fig 2.3

$f(x,y,z) = \frac{(x,y,z)}{\|(x,y,z)\|} \in$ unit sphere

i.e. $f(\underline{r}) = \frac{\underline{r}}{\|\underline{r}\|}$

vectors pointing directly away from $\underline{0}$

all of same length

draw 2d version

Example: gravitational force field

~~planet of mass~~

mass of M kg at $\underline{0}$

test mass of 1kg at dist d metres feels force of $\frac{GM}{d^2}$ Newtons in direction of the mass

i.e. $F(\underline{r}) := -\frac{GM}{\|\underline{r}\|^2} \frac{\underline{r}}{\|\underline{r}\|}$ is force felt by test mass at $\underline{r}$

Suppose a planet of mass M_p orbits a star of mass M_s

$M_p = 10^{24}$

$M_s = 10^{30}$

$d_p = 10^{12}$

Describe the resulting force field when the planet is d_p metres from the star.

Sol^n: Establish co-ordinates s.t. star is at $\underline{0}$ and planet is at $\underline{r}_p := (d_p, 0, 0)$

Force at $\underline{r}$ due to star on test mass of 1kg

is $F_s(\underline{r}) = -\frac{GM_s}{\|\underline{r}\|^2} \frac{\underline{r}}{\|\underline{r}\|}$

" " " " planet " "

is $F_p(\underline{r}) = -\frac{GM_p}{\|\underline{r}-\underline{r}_p\|^2} \frac{(\underline{r}-\underline{r}_p)}{\|\underline{r}-\underline{r}_p\|}$

total force is sum

$F(\underline{r}) = F_s(\underline{r}) + F_p(\underline{r})$ (no simplification!)

Graphs, & level sets

$f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$

Graph of f := $\{(x, y, f(x,y)) \mid (x,y) \in U\}$

sqdist.ogv

Level set of f at value c = $\{(x,y) \mid f(x,y) = c\} \subseteq U \subseteq \mathbb{R}^2$

limits & cont^y 2A03 Lec 2

- Balls
- ε-δ limits for $\underline{F}: \mathbb{R}^n \to \mathbb{R}^m$, cont^y
- specialisation to 1-var, $\mathbb{R}$-valued multivar
- $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{b}$ iff $\lim_{\underline{x}\to\underline{a}} F_i(\underline{x}) = b_i$
- $+$, $\cdot$ cont^s; hence limit laws
- cont^y of $\circ$s; hence of $\underline{F}+\underline{G}$, $f\cdot\underline{G}$

Notation: f, g, h are real-valued f^{ns} of 1-or-more var^s

$\underline{F}, \underline{G}, \underline{H}$ are $\mathbb{R}^n$-valued f^{ns} of $n \geq 1$

If $\underline{F}: U \to \mathbb{R}^n$ $U \subseteq \mathbb{R}^m$

write $F_i: U \to \mathbb{R}$ $i = 1, \dots, n$ for the comp^s

so $\underline{F}(\underline{x}) = (F_1(\underline{x}), \dots, F_n(\underline{x}))$

$= (F_1(x_1, \dots, x_m), \dots, F_n(x_1, \dots, x_m))$

e.g. $A \in \mathbb{R}_{n\times m}$ $: \mathbb{R}^m \to \mathbb{R}^n$

$(x_1, \dots, x_m) \mapsto \underline{A}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ just do 2×3

A_i ~ i^{th} row

$A_i(x_1, \dots, x_m) = a_{i1}x_1 + \dots + a_{im}x_m$

~~$\underline{a} \in \mathbb{R}^m$, $\varepsilon > 0$, ball of rad ε around $\underline{a}$~~

~~$B_{\underline{a}}(\varepsilon) \subseteq \mathbb{R}^m$ $B_{\underline{a}}(\varepsilon) = \{\underline{x} \mid \|\underline{x}-\underline{a}\| < \varepsilon\}$~~

$F: U \to \mathbb{R}^n$

- $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{b}$ iff $\forall \varepsilon > 0.\ \exists \delta > 0.$ for every $\underline{x} \in U$ s.t.
  $0 < \|\underline{x} - \underline{a}\| < \delta$,
  we have
  $\|\underline{F}(\underline{x}) - \underline{b}\| < \varepsilon$

In other words:

let $B_\varepsilon(\underline{b})$:= ball of radius ε around $\underline{b}$
$= \{\underline{y} \mid \|\underline{y} - \underline{b}\| < \varepsilon\}$

$\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{b}$ iff $\forall \varepsilon > 0.$ all $\underline{x} \in U$ sufficiently close to, but not equal to, $\underline{a}$, map under F into $B_\varepsilon(\underline{b})$

$f: (0, \infty) \to \mathbb{R}$, $f(x) = e^{-\frac{1}{x}}$

1d, 2d vector examples

- ~~Agrees with usual notion for real-valued f^n~~
- $\underline{F}$ is cont^s at $\underline{a}$ iff $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{F}(\underline{a})$
  (in particular, both exist)

so $\underline{F}$ is cont^s at $\underline{a}$ iff $\underline{x} \in \mathrm{dom}(\underline{F})$ and $\forall \varepsilon > 0.\ \exists \delta > 0.$ $\underline{x} \in B_\delta(\underline{a}) \cap U$
$\Rightarrow \underline{F}(\underline{x}) \in B_\varepsilon(\underline{F}(\underline{a}))$

Theorem (2.3): if $\underline{F}: U \to \mathbb{R}^n$ ($U \subseteq \mathbb{R}^m$), $\underline{G}: V \to \mathbb{R}^k$

ran $\underline{F} \subseteq V = \mathrm{dom}\,\underline{G}$ $\underline{F}$ cont^s at $\underline{a}$, $\underline{G}$ cont^s at $\underline{F}(\underline{a})$

then $\underline{G}\circ\underline{F}$ is cont^s at $\underline{a}$

$$\begin{array}{ccccc} U & \xrightarrow{F} & V & \xrightarrow{G} & \mathbb{R}^k \\ a & & F(a) & & G(F(a)) \end{array} \qquad \underbrace{\quad}_{G\circ F}$$

not on syllabus

pf: Let $\underline{a} \in U$, $\varepsilon > 0$

Since $\underline{G}$ is cont^s exists δ s.t.

G maps $B_\delta(\underline{F}(\underline{a})) \cap V$ into $B_\varepsilon(G(F(\underline{a})))$

but then since $\underline{F}$ is cont^s (using δ as its ε!)

exists $\eta > 0$ s.t. F maps $B_\eta(\underline{a}) \cap U$ into $B_\delta(\underline{F}(\underline{a}))$

$$B_\eta(\underline{a}) \xrightarrow{F} B_\delta(\underline{F}(\underline{a})) \xrightarrow{G} B_\varepsilon(G(F(\underline{a})))$$

so $\underline{G}\circ\underline{F}$ is cont^s! $\square$

Remark: Can also (book does) define $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x})$ componentwise:

$\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) := (\lim_{\underline{x}\to\underline{a}} F_1(\underline{x}), \dots, \lim_{\underline{x}\to\underline{a}} F_n(\underline{x}))$

when all exist,

and $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x})$ exists iff all $\lim_{\underline{x}\to\underline{a}} F_i(\underline{x})$ exist.

So $\underline{F}$ cont^s iff all F_i cont^s

Examples: vector addition

$+: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$
$\quad \mathbb{R}^n\times\mathbb{R}^n \cong \mathbb{R}^{2n}$

$+: (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (x_1+y_1, \dots, x_n+y_n)$

is cont^s, so is scalar mult^n

$\cdot: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$
$(c, \underline{x}) \mapsto c\underline{x}$

if $\underline{F}, \underline{G}$ are cont^s at $\underline{a}$, so is $\underline{F}+\underline{G}$

$$\left(\begin{array}{ccccc} \mathbb{R}^m & \xrightarrow{(F,G)} & \mathbb{R}^n\times\mathbb{R}^n & \xrightarrow{+} & \mathbb{R}^n \\ \underline{x} & \longmapsto & (\underline{F}(\underline{x}), \underline{G}(\underline{x})) & \longmapsto & \underline{F}(\underline{x}) + \underline{G}(\underline{x}) \end{array} \right)$$

and $c\underline{F}$, $f\underline{G}$ (f cont^s at $\underline{a}$)

$f(x, y) = \dfrac{xy}{x^2+y^2}$ has no limit at 0

indeed: $f(0, y) = 0$ any $y \neq 0$ so limit could only be 0

$f(x, x) = \dfrac{x^2}{2x^2} = \frac{1}{2}$ any $x \neq 0$ so limit could only be $\frac{1}{2}$

$\underline{F}(\underline{x}) = \dfrac{\underline{x}}{\|\underline{x}\|}$ has no limit at 0

but $\dfrac{\underline{x}}{\|\underline{x}\|^{1/2}}$ does

and $f(x, y) = (-y, x)$ does

Note: $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{b}$

means precisely that if we set $\underline{F}(\underline{a}) := \underline{b}$

then F becomes cont^s at $\underline{a}$

so we get limit laws -

$\lim_{\underline{x}\to\underline{a}} (\underline{F}+\underline{G})(\underline{x}) = \lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) + \lim_{\underline{x}\to\underline{a}} \underline{G}(\underline{x})$

$\lim_{\underline{x}\to\underline{a}} f(\underline{x})\underline{G}(\underline{x}) = (\lim_{\underline{x}\to\underline{a}} f(\underline{x}))(\lim_{\underline{x}\to\underline{a}} \underline{G}(\underline{x}))$

when RHS lims exist

2A03 Lec 3 Diff^n

Plan: · Linear approximations of f

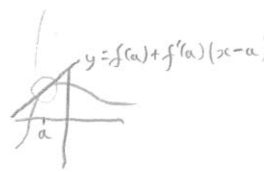
· " " " " F

· Jacobian

Recall: for $f: \mathbb{R} \to \mathbb{R}$

$f'(a) = b$

means $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = b$



i.e. $\lim_{h \to 0} \frac{|f(a+h) - (f(a) + bh)|}{|h|} = 0$

i.e. error in the approximation $f(a) + bh$ ("tangent line")

to $f(a+h)$

tends to 0 faster than $|h|$ does

as $h \to 0$

call this a "good linear approximation" to f at a

so f diff^{ble} at a $\Leftrightarrow$ exists good linear approx

~~and~~ derivative $f'(a) = b$ ~~is the linear approx~~ is a coefficient for the approx

Now consider $f: \mathbb{R}^2 \to \mathbb{R}$

Again, say ~~...~~ $f(\underline{a}) + b_1 h_1 + b_2 h_2$

is a good linear approx

if $\lim_{\underline{h} \to 0} \frac{|f(\underline{a} + \underline{h}) - (f(a) + b_1 h_1 + b_2 h_2)|}{\|h\|} = 0$

we say f is diff^{ble} at $\underline{a}$ if such an approx exists,

and then $b_1 = \frac{\partial f}{\partial x}(a)$, $b_2 = \frac{\partial f}{\partial y}(\underline{a})$

e.g. $f(x,y) = xy$ $\quad \frac{\partial f}{\partial x} = y \quad \frac{\partial f}{\partial y} = x$

$\underline{a} = (1,2)$

$(1+h_1)(2+h_2) = 2 + 2h_1 + h_2 + \underbrace{h_1 h_2}_{\to 0 \text{ faster than } \|h\| \text{ as } h \to 0}$

$f(1+h_1, 2+h_2) \approx (1)(2) + \frac{\partial f}{\partial x}(1,2)\, h_1 + \frac{\partial f}{\partial y}(1,2)\, h_2 = 2 + 2h_1 + h_2$

$= 2 + \nabla f(\underline{a}) \cdot \underline{h}$

Let $\nabla f(\underline{a}) := \left(\frac{\partial f}{\partial x}(\underline{a}), \frac{\partial f}{\partial y}(\underline{a})\right)$ "grad" $\quad \nabla f: \mathbb{R}^2 \to \mathbb{R}^2$

f diff^{ble} at $\underline{a}$ $\Leftrightarrow$ $\frac{\partial f}{\partial x_i}(\underline{a})$ exist

and $\lim_{\underline{h} \to 0} \frac{|f(\underline{a}+\underline{h}) - (f(\underline{a}) + \nabla f \cdot \underline{h})|}{\|\underline{h}\|} = 0$

$\underline{F}: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$

again, same game, but with matrices

linear approx to $\underline{F}$ at $\underline{a}$ will be of form

$\underline{F}(a) + M\underline{h} \qquad M \in Mat_{n,m} \qquad M: \mathbb{R}^m \to \mathbb{R}^n$

treated as col vec

Again, can say what M will be if it exists;

$$D\underline{F}(a) := \left(\frac{\partial F_i}{\partial x_j}\right)_{i,j} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}$$

"Jacobian matrix"

So

$\underline{F}$ is diff^{ble} at $\underline{a}$

iff all $\frac{\partial F_i}{\partial x_j}$ exist

and $\frac{\|\underline{F}(\underline{a}+\underline{h}) - (\underline{F}(\underline{a}) + D\underline{F}(\underline{a})\,\underline{h})\|}{\|\underline{h}\|} \xrightarrow[\underline{h} \to 0]{} 0$

then call $D\underline{F}$ the derivative of $\underline{F}$

Fact: if all $\frac{\partial F_i}{\partial x_j}$ cont^s at $\underline{a}$, then $\underline{F}$ diff^{ble} at $\underline{a}$

Fact: diff^{ble} at $\underline{a}$ $\Rightarrow$ cont^s at $\underline{a}$

e.g. $\underline{F}(x,y) = (-y, x)$: already linear!

$D\underline{F}(\underline{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ constant

~~$F(\underline{x}) = \frac{\underline{x}}{\|\underline{x}\|}$ $D\underline{F}(\underline{x}) =$~~

Tutorial example

$\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$F(x,y) = (xy, y) \quad D\underline{F}(\underline{x}) = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$

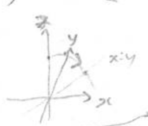
2A03 Lec 4,5 Derivatives cont'd

$\mathbb{R} \to \mathbb{R}^3$

Suppose an object moves in space and is at position $\underline{c}(t) \in \mathbb{R}^3$ at time t.

e.g. if we fire a cannon ball from $(0,0,10)$ at $t=0$ such that it has initial velocity $\frac{dc}{dt}$ $(1,1,0)$, then its position is given by

$\underline{c}(t) = (t, t, 10 - \frac{gt^2}{2})$



$\underline{c}$ traces out a curve

$\underline{c}$ gives a $\underline{\text{parametrisation}}$ of the curve (more on which later).

$\underline{c} : U^{\subseteq \mathbb{R}} \to \mathbb{R}^3$

$\frac{dc}{dt} = \underline{c}' = D\underline{c} \in Mat_{3,1}$ column vector

$D\underline{c}(t)$ is $\underline{\text{velocity}}$ at time t

e.g. $D\underline{c}_{cb} = (1, 1, -gt)$.

$\underline{c}'' = D(D\underline{c})(t)$ is $\underline{\text{accel}}$

$D^2 \underline{c}_{cb} = (0, 0, -g)$

Notation: $\underline{s}, \underline{v}, \underline{w}, \ldots : \mathbb{R} \to \mathbb{R}^n$

Chain rule:

$B(A\underline{x}) = (BA)\underline{x}$ i.e.

Recall if $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^m \xrightarrow{T} \mathbb{R}^k$, $\mathbb{R}^n \xrightarrow{T \circ S} \mathbb{R}^k$

S, T linear transformations

$S(\underline{x}) = A\underline{x}$ $A \in Mat_{m,n}$

$T(\underline{y}) = B\underline{y}$ $B \in Mat_{k,m}$

then $T \circ S$ is the linear transformation $(BA)\underline{x}$ $B \cdot A \in Mat_{k,n}$

i.e. composition $\rightsquigarrow$ matrix mult^n for linear f^{ns}

sim for affine linear - $S(\underline{x}) = A\underline{x} + \underline{a}$ $T(\underline{y}) = B \cdot \underline{y} + \underline{b}$

$T \circ S = B(A\underline{x} + \underline{a}) + \underline{b} = (BA)\underline{x} + B\underline{a} + \underline{b}$

chain rule says: composition of good linear approx^{ns} is a good linear approx^n

i.e. $\underline{\text{Fact [chain rule]}}$: if $U \xrightarrow{F} V \xrightarrow{G} \mathbb{R}^k$, $G \circ F : U \to \mathbb{R}^k$

F diff^{ble} at $\underline{a}$, G diff^{ble} at $F(\underline{a})$

then $G \circ F$ diff^{ble} at $\underline{a}$

and $D(G \circ F)(\underline{a}) = DG(F(\underline{a})) \cdot DF(\underline{a})$ $\boxed{D(G \circ F) = (DG \circ F)(DF)}$

Formulae for derivatives

F, G diff^{ble}, $c \in \mathbb{R}$

- linearity: $D(\underline{F} + \underline{G}) = D\underline{F} + D\underline{G}$
  $D(c\underline{F}) = cD\underline{F}$
- product rules: $\nabla(fg) = f\nabla g + g\nabla f$ $\quad f, g : \mathbb{R}^n \to \mathbb{R}$
  $(\underline{v} \cdot \underline{w})' = \underline{v} \cdot \underline{w}' + \underline{v}' \cdot \underline{w}$ $\quad \underline{v}, \underline{w} : \mathbb{R} \to \mathbb{R}^n$
  $(\underline{v} \times \underline{w})' = \underline{v} \times \underline{w}' + \underline{v}' \times \underline{w}$

(all hold pointwise
e.g. $\underline{F}, \underline{G}$ diff^{ble} at $\underline{a}$
$\Rightarrow D(\underline{F} + \underline{G})(\underline{a}) = D\underline{F}(\underline{a}) + D\underline{G}(\underline{a})$)

Example:

$\underline{r}(t)$ = position of object; assume diff^{ble}.

$$\frac{d}{dt}\|\underline{r}(t)\|^2 = \frac{d}{dt}(\underline{r}(t) \cdot \underline{r}(t)) = \underline{r}'(t) \cdot \underline{r}(t) + \underline{r}(t) \cdot \underline{r}'(t) = 2\underline{r}(t) \cdot \underline{v}(t)$$

so velocity is orthogonal to position vector

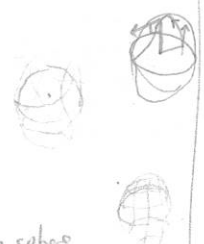
iff $\underline{r}(t) \cdot \underline{v}(t) = 0$

iff $\frac{d}{dt}\|\underline{r}(t)\|^2 = 0$

iff $\|\underline{r}(t)\|^2$ is constant

iff $\|\underline{r}(t)\|$ is constant

iff object moves on surface of a sphere



Example 2: let's prove $(\underline{v} \cdot \underline{w})' = \underline{v} \cdot \underline{w}' + \underline{v}' \cdot \underline{w}$.

$(\underline{v} \cdot \underline{w})(t) = (\underline{G} \circ \underline{F})(t)$

$\underline{F}(t) = (v_1(t), v_2(t), v_3(t), w_1(t), w_2(t), w_3(t))$

$\underline{G}(x_1, y_1, z_1, x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$

$t \xmapsto{F} (\underline{v}(t), \underline{w}(t)) \xmapsto{G} \underline{v}(t) \cdot \underline{w}(t)$

$\mathbb{R} \xrightarrow{F} \mathbb{R}^6 \xrightarrow{G} \mathbb{R}$

$D\underline{F} = (v_1', \ldots, w_3')^T = (\underline{v}', \underline{w}')^T$

$D\underline{G} = (x_2, y_2, z_2, x_1, y_1, z_1)$

so $D(G \circ F) = (DG \circ F)(DF) = (\underline{w}, \underline{v})(\underline{v}', \underline{w}')^T = \underline{w} \cdot \underline{v}' + \underline{v} \cdot \underline{w}'$

Example 1:

$\mathbb{R}^2 \xrightarrow{f} \mathbb{R} \xrightarrow{v} \mathbb{R}^2$, $\underline{v} \circ f$

$f(\underline{x}) = \|\underline{x}\|$

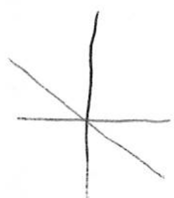
$\underline{v}(t) = (t, t^2)$

$D(\underline{v} \circ f) = (D\underline{v} \circ f) Df$

$$= \begin{pmatrix} 1 \\ 2\|\underline{x}\| \end{pmatrix} \left(\frac{x}{\|\underline{x}\|}, \frac{y}{\|\underline{x}\|} \right) = \begin{pmatrix} \frac{x}{\|\underline{x}\|} & \frac{y}{\|\underline{x}\|} \\ 2x & 2y \end{pmatrix}$$

Sol^n: (a) parabola

(b) slope



2A03 Lec 6

change of variables;

$\theta(x,y) = \arctan$ ...

$x = r\cos\theta$
$y = r\sin\theta$

$P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(r,\theta) \longmapsto (r\cos\theta, r\sin\theta)$

Let f a function on the plane expressed in cartesian co-ords
e.g. $f(x,y) = x^2+xy+y^2$

so in terms of polar co-ords, it's the function
$(r,\theta) \longmapsto f(r\cos\theta, r\sin\theta)$
$= r^2\cos^2\theta + r^2\cos\theta\sin\theta + r^2\sin^2\theta$
$\left(= r^2\left(1+\frac{\sin 2\theta}{2}\right)\right)$

call it $g(r,\theta)$
so $g = f\circ P$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "P"] \arrow[dr, "g"'] & \mathbb{R}^2 \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

Warning: often, people will write $f(r,\theta)$ to mean $g = f\circ P$
Confusing!

Now, what are $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial\theta}$? $\left(\text{"}\frac{\partial f}{\partial r}, \frac{\partial f}{\partial\theta}\text{"}\right)$

$\nabla g = (\nabla f\circ P)\, DP$

$$\left(\frac{\partial g}{\partial r}, \frac{\partial g}{\partial\theta}\right) = \nabla f(r\cos\theta, r\sin\theta)\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}$$

if $f(x,y) = x^2+xy+y^2$
then $\nabla f(x,y) = (2x+y,\ x+2y)$
so $\nabla g(r,\theta) = (2r\cos\theta + r\sin\theta,\ r\cos\theta + 2r\sin\theta)\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}$

$= (2r\cos^2\theta + r\sin\theta\cos\theta + r\cos\theta\sin\theta + 2r\sin^2\theta,\ \ldots) = \begin{pmatrix}2r(1+\sin\theta\cos\theta)\\ r^2(\cos^2\theta-\sin^2\theta)\end{pmatrix} = \begin{pmatrix} r(2+\sin(2\theta)), \\ r^2\cos(2\theta)\end{pmatrix}$

Example 2 simplified

$\underline{v},\underline{w} : \mathbb{R} \longrightarrow \mathbb{R}^2$

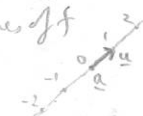
$\underline{v}\cdot\underline{w} = f\circ\underline{u}$
where $\underline{u}(t) = (v_1(t), v_2(t), w_1(t), w_2(t))$
$f(x,y,z,w) = xz + yw$

$\underline{u}' = (v_1', v_2', w_1', w_2')$
$\nabla f(\underline{x}) = (z, w, x, y)$
$(\underline{v}\cdot\underline{w})' = (f\circ\underline{u})' = (\nabla f\circ\underline{u})\cdot\underline{u}'$
$= (w_1, w_2, v_1, v_2)\begin{pmatrix}v_1'\\ v_2'\\ w_1'\\ w_2'\end{pmatrix}$

$= w_1v_1' + w_2v_2' + v_1w_1' + v_2w_2'$
$= \underline{w}\cdot\underline{v}' + \underline{v}\cdot\underline{w}'$

Directional derivatives;

$f : U \subseteq \mathbb{R}^n \to \mathbb{R}$
$\underline{a} \in U$
$\underline{u} \in \mathbb{R}^n$ a unit vector $(\|\underline{u}\| = 1)$
let $\underline{c}(t) := \underline{a} + t\underline{u}$
so $f(\underline{c}(t)) = (f\circ\underline{c})(t)$ gives values of f as we "walk" at constant speed 1 in direction of $\underline{u}$ from $\underline{a}$.



Def^n; $D_{\underline{u}}f(\underline{a}) := \frac{d}{dt}(f\circ\underline{c})(t)$

so by chain rule
if f is diff^{ble} at $\underline{a}$
then $D_{\underline{u}}f(\underline{a})$ exists and
$D_{\underline{u}}f(\underline{a}) = \nabla f(\underline{c}(0))\cdot\underline{c}'(0)$
$= \nabla f(\underline{a})\cdot\underline{u}$ dot product

Recall; $\nabla f(\underline{a})\cdot\underline{u} = \|\nabla f(\underline{a})\|\cdot\|\underline{u}\|\cdot\cos\theta$
so $D_{\underline{u}}f(\underline{a}) = \|\nabla f(\underline{a})\|\cos\theta$
where θ is angle between $\nabla f(\underline{a})$ and $\underline{u}$

so $\nabla f(\underline{a})$ is in direction of maximal rate of change of f from $\underline{a}$
and $\|\nabla f(\underline{a})\|$ is that maximum.

if $\underline{u}\cdot\nabla f(\underline{a}) = 0$
i.e. $\underline{u} \perp \nabla f(\underline{a})$
then $D_{\underline{u}}f(\underline{a}) = 0$
i.e. f is (instantaneously) constant in dir $\underline{u}$

so for diff^{ble} f
$\nabla f(\underline{a})$ is perpendicular to the level set
$\{\underline{x} \mid f(\underline{x}) = f(\underline{a})\}$

e.g. $f(\underline{x}) = \|\underline{x}\|$ $\underline{x} = (x,y,z)$
level sets are spheres
$\nabla f(\underline{x})$ is normal
indeed, $\nabla f(\underline{x}) = \left(\frac{x}{\|\underline{x}\|}, \frac{y}{\|\underline{y}\|}, \frac{z}{\|\underline{z}\|}\right) = \frac{\underline{x}}{\|\underline{x}\|}$
("steepest" direction is straight out from $\underline{0}$
constant "steepness")

2A03 Lec 7 Higher Order Derivatives

$f : U^{\subseteq \mathbb{R}^n} \to \mathbb{R}^m$

$\nabla f : U^{\subseteq \mathbb{R}^n} \to \mathbb{R}^n$

Recall: if f is diff^{ble} at $\underline{a}$ and $\|\underline{u}\|=1$,

$\underline{u} \cdot \nabla f(\underline{a}) = D_{\underline{u}} f(\underline{a})$ = directional derivative at $\bar{a}$ in dir^n $\underline{u}$

= rate of change of f at $\bar{a}$ as move from $\underline{a}$ in dir^n $\underline{u}$

$= \frac{d}{dt} f(\underline{a}+t\underline{u})$

Remark: if f is diff^{ble} and $\underline{u}$ "points along" (i.e. tangent to) a level set $f(\underline{x}) = c$

then $\underline{u} \cdot \nabla f(\underline{a}) = \vec{0}$

so $\nabla f(\underline{a})$ is $\perp \underline{u}$

so $\nabla f(\underline{a})$ is perpendicular (normal) to the level set at $\underline{a}$ $\{f(\underline{x}) = f(\underline{a})\}$

e.g. $f(\underline{x}) = \|\underline{x}\|$ level sets are spheres

$\underline{x}$ is indeed normal to the sphere of radius $\|\underline{x}\|$

Now $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

each $\frac{\partial f}{\partial x_i}$ is a f^n $U \to \mathbb{R}$

so we can try to differentiate it

$\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i} = \frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j}$ "second derivative"

e.g. $f(x,y) = xe^{xy}$

$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x} x^2 e^{xy} = 2xe^{xy}$ ~~…~~

$f_{xy} = \frac{\partial}{\partial y}\left(e^{xy} + xye^{xy}\right) = xe^{xy} + xe^{xy} = 2xe^{xy}$

$f_{xy} = f_{yx}$!

$\frac{\partial^2 f}{\partial x \partial x}$ $f_{xx} = ye^{xy} + ye^{xy} + xy^2e^{xy} = (y+xy^2)e^{xy}$

$f_{yy} = x^3 e^{xy}$

Note. $f : \mathbb{R}^3 \to \mathbb{R}$ has $\underline{9}$ second derivatives

$f : \mathbb{R}^m \to \mathbb{R}$ " m^2 " "

Jacobian of ∇f = matrix of 2nd deriv^s of f

$f : U^{\subseteq \mathbb{R}^m} \to \mathbb{R}$

Def^n: f is $\underline{C^1}$ if all 1st partial derivatives exist and are cont^s on U

f is $\underline{C^2}$ " " " 2nd "

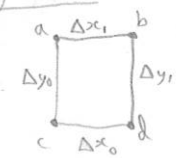
$C^2 \Rightarrow C^1 \Rightarrow$ diff^{ble}

Fact: if f is C^2

then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ (all i,j)

(as we saw with xe^{xy} above)

Rough idea:



$\Delta x_1 := f(b) - f(a)$

$\Delta x_0 := f(d) - f(c)$

$\Delta y_1 := f(b) - f(d)$

$\Delta y_0 := f(a) - f(c)$

$\Delta_x \Delta y := \Delta y_1 - \Delta y_0$

$\Delta_y \Delta x := \Delta x_1 - \Delta x_0$

then $\Delta_x \Delta y = \Delta_y \Delta x$

now shrink and take limits.. if all cont^s, works nicely.

Heat Equation: (skipped in lecture)

Rod of conductive metal $\longrightarrow x$

$T(x,t) :=$ heat at x along rod at time t

no external heating or cooling

Fact (1d heat equation):

$\frac{\partial T}{\partial t} = \sigma \frac{\partial^2 T}{\partial x^2}$

Rough idea why: discrete model: divide rod into cells and time into steps, and say at each time step each cell gives $\frac{1}{3}$ its heat to its left neighbour, $\frac{1}{3}$ to right, keeps $\frac{1}{3}$

so the heat gained by a cell is $\frac{1}{3}$ the difference between the differences

$t_{x,t+1} := \frac{1}{3}(t_{x-1,t} + t_{x,t} + t_{x+1,t})$

$t_{x,t+1} - t_{x,t}$

$= \frac{1}{3}(t_{x+1,t} - t_{x,t}) - \frac{1}{3}(t_{x,t} - t_{x-1,t})$

e.g. $T(x,t) = 7x$

is a "steady state sol^n"

$\frac{\partial T}{\partial t} = 0 = \frac{\partial^2 T}{\partial x^2}$

dynamic sol^n: $T(x,t) = \sin x\, e^{-\sigma t}$

2d heat eq^n:

$T(x,y,t)$ = temp in a sheet of metal at (x,y) at time t

$\frac{\partial T}{\partial t} = \sigma\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$

steady state sol^{ns}:

xy

$x^2 - y^2$

~~More conventional explanation of heat eq^n:~~

2A03 Lec 8 Taylor's Th^m

Def^n: Hessian of f
$Hf := D\nabla f = (f_{x_i x_j})_{ij}$

Let $f:\mathbb{R}\to\mathbb{R}$ be c^2

$$\int_0^h (h-t) f''(t)\,dt$$

$$= \int_0^h \frac{d}{dt}\left((h-t)f'(t) + f(t)\right) dt$$

$$= \left[(h-t)f'(t) + f(t)\right]_0^h$$

$$= f(h) - (hf'(0) + f(0))$$

$$f(h) = f(0) + hf'(0) + \int_0^h (h-t) f''(t)\,dt$$

$$f(x_0+h) = \underbrace{f(x_0) + hf'(x_0)}_{\substack{\text{linear approx} \\ T_1(x_0,h)}} + \underbrace{\int_{x_0}^{x_0+h} (x_0+h-t) f''(t)\,dt}_{\substack{\text{remainder} \\ R_1(x_0,h)}}$$

f is c^2, so $f''(t)$ is cont^s on $[x_0-c, x_0+c]$
hence bdd
so $|f''(t)| \le M$

then $|R_1(x_0,h)| = \left|\int_{x_0}^{x_0+h} (x_0+h-t) f''(t)\,dt\right|$
$\le \left|\int_{x_0}^{x_0+h} |h| M\,dt\right| = M|h|^2$ for any $-c \le h \le c$

f c^3

$$R_2(0,h) = \int_0^h \frac{(h-t)^2}{2} f'''(t)\,dt = \int_0^h \frac{d}{dt}\left(\frac{(h-t)^2}{2} f''(t) + (h-t)f'(t) + f(t)\right) dt$$

$$= f(h) - \underbrace{\left(\frac{h^2}{2} f''(0) + hf'(0) + f(0)\right)}_{T_2(0,h)}$$

$f(x_0+h) = \ldots$

$R_2(x_0,h) \le M|h|^3$

$$\Rightarrow \frac{f(x_0+h) - T_2(x_0,h)}{|h|^2} \xrightarrow[h\to 0]{} 0$$

T_2 is quad^c approx

$f:\mathbb{R}^m \to \mathbb{R}$ c^3

want linear & quad^c approx^{ns} with error estimates

$f(\underline{h}) = F_{\underline{h}}(1)$ $\quad F_{\underline{h}}(t) := f(t\underline{h})$ $\quad F_{\underline{h}}:\mathbb{R}\to\mathbb{R}$ $\quad F'_{\underline{h}}(t) = \nabla f(t\underline{h})\cdot\underline{h}$

$= F_{\underline{h}}(0) + F'_{\underline{h}}(0) + [\text{remainder}]$

$= f(\underline{0}) + \underline{h}\cdot\nabla f(\underline{0}) + R_1(\underline{h})$

again $c^2 \Rightarrow 2^{nd}$ derivs bdd on disc of radius c
$\Rightarrow R_1(\underline{h}) \le \|\underline{h}\|^2 M_c$

so $\frac{R_1(\underline{h})}{\|\underline{h}\|} \xrightarrow[\underline{h}\to 0]{} 0$

We knew this.

$F''_{\underline{h}}(t) = \frac{d}{dt}(\nabla f(t\underline{h})\cdot\underline{h}) = \left(\frac{d}{dt}\nabla f(t\underline{h})\right)\cdot\underline{h}$ (product rule)
$= D\nabla f(t\underline{h})\cdot\underline{h}\cdot\underline{h}$

$f(\underline{h}) = F_{\underline{h}}(1)$
$= F_{\underline{h}}(0) + F'_{\underline{h}}(0) + \frac{1}{2}F''_{\underline{h}}(0) + R_2(\underline{h})$
$= f(\underline{0}) + \nabla f(\underline{0})\cdot\underline{h} + (D\nabla f(\underline{0})\cdot\underline{h})\cdot\underline{h} + R_2(\underline{h})$
$\frac{1}{2}\underline{h}^t D\nabla f(\underline{0})\underline{h}$

$$F''_{\underline{h}}(t) = \frac{d}{dt}(\nabla f(t\underline{h})\cdot\underline{h})$$
$$= \left(\frac{d}{dt}\nabla f(t\underline{h})\right)\cdot\underline{h}$$
$$= (D\nabla f(t\underline{h})\cdot\underline{h})\cdot\underline{h}$$
$$= (Hf(t\underline{h})\cdot\underline{h})\cdot\underline{h}$$
$$= \underline{h}^t Hf(t\underline{h})\,\underline{h}$$

$$\frac{d}{dt}\left(\begin{pmatrix} f_x \\ f_y \end{pmatrix}(t\underline{h})\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right)$$
$$= \left(\frac{d}{dt}\begin{pmatrix} f_x \\ f_y \end{pmatrix}\right)\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$
$$= \left(\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}(t\underline{h})\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right)\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$
$$= (h_1\ h_2)\left(\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}(t\underline{h})\right)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

e.g. $m=2$. $(D\nabla f(\underline{0})\cdot\underline{h})\cdot\underline{h}$
$= (D(f_x, f_y)\cdot\underline{h})\cdot\underline{h}$
$= \begin{pmatrix} f_{xx}(0) & f_{xy}(0) \\ f_{yx}(0) & f_{yy}(0) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \underline{h} = (x,y)$
$=$
$= x^2 f_{xx}(0) + 2xy f_{xy}(0) + y^2 f_{yy}(0)$

So 2^{nd} order Taylor's formula in two vars

$f:\mathbb{R}^2\to\mathbb{R}$ c^3

$$f(\underline{x}_0+\underline{h}) = f(x_0+x, y_0+y) = f(\underline{x}_0) + xf_x(\underline{x}_0) + yf_y(\underline{x}_0)$$
$$+ x^2 f_{xx}(\underline{x}_0) + 2xy f_{xy}(\underline{x}_0) + y^2 f_{yy}(\underline{x}_0)$$
$$+ R_2(\underline{x}_0,\underline{h})$$

where $\frac{R_2(\underline{x}_0,\underline{h})}{\|\underline{h}\|^2} \xrightarrow[\underline{h}\to 0]{} 0$

Notation: $Hf(\underline{x}) := D\nabla f(\underline{x})$ "Hessian"

2^{nd} order in m vars: $\boxed{f(\underline{x}_0+\underline{h}) = f(\underline{x}_0) + \underline{h}\cdot\nabla f(\underline{x}_0) + \underline{h}^t\cdot Hf(\underline{x}_0)\cdot\underline{h} + R_2(\underline{x}_0,\underline{h})}$

2A03 Lec 9 Examples of Taylor

$f(x,y) = e^{xy}$

$\nabla f = (f_x, f_y) = (ye^{xy}, xe^{xy})$

$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} y^2 e^{xy} & (1+xy)e^{xy} \\ (1+xy)e^{xy} & x^2 e^{xy} \end{pmatrix}$

So 2nd order Taylor approx^n at $\underset{\sim}{x}_0 = (x_0, y_0)$

$f(\underset{\sim}{x}_0 + \underset{\sim}{h}) = f(\underset{\sim}{x}_0) + \nabla f \cdot \underset{\sim}{h} + \frac{1}{2} \underset{\sim}{h}^t Hf \underset{\sim}{h} + R_2$

$= f(\underset{\sim}{x}_0) + h_1 y_0 e^{x_0 y_0} + h_2 x_0 e^{x_0 y_0}$

at $\underset{\sim}{x}_0 := (0,1)$:

$f(x, 1+y) = f(0,1) + \nabla f(0,1) \cdot \underset{\sim}{h} + \frac{1}{2} \underset{\sim}{h}^t Hf(0,1) \underset{\sim}{h} + R_2(\underset{\sim}{h})$

$= 1 + (1, 0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x, y) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + R_2(\underset{\sim}{h})$

$= 1 + x + \frac{1}{2}(x\ y) \begin{pmatrix} x+y \\ x \end{pmatrix} + R_2(\underset{\sim}{h})$

$= 1 + x + \frac{1}{2}(x^2 + 2xy) + R_2(\underset{\sim}{h})$

$f(x,y,z) = x^3 + y^3 + z^3$

$\nabla f(\underset{\sim}{x}) = (3x^2\ \ 3y^2\ \ 3z^2)$

$Hf = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{pmatrix}$

$f(\underset{\sim}{x}_0 + \underset{\sim}{h}) = f(x_0) + 3x_0^2 x + 3y_0^2 y + 3z_0^2 z$
$+ \frac{1}{2}(6x_0 x^2 + 6y_0 y^2 + 6z_0 z^2)$
$+ R_2$

2A03 Lec 10,11 Optimisation & Extreme values

$f : X^{\subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}$

f has a local minimum at $\underline{x}_0$

if for some open ball B at $\underline{x}_0$

$f(\underline{x}_0) \leq f(\underline{x})$ for all $\underline{x} \in B$

then $f(\underline{x}_0)$ is a local min val of f.

x_0 is a global/absolute minimum of f

if $f(\underline{x}_0) \leq f(\underline{x})$ for all $\underline{x} \in X$

Obviously, global min $\Rightarrow$ local min.

"extreme" means "min or max"

~~Thm: If f has a local extremum at $\underline{x}_0$ and f is diff'ble in an open ball around $\underline{x}_0$ then $\nabla f(\underline{x}_0) = 0$~~

Defn: Let $X \subseteq \mathbb{R}^2$.

$\underline{x} \in X$ is interior to X

if some open ball around $\underline{x}$ is contained in X.

Thm: Suppose $\underline{x}_0$ is interior to $X = \operatorname{dom} f$

and f is diff'ble at $\underline{x}_0$

and f has a local extremum at $\underline{x}_0$.

Then $\nabla f(\underline{x}_0) = \underline{0}$

Pf: By the 1-var version, the directional derivs are 0 $\square$

~~$\nabla f = 0 \not\Rightarrow x_0$ extreme~~

e.g. $x^2 - y^2$

Defn: $\underline{x}$ interior to X is critical

if $\nabla f(\underline{x}) = 0$

or f is not diff'ble at $\underline{x}$.

So for interior points,

extreme $\Rightarrow$ critical

but critical $\not\Rightarrow$ extreme

e.g. $x^2 - y^2$ "saddle point"

Suppose f C^2 at $\underline{x}_0$ and $\underline{x}_0$ critical ($\nabla f(\underline{x}_0) = \underline{0}$)

Then by Taylor, f has good quad^c approx near $\underline{x}_0$:

$f(\underline{x}_0 + \underline{x}) \approx f(\underline{x}_0) + f_{xx}(\underline{x}_0) x^2 + 2 f_{xy}(\underline{x}_0) xy + f_{yy}(\underline{x}_0) y^2$

$= f(\underline{x}_0) + \underline{x}^t Hf(\underline{x}_0) \underline{x}$

$H := Hf(\underline{0}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $\quad |H| = ac - b^2$

~~Eigenvalues: $\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = (a-\lambda)(c-\lambda) - b^2 = 0$~~

~~$\Leftrightarrow \lambda^2 - (a+c)\lambda + ac - b^2 = 0$~~

~~$\Leftrightarrow \lambda = \frac{a + c \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2}$~~

~~$D := ac - b^2 = f_{xx}(\underline{x}_0) f_{yy}(\underline{x}_0) - f_{xy}(\underline{x}_0)^2$~~

~~Exercise: e-values are of same sign if $D > 0$, diff $D < 0$~~

~~and are distinct in either case.~~

~~Fact: after rotating co-ordinates, H becomes of form $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and it follows that $f(x,y) = f(x_0) + g(u,v)$ where $(u,v) \leftrightarrow (x,y)$ is a rotation centred at $\underline{x}_0$~~

~~2nd deriv test: Fact: If $|H| > 0$, e.g. $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ with $a, c > 0$ or $a, c < 0$ then~~

Fact [2nd deriv test]: Suppose f is C^2 on a ball around $\underline{x}_0$, and $\nabla f(\underline{x}_0) = 0$.

Let $D := \det(Hf(\underline{x}_0)) = f_{xx}(\underline{x}_0) f_{yy}(\underline{x}_0) - f_{xy}(\underline{x}_0)^2$

Then if $D > 0$

then f has an extreme value at $\underline{x}_0$,

min if $f_{xx}(\underline{x}_0) < 0$

max if $f_{xx}(\underline{x}_0) > 0$

and if $D < 0$,

then f does not have an extreme value at $\underline{x}_0$.

Very Sketchy Proof: Easily correct for $f(x,y) = ax^2 + cy^2$; after rotation of co-ords every $\underline{h}^t Hf \underline{h}$ is of that form (not obvious!)

by Taylor, behaviour of f near a crit point ... $h^t Hf\, h$

Remark: In higher dimensions, similar ideas work, but no longer just about $\det Hf$.

Defn: $\underline{x}$ is in the closure of $X \subseteq \mathbb{R}^m$

if every open ball around $\underline{x}$ contains some point of X

X is closed if $X = \text{closure}(X)$

i.e. if every point in the closure of X is in X.

X is open if every point of X is interior to X

Exercise: $X \subseteq \mathbb{R}^m$ is closed iff $\mathbb{R}^3 \setminus X$ is open

e.g. an open ball $\{\underline{x} \mid \|\underline{x} - \underline{a}\| < r\}$ is open

a closed ball $\{\underline{x} \mid \|\underline{x} - \underline{a}\| \leq r\}$ is closed.

Defn: $X \subseteq \mathbb{R}^m$ is bounded if it is contained in some ball

Thm: Suppose f is continuous on its domain X, and X is closed and bounded

Then f has a global max and a global min on X

Defn: The boundary ∂X of a closed set X is the set of points of X which are not interior to X.

e.g. $C :=$ closed ball $= \{\|\underline{x} - \underline{a}\| \leq r\}$

$\partial C =$ bounding circle $= \{\|\underline{x} - \underline{a}\| = r\}$

Finding global extrema

$f : X^{\subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}$

[If X is not closed or if it is not bounded f might not have a min on X and/or max]

Interior points: check critical points, if C^2, can use 2nd deriv test for this

The rest: can often parametrise the rest by curves, giving 1-variable f^{ns} to find extrema of.

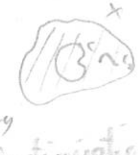
Always be aware that if X is not closed or not bdd f might not have a global min and/or max.

Example: $f(x,y) = x^3 - x^2 - y^2$ on

2A03 Lec 12 Lagrange Multipliers

Def^n: If $X \subseteq \mathrm{dom}\, f$, the restriction of f to X is the function $f|_X$ whose domain is X and whose values are those of f:
$(f|_X)(\underline{x}) = f(\underline{x})$ for $\underline{x} \in X$, undefined elsewhere.

Let $f : X^{\subseteq \mathbb{R}^2} \to \mathbb{R}$ C^1



Let C be a curve in X.
Consider the problem of finding extreme values of $f|_C$ ("optimisation of f with constraint C")

e.g. if X is closed, let ∂X be the boundary of X, being the points of X not interior to X.
For "nice" X, ∂X is a curve.
f may have extrema on ∂X and these will be extrema of $f|_{\partial X}$

e.g. Suppose we want to 3d-print a cylindrical cup, but we only have so much plastic to extrude. What radius and height should we choose to maximize the volume?
constraint on materials $\Rightarrow$ (r,h) lies on a curve, say
$$\pi r^2 + 2\pi r h = 10$$
want to maximise the restriction of $V(r,h) = \pi r^2 h$ to this curve.

Approach 1: parametrise the curve as $\underline{c}(t)$, $\underline{c} : I^{\subseteq \mathbb{R}} \to \mathbb{R}^2$
so every point of C is $\underline{c}(t)$ for some $t \in I$

e.g. parametrise semicircle as

$\underline{c}(t) = (\sqrt{1-t^2}, t)$ $t \in [-1,1]$

then $f \circ \underline{c} : \mathbb{R} \to \mathbb{R}$
use 1-var techniques to find extrema

Approach 2: If say C is a level curve of a C^1 function $g(x,y)$
i.e. $C = \{(x,y) \mid g(x,y) = k\}$
can use Lagrange multipliers:

Th^m [Lagrange multipliers]: f, g, C as above
suppose $\underline{x} \in C$ is a local extremum of $f|_C$, and $\nabla g(\underline{x}) \neq \underline{0}$
then $\nabla f(\underline{x})$ is collinear with $\nabla g(\underline{x})$
i.e. for some $\lambda \in \mathbb{R}$, $\nabla f(\underline{x}) = \lambda \nabla g(\underline{x})$

e.g. $C = \{\|\underline{x}\| = 1\}$, $f\binom{x}{y} = x + y$
$g(x) = \|\underline{x}\|$
$\nabla g(x) = \hat{\underline{x}} = \frac{\underline{x}}{\|\underline{x}\|}$ $\nabla f(\underline{x}) = (1,1)$

Idea of Pf
Recall: a level curve of g is everywhere $\perp$ to ∇g
so STS that at a local extremum of $f|_C$, ∇f is $\perp$ to the curve.
Fact: we can find a C^1 parametrisation of part of the curve near $\underline{x}$
$\underline{c} : (-1,1) \to C$ s.t. $\underline{c}(0) = \underline{x}$, $\underline{c}'(0) \neq \underline{0}$
$\underline{x}$ local extremum
$\Rightarrow \frac{d}{dt} f(\underline{c}(0)) = 0$
$\Rightarrow \nabla f(\underline{c}(0)) \cdot \underline{c}'(0) = 0$

so $\nabla f(\underline{x}) \perp \underline{c}'(0)$ which points along the curve at $\underline{x}$
but also $g(\underline{c}(t)) = k$ so $\frac{d}{dt} g(\underline{c}(t)) = 0$
so $\nabla g(\underline{x}) \cdot \underline{c}'(0) = 0$ (level curves $\perp$ to grad)
2 vectors $\perp$, so $\nabla f(\underline{x})$, $\nabla g(\underline{x})$ collinear □

Example:
maximise $V(r,h) = \pi r^2 h$
subject to the constraint
$A(r,h) = \pi r^2 + 2\pi r h = 10$, $r \geq 0$, $h \geq 0$

~~Note: ... closed and bdd so has a global max~~

Let (r,h) be a local max$_{\text{pt}}$
$\nabla V = (2\pi r h, \pi r^2)$
~~$\nabla A = (2\pi r, 2\pi r)$~~ $\nabla A(r,h) = (2\pi r + 2\pi h, 2\pi r)$
$r = 0$ blatantly not max,
so $\nabla A(r,h) \neq 0$
A, V C^1
so by Lagrange multipliers
$\nabla V(r,h) = \lambda \nabla A(r,h)$ some λ
i.e. (1) $2\pi r h = \lambda(2\pi r + 2\pi h)$ ~~$\Rightarrow \lambda = h$~~
and (2) $\pi r^2 = \lambda 2\pi r$ ~~$\Rightarrow r = 2\lambda = 2h$~~
meanwhile, (3) $A(r,h) = \pi r^2 + 2\pi r h = 10$
~~$A(r,h) = A(2h,h) = 0$~~
~~$\pi(2h)^2 + 2\pi(2h)h = 10$~~
~~$h^2(4\pi + 4\pi) = 10$~~
~~$h = \sqrt{10/8\pi}$~~
~~$r = 2\sqrt{10/8\pi}$~~
~~only local max and closed & bdd~~
~~so must be global max~~ ~~□~~

Solve: (2) $\Rightarrow$ $r = 2\lambda$
so (1) $\Rightarrow 2\pi r h = \pi(r^2 + rh)$
$\Rightarrow rh = r^2$
$\Rightarrow r = h$
then (3) $\Rightarrow \pi r^2 + 2\pi r^2 = 10$
$\Rightarrow r^2 = \frac{10}{3\pi}$
$\Rightarrow r = \sqrt{10/3\pi}$ (+ve square root since we know $r \geq 0$!)
so $\boxed{(r,h) = (\sqrt{10/3\pi}, \sqrt{10/3\pi})}$
is a local extremum, and the only one with $r > 0$
so this must be the global max,
since $V(r,h) \xrightarrow[r \to 0]{} 0$ ...

2A03 Lecs 13, 14

Def^n: a path in $\mathbb{R}^n$ is a function
$\underline{c} : I^{\subseteq \mathbb{R}} \to \mathbb{R}^n$ where $I \subseteq \mathbb{R}$ is an interval

e.g. $\underline{c}(t) = (t, \sin t, \cos t)$ $\quad t=0$, t increasing
dom $\underline{c} = [0, \infty)$

a path $\underline{c}$ parametrises the curve it traces out

Not all curves come parametrised!
e.g. level curves
(book graphs)

The graph of a 1-var f^n
$\{(x,y) \mid y = f(x)\}$ is a curve with an obv parametrisation: $\underline{c}(t) = (t, f(t))$

Implicit f^n Th^m for plane curves

If $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1, $f(a,b) = 0$, $f_y(a,b) \neq 0$
then $C := \{(x,y) \mid f(x,y) = 0\}$ "looks locally like the graph of a $C^1 f^n$"
i.e. there exists an open interval $U \ni a$ and $g : U \to \mathbb{R}$ C^1
s.t. $g(a) = b$
$(x, g(x)) \in C \quad \forall x \in U$
$g'(x) = -\dfrac{f_x(x, g(x))}{f_y(x, g(x))}$ for $x \in U$

e.g. $y^2 = x^3 + x^2$

$f(x,y) = y^2 - x^3 - x^2$
$\nabla f = (-3x^2 - 2x, 2y)$
at e.g. $\left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$
$f_y \neq 0$ $\quad$ graph of g
$\nabla f = \left(\frac{3}{4} + 1, -\frac{1}{\sqrt{2}}\right)$
$g'(x) = -\dfrac{7/4}{-1/\sqrt{2}} = \dfrac{7\sqrt{2}}{4}$

Smooth Parametisations and arc-length

A curve can have many parametisations
e.g. $\underline{c}_1(\theta) = (\cos\theta, \sin\theta) \quad \theta \in [0, 2\pi]$
$\underline{c}_2(\theta) = (\cos\theta^2, \sin\theta^2) \quad \theta \in [0, \sqrt{2\pi}]$
$\underline{c}_3(\theta) = (\cos\theta, \sin\theta) \quad \theta \in [0, 3\pi]$
$\underline{c}_4(x) = \begin{cases} (x, \sqrt{1-x^2}) & \text{if } x \leq 1 \\ (x-2, -\sqrt{1-(x-2)^2}) & \text{if } x > 1 \end{cases} \quad x \in [-1, 3]$

A smooth parametrisation of a curve C is a path $\underline{c} : [a,b] \to \mathbb{R}$
- each point of C appears as $\underline{c}(t)$ for a unique t ("no doubling up")
- $\underline{c}$ is C^1
- $\underline{c}'(t) \neq 0$ for any t

A smooth curve is a curve which has a smooth parametrisation.

e.g. · a circle is smoothly parametrized by $\underline{c}_1$
· the graph of a C^1 f^n $f : [a,b] \to \mathbb{R}$ is sm^t param^{ed} by $(x, f(x))$
· $(t, \sin t, \cos t)$ $\quad t \in [0, 5]$

Recall: If we think of the parameter as time,
$\underline{c}(t)$ = position on curve at time t,
then $\underline{c}'(t)$ = velocity
$\underline{c}''(t)$ = acceleration
$\|\underline{c}'(t)\|$ = speed

Def^n: The length of a smooth curve is $\displaystyle\int_a^b \|\underline{c}'(t)\|\, dt$
where $\underline{c} : [a,b] \to \mathbb{R}$ is any smooth param^n

Example:

arc of a circle

could parametise as
$\underline{c}(\theta) = (\cos\theta, \sin\theta) \quad \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$
$\|\underline{c}'(\theta)\| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = 1$
$\displaystyle\int_{-\pi/4}^{\pi/4} 1\, d\theta = \frac{\pi}{2}$

or as $\underline{c}(y) = (\sqrt{1-y^2}, y) \quad y \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$
$\underline{c}'(y) = \left(\dfrac{-y}{\sqrt{1-y^2}}, 1\right)$
$\|\underline{c}'(y)\| = \sqrt{\dfrac{y^2}{1-y^2} + 1^2} = \sqrt{\dfrac{1}{1-y^2}}$

$\displaystyle\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{dy}{\sqrt{1-y^2}} = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{dy}{\sqrt{1-\sin^2\theta}}$ $\qquad y = \sin\theta$, $\dfrac{dy}{d\theta} = \cos\theta$
$\displaystyle = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{dy}{\cos\theta} = \int_{\operatorname{asin}(-1/\sqrt{2})}^{\operatorname{asin}(1/\sqrt{2})} d\theta = \int_{-\pi/4}^{\pi/4} d\theta = \frac{\pi}{2}$ phew!

pf of Lagrange: If $\underline{a} = (a,b)$ local extr^m of $f(x,y)$ on $g(x,y) = k$ and $\nabla g(a,b) \neq 0$
by implicit f^n th^m have local param^n
$\underline{c} : [-1,1] \to \mathbb{R}^2 \quad g(\underline{c}(t)) = k, \ \underline{c}(0) = \underline{a}, \ \underline{c}'(0) \neq 0$
$\underline{a}$ local extremum $\Rightarrow \frac{d}{dt} f(\underline{c}(0)) = 0$
$\Rightarrow \nabla f(\underline{c}(0)) \cdot \underline{c}'(0) = 0$
$g(\underline{c}(t)) = 0 \Rightarrow \frac{d}{dt} g(\underline{c}(t)) = 0$
$\Rightarrow \nabla g(\underline{c}(0)) \cdot \underline{c}'(0) = 0$
$\underline{c}'(0) \neq 0$, in plane, so $\nabla f(\underline{a})$ collinear with $\nabla g(\underline{a})$ $\square$

2A03 Lec 15

Canonical way to parametrise a smooth curve;

<u>Def</u>: If $\underline{c}$ is a smooth curve, the <u>parametrisation by arc-length</u> is $\underline{k}(s) = \underline{c}(\ell^{-1}(s))$

where $\underline{c}: [a,b] \to \mathbb{R}^n$ is a smooth param^n of C

and $\ell(t) = \int_a^t \|\underline{c}'(x)\|\, dx$

(<u>Note</u>: not quite unique - need to pick an orientation and a start point)

<u>Note</u>: $\underline{c}(t) = \underline{k}(\ell(t))$

never 0 by smoothness

so $\underline{c}'(t) = \underline{k}'(\ell(t))\,\ell'(t) = \underline{k}'(\ell(t))\,\|\underline{c}'(t)\|$

so $\underline{k}'(s) = \dfrac{\underline{c}'(\ell^{-1}(s))}{\|\underline{c}'(\ell^{-1}(s))\|} = \widehat{\underline{c}'(\ell^{-1}(s))}$

so <u>length from $\underline{k}(0) = \underline{c}(a)$ to $\underline{k}(s)$</u> $= \int_0^s \|\underline{k}'(x)\|\, dx = \int_0^s dx = \underline{s}$

<u>Example</u>: $\underline{a}(t) = (\sqrt{1-t^2}, t)$, $t \in [-1/\sqrt{2}, 1/\sqrt{2}]$

$$\ell(t) = \int_{-1/\sqrt{2}}^{t} \frac{dy}{\sqrt{1-y^2}} = \int_{\operatorname{asin}(-1/\sqrt{2})}^{\operatorname{asin} t} d\theta = \operatorname{asin} t - (-\pi/4)$$

so $\ell^{-1}(s) = \sin(s - \pi/4)$ (solve $s = \ell(t) = \operatorname{asin} t + \pi/4$, $t = \sin(s - \pi/4)$)

so arc-length param^n is

$$\underline{c}(s) = \underline{a}(\ell^{-1}(s)) = (\sqrt{1-\sin^2(s-\pi/4)}, \sin(s-\pi/4))$$
$$= (\cos(s-\pi/4), \sin(s-\pi/4))$$

<u>Warning</u>: $\underline{k}(s)$ often written "$\underline{c}(s)$"

<u>Curvature</u>

If $\underline{k}$ is an arclength param^n of C

$\|\underline{k}'(s)\| = 1$ for all s

$\underline{k}'(s)$ is <u>unit tangent vector</u> at $\underline{k}(s)$

$\|\underline{k}'(s)\| = 1 \Rightarrow 0 = \frac{d}{ds}\, \underline{k}'(s)\cdot\underline{k}'(s) = 2\,\underline{k}'(s)\cdot\underline{k}''(s)$

so $\underline{k}''(s) \perp \underline{k}'(s)$

$K(s) := \|\underline{k}''(s)\|$ is the <u>curvature</u> at $\underline{k}(s)$ of C

<u>Example</u>;

Helix $\underline{c}(t) = (t, \cos t, \sin t)$ $[0, 6\pi]$

$\ell(t) = \int_0^t \|\underline{c}'(x)\|\, dx = \int_0^t \sqrt{2}\, dx = \sqrt{2}\, t$

so $\underline{k}(s) = \underline{c}(\ell^{-1}(s)) = \underline{c}(s/\sqrt{2})$

$\underline{k}'(s) = \widehat{\underline{c}'(s/\sqrt{2})} = \dfrac{(1, -\sin(s/\sqrt{2}), \cos(s/\sqrt{2}))}{\sqrt{2}}$

$\underline{k}''(s) = \frac{d}{ds}\underline{k}'(s) = \left(0, -\frac{\cos(s/\sqrt{2})}{2}, -\frac{\sin(s/\sqrt{2})}{2}\right)$

$K(s) = \|\underline{k}''(s)\| = 1/2$

Often convenient to find curvature in terms of $\underline{c}(t)$ without going via $\underline{k}(s)$

$\underline{c}(t) = \underline{k}(\ell(t))$

$+\,\underline{k}'(\ell(t))$

$\underline{T}_c(t) := \widehat{\underline{c}'(t)} = \underline{k}'(\ell(t))$ = unit tangent vector at $\underline{c}(t)$

so $\underline{T}_c'(t) = \underline{k}''(\ell(t))\,\|\underline{c}'(t)\|$

Let $K_c(t) := K(\ell(t))$ = curvature at $\underline{c}(t)$

so $K_c(t) = \dfrac{\|\underline{T}_c'(t)\|}{\|\underline{c}'(t)\|}$ (note need smoothness!)

<u>Warning</u>: $\underline{T}_c(t)$ and $K_c(t)$ often just written $\underline{T}(t)$ and $K(t)$

<u>Example</u>

what is the curvature of the graph of sin?

Parametrise by x: $\underline{c}(x) = (x, \sin x)$

$\underline{c}'(x) = (1, \cos x)$ $\quad \|\underline{c}'(x)\| = \sqrt{1+\cos^2 x}$

$\underline{T}_c(x) = \dfrac{(1, \cos x)}{\sqrt{1+\cos^2 x}}$ $\quad \underline{T}_c'(x) = \sin x \cos x (1+\cos^2 x)^{-3/2} (1, \cos x) + \dots = (-\sin x \cos x (1+\cos^2 x)^{-3/2}, -\sin x (1+\cos^2 x)^{-1/2} - \sin x \cos^2 x (1+\cos^2 x) \dots)$

$\|\underline{T}_c'(x)\| = \left(\sin^2 x \cos^2 x (1+\cos^2 x)^{-3} + \dfrac{\sin^2 x}{1+\cos^2 x} + \dfrac{\sin^2 x \cos^2 x}{(1+\cos^2 x)^2} + \dfrac{\sin^2 x \cos^4 x}{\dots}\right)^{1/2}$

$\dfrac{d}{dt}\|\underline{b}(t)\| = \dfrac{d}{dt}\sqrt{\underline{b}(t)\cdot\underline{b}(t)} = \dfrac{\underline{b}'(t)\cdot\underline{b}(t)}{\|\underline{b}(t)\|}$

$= \dfrac{\sin^2 x \cos^2 x}{(1+\cos^2 x)^3} \dots$

$- \dfrac{\sin^2 x \cos^2 x}{(1+\cos^2 x)^2}$

$+ \dfrac{\sin^2 x}{1+\cos^2 x} \Big)^{1/2}$

$= \sqrt{\dfrac{\sin^2 x}{(1+\cos^2 x)^2}}$

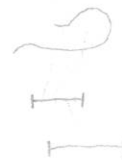
$= \dfrac{\sin x}{1+\cos^2 x}$

so $K_c(x) = \dfrac{\|\underline{T}_c'(t)\|}{\|\underline{c}'(t)\|} = \dfrac{\sin x}{(1+\cos^2 x)^{3/2}}$

2A03 Lec 16 Reparametrisations, Orientation

Sean Fraser 11·5

Def^n If $\underline{c}:[a,b]\to\mathbb{R}^n$ is a C^1 path,
a reparametrisation of $\underline{c}$
is a path $\gamma = \underline{c}\circ\phi : [c,d]\to\mathbb{R}^n$
where $\phi:[c,d]\to[a,b]$ is a C^1 bijection.



Example. $\underline{c} = \underline{k}\circ \underline{\ell}_c$

any smooth path is a reparametrisation of a parametrisation by arc length.

Note: two arc-length param^{n}s

If $\phi:[a,b]\to[c,d]$ is a bij^n, then
either (i) ϕ is strictly increasing, $\phi(a)=c$, $\phi(b)=d$
or (ii) " " decreasing, $\phi(a)=d$, $\phi(b)=c$

in case (i), we say the reparam^n $\gamma=\underline{c}\circ\phi$ is orientation-preserving
" " (ii) orientation-reversing

Example $\underline{c}(t)=(\cos t, \sin t)$ $(=k(t))$

$\phi_1(t)=2t$
$\phi_2(t)=-t$
$\phi_3(t)= (t-\pi)^3$

Note: ℓ is orientation-preserving

Def^n: a smooth curve is closed if $\underline{c}(a)=\underline{c}(b)$ where $\underline{c}:[a,b]\to\mathbb{R}^n$ is a (any) smooth param^n.

Fact: any two smooth param^{n}s of a smooth curve (assuming, in the case of a closed curve, that they have the same endpoints) are reparam^{n}s of each other.
(pf: go via arc-length params)

§5.2 4, 12, 28
§5.1 20
§3.4 24
§3.3 26
§3.1 20

2A03 Lec 17 Path integrals

Example:

Suppose a straight thin rod, placed in the plane along the x-axis, $0 \le x \le 10$, has (cross-sectional) density $\rho(x)$ $kg\,m^{-1}$. What is its mass?

Answer: $\int_0^{10} \rho(x)\,dx$.

Now suppose a curved thin rod, describing a parabolic arch $y = x^2$, $-5 \le x \le 5$, has (cross-sectional) density $\rho(x,y)$ at (x,y), e.g. $\rho(x,y) = 1+y$. What is its mass?

Or: given a loop of wire and its temp at (x,y), what is its average temp?

Def^n: If $\underline{c}:[a,b] \to \mathbb{R}^n$ is a C^1 path and if f is a $\mathbb{R}$-val^d f^n s.t. $f \circ \underline{c} : [a,b] \to \mathbb{R}$ is cont^s then the path integral of f along $\underline{c}$ is

$$\int_{\underline{c}} f\,ds := \int_a^b f(\underline{c}(t))\|\underline{c}'(t)\|\,dt$$

Def^n: If $C \subset \mathbb{R}^n$ is a smooth curve and if $f: C \to \mathbb{R}$ is cont^s then the integral of f on C is

$$\int_C f\,ds = \int_{\underline{c}} f\,ds$$

where $\underline{c}$ is any smooth para^n of C.

e.g. if $\rho(\underline{x})$ = density at x, $\int_C \rho\,ds$ is total mass

- $\int_C 1\,ds = \int_C ds = \int_{\underline{c}} \|\underline{c}'(t)\| =$ length
- Average value of f on C = $\dfrac{\int_C f\,ds}{\text{length}}$.

Explanation

Say $[a,b] = [0,1]$. f = density. Then $\int_{\underline{c}} f\,ds$

$$= \int_0^1 f(\underline{c}(t))\|\underline{c}'(t)\|\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\underline{c}(\tfrac{i}{n}))\|\underline{c}'(\tfrac{i}{n})\|\tfrac{1}{n}$$

consider $[\frac{i}{n}, \frac{i+1}{n}]$ for n large, corresponding to segment of curve between $\underline{c}(i/n)$ and $\underline{c}(\frac{i+1}{n})$.

For large n approx straight, of length $\|\underline{c}'(i/n)\| \cdot \frac{1}{n}$
↑ "speed" ↑ "time"

and value of f is approx constant.

so $f(\underline{c}(i/n))\|\underline{c}'(i/n)\|\frac{1}{n}$ ≈ mass of this segment.

Example: $C = \{y = x^2, x \in [-5,5]\}$, $\rho(x,y) = 1+y$

$\underline{c}(t) = (t, t^2)$ $t \in [-5,5]$

$\underline{c}'(t) = (1, 2t)$

$\|\underline{c}'(t)\| = \sqrt{1+4t^2}$

$$\int_C \rho\,ds = \int_{\underline{c}} \rho\,ds = \int_{-5}^{5} \rho(\underline{c}(t))\|\underline{c}'(t)\|\,dt = \int_{-5}^{5} (1+t^2)\sqrt{1+4t^2}\,dt = \ldots$$

Th^m: If γ is a reparametrisation of $\underline{c}$ then $\int_\gamma f\,ds = \int_{\underline{c}} f\,ds$

(so $\int_C f\,ds$ is well-defined)

Pf: Say $\underline{c}:[a,b] \to \mathbb{R}^n$, $\phi:[c,d] \xrightarrow{\text{bij}^n} [a,b]$

$\gamma = \underline{c} \circ \phi : [c,d] \to \mathbb{R}^n$

$$\int_\gamma f\,ds = \int_c^d f(\gamma(t))\|\gamma'(t)\|\,dt = \int_c^d f(\underline{c}(\phi(t)))\cdot\|\underline{c}'(\phi(t))\||\phi'(t)|\,dt$$

Suppose ϕ increasing so $|\phi'(t)| = \phi'(t)$, $\phi(c) = a$, $\phi(d) = b$

$$= \int_a^b f(\underline{c}(u))\cdot\|\underline{c}'(u)\|\,du \qquad \left(u = \phi(t),\ \frac{du}{dt} = \phi'(t)\right)$$

$$= \int_{\underline{c}} f\,ds$$

else ϕ is decreasing $|\phi'(t)| = -\phi'(t)$, $\phi(c) = b$, $\phi(d) = a$

$$= \int_b^a f(\underline{c}(u))\|\underline{c}'(u)\|(-1)\,du = \int_a^b f(\underline{c}(u))\|\underline{c}'(u)\|\,du = \int_{\underline{c}} f\,ds \qquad \square$$

Lec 18 Path integrals of vector fields

Def^n: If $\underline{c}:[a,b] \to \mathbb{R}^n$ is a C^1 path and $\underline{F}$ is a vector field s.t. $\underline{F} \circ \underline{c} : [a,b] \to \mathbb{R}^n$ is cont^s then the path integral (or line integral) of $\underline{F}$ along $\underline{c}$ is

$$\int_{\underline{c}} \underline{F}\cdot d\underline{s} := \int_a^b \underline{F}(\underline{c}(t))\cdot \underline{c}'(t)\,dt$$

Example:

Let $\underline{F}(\underline{x}) = \dfrac{-\underline{x}}{\|\underline{x}\|^3} k$

Let $\underline{c}(t) = (\cos t, \sin t)$ $t \in [0, \pi/2]$

$$\int_{\underline{c}} \underline{F}\cdot d\underline{s} = k\int_0^{\pi/2} \frac{-\underline{c}(t)}{\|\underline{c}(t)\|^3}\cdot \underline{c}'(t)\,dt$$

$$= k\int_0^{\pi/2} 0\,dt \quad \text{since } \underline{c}(t)\cdot\underline{c}'(t) = 0\ \forall t$$

$$= 0$$

Let $\gamma(t) = (2\cos t, \sin t)$ $t \in [0, \pi/2]$

$\gamma'(t) = (-2\sin t, \cos t)$

$\gamma(t)\cdot\gamma'(t) = -3\cos t \sin t$

$\|\gamma(t)\| = \sqrt{4\cos^2 t + \sin^2 t} = \sqrt{3\cos^2 t + 1}$

$$\int_\gamma \underline{F}\cdot d\underline{s} = k\int_0^{\pi/2} \frac{-\gamma(t)}{\|\gamma(t)\|^3}\cdot\gamma'(t)\,dt$$

$$= k\int_0^{\pi/2} \frac{3\cos t \sin t}{(3\cos^2 t + 1)^{3/2}}\,dt$$

$$= (6.2)k$$

Physics terminology:

If $\underline{F}(\underline{x})$ is a force field and $\underline{c}(t)$ is the trajectory of a particle, $\int_{\underline{c}} \underline{F}\,d\underline{s}$ is the "work done" by the force on the particle along the path.

= "change in energy"

If $\underline{F}$ is the only force acting on a rigid body, work done = change in kinetic energy ($= \frac{1}{2}m\|\underline{v}\|^2$)

Indeed $\int_{\underline{c}} \underline{F}\,d\underline{s} = \int_a^b \underline{F}(\underline{c}(t))\cdot\underline{c}'(t)\,dt$ $\qquad$ $\underline{c}(t)$ = pos^n at time t

$= \int_a^b m\underline{a}(t)\cdot\underline{V}(t)\,dt$ $\qquad$ (N2L)

$= \int_a^b \frac{m}{2}\frac{d}{dt}(v\cdot v)\,dt = \frac{1}{2}m(\|\underline{v}(b)\| - \|\underline{v}(a)\|)$

2A03 Lec 17 cont'd

Th^m: If $\underline{\gamma} = \underline{c} \circ \phi$ is a reparam^n of $\underline{c}$

then $\displaystyle\int_{\underline{\gamma}} \underline{F}\cdot d\underline{s} = \begin{cases} \int_{\underline{c}} \underline{F}\cdot d\underline{s} & \text{if } \phi \text{ is orientation preserving} \\ -\int_{\underline{c}} \underline{F}\cdot d\underline{s} & \text{reversing} \end{cases}$

Pf:
$$\int_{\underline{\gamma}} \underline{F}\cdot d\underline{s} = \int_c^d \underline{F}(\underline{\gamma}(t))\cdot \underline{\gamma}'(t)\,dt$$
$$= \int_c^d \underline{F}(\underline{c}(\phi(t)))\cdot \underline{c}'(\phi(t))\,\phi'(t)\,dt \qquad u = \phi(t)$$
$$= \int_{\phi(c)}^{\phi(d)} \underline{F}(\underline{c}(u))\cdot \underline{c}'(u)\,du$$
$$= \begin{cases} \int_a^b \underline{F}(\underline{c}(u))\cdot \underline{c}'(u)\,du = \int_{\underline{c}} \underline{F}\cdot d\underline{s} & \text{if } \phi \text{ is orientation preserving} \\ \int_b^a \cdots = -\int_{\underline{c}} \underline{F}\cdot d\underline{s} & \text{reversing} \end{cases}$$
$\square$

So e.g. in elliptic orbit example; would have got same answer if parametrised with time
So change in kinetic energy is $6\cdot 2$ J

Def^n: An oriented smooth curve is a smooth curve with a choice of orientation. So $\int \underline{F}\cdot d\underline{s}$ makes sense for $\underline{F}$ a cont^s vector field. §4.6, 5.3

Lec 18 curl and circulation

Def^n: The scalar curl of a diff^{ble} vector field $\underline{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is the f^n $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} : \mathbb{R}^2 \to \mathbb{R}$

e.g. if $\underline{F}(x,y) = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} = (ax+by, cx+dy)$
the scalar curl is constantly $c-b$

Def^n The curl of a diff^{ble} vector field $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field
$$\text{curl}\,\underline{F} = \nabla\times\underline{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Mnemonic: "$\nabla\times\underline{F} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$" (c.f. "$\underline{a}\times\underline{b} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$")

Examples: If $\underline{F}(x,y,z) = (G_1(x,y), G_2(x,y), 0)$
then $\text{curl}\,\underline{F} = (0, 0, \text{scalarcurl}(\underline{G}))$

Example: If a rigid body is spinning around an axis through the origin
then the velocity at a point is $\underline{v}(\underline{x}) = \underline{w}\times\underline{x}$
where $\underline{w}$ (the "angular velocity") points along the axis of rotation

then $\text{curl}\,\underline{v}(\underline{x}) = \text{curl}(w_2x_3 - w_3x_2,\ w_3x_1 - w_1x_3,\ w_1x_2 - w_2x_1)$
$$= (w_1 - (-w_1),\ w_2 - (-w_2),\ w_3 - (-w_3))$$
$$= 2\underline{w}$$

Example: If $\underline{F}$ is a force field it exerts a torque on a small object placed at $\underline{x}$ proportional to curl $\underline{F}$

Circulation

Def^n: A smooth curve is closed if it has a smooth param^n $\underline{c}: [a,b] \to \mathbb{R}^n$ with $\underline{c}(a) = \underline{c}(b)$.

$\underline{c}(a) = \underline{c}(b)$

Def^n: Let C be a smooth closed oriented curve.
The integral $\int_C \underline{F}\cdot d\underline{s}$ of a cont^s vector field is called the circulation of $\underline{F}$ around C.

Fact: "curl is infinitesimal circulation":
If $\hat{\underline{u}}$ is a unit vector, $\underline{x}\in\mathbb{R}^3$, $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ C^1
if we take smooth closed curves C_i in the plane through $\underline{x}$ orthogonal to $\hat{\underline{u}}$, oriented right-handedly around $\underline{u}$, with the maximal distance from $\underline{x}$ on C_i tending to 0 as $i\to\infty$
then $\displaystyle\lim_{i\to\infty}\left(\frac{1}{\text{area}(C_i)}\int_{C_i}\underline{F}\cdot d\underline{s}\right) = \hat{\underline{u}}\cdot\text{curl}(\underline{F})$

$\hat{u}$

Fact: For s.c. domain curl$(\underline{F}) = \underline{0}$ everywhere "irrotational"
iff all circulations are 0.

These are consequences of Stokes' Th^m

2A03 Lec 19

Def^n: A vector field $\underline{F}: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a gradient vector field if $\underline{F} = \nabla f$ for some $f : U \to \mathbb{R}^n$

Example A force field $\underline{F}$ is conservative iff it's a gradient vector field.

Then $\underline{F} = -\nabla \phi$ for some ϕ; such a ϕ is called a potential for the force

e.g. $\underline{F} = -k\frac{\underline{x}}{\|\underline{x}\|^3} = -\nabla\left(\frac{-k}{\|\underline{x}\|}\right)$

$\frac{-k}{\|\underline{x}\|}$ is "gravitational potential energy"

Example: Let $\underline{v}(\underline{x}, t)$ be the velocity of a fluid at $\underline{x}$ at time t.
Understanding such $\underline{v}$ is the subject of fluid mechanics.
A fluid is said to be under "potential flow" if $\underline{v}(\underline{x}, t) = -\nabla\phi(t)$ for some $\phi(t)$
i.e. if for any t, $\underline{v}(\underline{x}, t)$ is a gradient vector field
(so problem reduced to understanding evolution of ϕ...)

Lemma 2: If $\underline{F}: U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 gradient vector field then $\operatorname{curl} \underline{F} = \underline{0}$

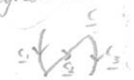
Proof: Say $\underline{F} = \nabla f$
Then e.g. $(\operatorname{curl} \underline{F})_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial f}{\partial y \partial z} - \frac{\partial f}{\partial z \partial y} = 0$
(since f is C^2, since $\underline{F}$ is C^1) $\square$

Converse not quite true, e.g.
$\underline{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$; $\underline{F}: \mathbb{R}^3 \setminus \{x=0=y\} \to \mathbb{R}^3$

$$\operatorname{curl} \underline{F} = \left(0, 0, \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)$$
$$= \left(0, 0, \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right)$$
$= \underline{0}$ everywhere on $\operatorname{dom} \underline{F} = \mathbb{R}^3 \setminus \{x=0=y\}$

but let $\underline{c}(t) = (\cos t, \sin t, 0)$ $t \in [0, 2\pi]$
then $\int_c \underline{F} \cdot d\underline{s} = \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt = 2\pi$

Def^n: A path is piecewise smooth (p.s.) if it is formed by joining end-to-end finitely many smooth paths. A curve is p.s. if it has a p.s. parametrisation, or is a finite union of such curves.
An Integral over a p.s. path is the sum of the integrals over the component smooth paths.



$$\int_c = \int_{c_1} + \int_{c_2} + \int_{c_3}$$

From now on, unless otherwise specified, we will assume all paths/curves over which we integrate to be p.s.

Lemma 1: If $\underline{F} = \nabla f$ is a gradient vector field and $\underline{c}: [a,b] \to \mathbb{R}^n$ is a smooth path,
$\int_c \underline{F} \cdot d\underline{s} = f(\underline{c}(b)) - f(\underline{c}(a))$.

In particular, all circulations are 0.

Pf: $\int_c F \cdot d\underline{s} = \int_c \nabla f \cdot d\underline{s} = \int_a^b \nabla f \cdot \underline{c}'(t)\, dt$
$= \int_a^b \frac{d}{dt}(f(\underline{c}(t)))\, dt$
$= f(\underline{c}(b)) - f(\underline{c}(a))$ (FTC) $\square$

Def^n: $X \subseteq \mathbb{R}^n$ is simply connected (s.c.) if (i) any two points of X are connected by a path in X ("path-connected")
and (ii) any closed path in X can be "continuously contracted" to a point in X without leaving X

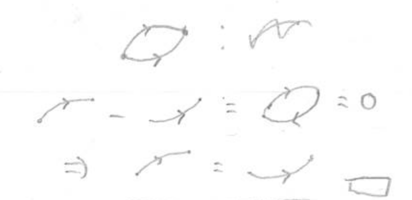
Examples: $\mathbb{R}^n$ is s.c.
$\mathbb{R}^2 \setminus \{0\}$ is not s.c.
$\mathbb{R}^3 \setminus \{x=0=y\}$ is not s.c.
$\mathbb{R}^3 \setminus \{0\}$ is s.c.
A torus is not s.c.
A sphere is s.c.
$\mathbb{R}^3 \setminus$ cylinder is s.c.

Fact: If $\underline{F}: U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ and $U = \operatorname{dom} \underline{F}$ is open and s.c.
then if $\operatorname{curl} \underline{F} = \underline{0}$ on U
then any circulation $\oint_c \underline{F} \cdot d\underline{s}$, c a closed oriented p.s. curve, is 0.
("irrotational on s.c. $\Rightarrow$ circulation-free")

This is a consequence of Stokes' theorem, which we'll hopefully eventually (mostly) prove.

Example
If $\underline{F}$ is a conservative force field say $\underline{F} = -\nabla\phi = \nabla(-\phi)$
then if a particle is acted on by $\underline{F}$ as it moves along a trajectory c from $\underline{\alpha} = \underline{c}(a)$ to $\underline{\beta} = \underline{c}(b)$, then the work done is
$\int_c \underline{F} \cdot d\underline{s} = -\phi(\underline{\beta}) - (-\phi(\underline{\alpha})) = \phi(\underline{\alpha}) - \phi(\underline{\beta})$
(doesn't depend on the curve!)
Recall (ignoring other forces) this is the change in kinetic energy, so if we interpret ϕ as potential energy, we have
change in energy = change in kin. energy + change in pot energy
$= (\phi(\underline{\alpha}) - \phi(\underline{\beta})) + (\phi(\underline{\beta}) - \phi(\underline{\alpha})) = 0$
so for conservative forces, (potential energy + kin energy) is conserved.

Lemma 3: If all circulations of $\underline{F}$ are 0 then any $\int_c \underline{F} \cdot d\underline{s}$ depends only on the end-points of c
("circulation-free $\Leftrightarrow$ path-independent")
"Pf" by diagram.



Lemma 4 "path-independent $\Leftrightarrow$ gradient": If any $\int_c \underline{F} \cdot d\underline{s}$ depends only on the end points then $\underline{F}$ is a gradient vector field

2A03 Lec 21

Outline:

- Stokes' Motivation
- Integrability of $f : R \to \mathbb{R}$ rectangles
- Fubini
- Integrability of $f : D \to \mathbb{R}$ D bounded
- Elementary regions, integration thereon
- piecewise elementary regions
- 3d analogues

<u>Stokes' Th^m</u> (rough version):

Let $\underline{F} : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ C^1 vector field

Let $S \subseteq \mathbb{R}^3$ be a "nice" surface with boundary ∂S a closed p.s. curve

Then

$$\int_{\partial S} \underline{F} \cdot d\underline{s} = \text{"the integral of curl } \underline{F} \text{ on } S\text{"}$$

To make sense of such integrals on surfaces, we'll do as we did for curves:

- parametrise the surface, putting it (piecewise) in correspondence with a region in the plane
- define the integral on the surface as an integral on that region

So first, we should understand

<u>Integrals on regions in the plane, and in $\mathbb{R}^3$</u>

<u>Rectangles</u>:

Let $R = [a,b] \times [c,d] \subseteq \mathbb{R}^2$, a closed rectangle.

Let $f : R \to \mathbb{R}$

<u>Def^n</u>: f is <u>integrable</u> if the limit of Riemann sums

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(x_{i,n}, y_{j,n}) \frac{\Delta x \Delta y}{n^2}$$

has a finite value which does not depend on the choice of

$x_{i,n} \in [a + i\Delta x, a + (i+1)\Delta x]$

$y_{j,n} \in [c + j\Delta y, c + (j+1)\Delta y]$

We then write $\iint_R f \, dA$ for this value.

<u>Fact [Fubini]</u>:

For $f : R \to \mathbb{R}$ integrable

$$\iint_A f \, dA = \int_c^d \int_a^b f(x,y) \, dx \, dy = \int_a^b \int_c^d f(x,y) \, dy \, dx$$

e.g. $\iint_R xy \, dA = \int_c^d \int_a^b xy \, dx \, dy = \int_c^d y\left(\frac{b^2 - a^2}{2}\right) dy$

$= \frac{(d^2 - c^2)}{2} \frac{(b^2 - a^2)}{2}$

Actually, recall: for f and g integrable,

$$\iint_R f(x) g(y) \, dA = \int_c^d \int_a^b f(x) g(y) \, dx \, dy = \int_c^d g(y) \left(\int_a^b f(x) \, dx \right) dy$$

$$= \left(\int_a^b f(x) \, dx \right) \left(\int_c^d g(y) \, dy \right)$$

Now suppose D is a bounded region in $\mathbb{R}^2$, $f : D \to \mathbb{R}$

Let $R = [a,b] \times [c,d]$ be a closed rectangle containing D

Extend f to $f_0 : R \to \mathbb{R}^2$

$$f_0(\underline{x}) = \begin{cases} f(\underline{x}) & \text{if } \underline{x} \in D \\ 0 & \text{else} \end{cases}$$

<u>Def^n</u>: $f : D \to \mathbb{R}$ is integrable if $f_0 : R \to \mathbb{R}$ is, and then $\iint_D f \, dA := \iint_R f_0 \, dA$

(easily, this doesn't depend on R)

But this is hard to calculate directly.

<u>Def^n</u>: $D \subseteq \mathbb{R}^2$ of the form

$D = \{(x,y) \mid x \in [a,b]$
$\phi(x) \le y \le \psi(x)\}$

where $\phi, \psi : [a,b] \to \mathbb{R}$ cont^s

$\phi(x) \le \psi(x)$ for $x \in [a,b]$

("region between graphs of two f^{ns} of x on a closed interval")

is called an <u>x-simple</u> region.

y-simple: same with vars swapped:

$\{(x,y) \mid y \in [c,d], \phi(y) \le x \le \psi(y)\}$

$\phi, \psi : [c,d] \to \mathbb{R}$ cont^s, $\phi \le \psi$.

<u>Lec 22</u>

<u>Def^n</u>: An <u>elementary region</u> is a $D \subseteq \mathbb{R}^2$ which is either x-simple or y-simple.

- A <u>piecewise elementary region</u> is a $D \subseteq \mathbb{R}^2$ which is a finite union $D = D_1 \cup \ldots \cup D_n$ of elementary regions

which are "almost disjoint" - intersect only on the boundaries

i.e. for any $i \ne j$, any point in the intersection $D_i \cap D_j$ is in the boundary of D_i and in the boundary of D_j

i.e. $D_i \cap D_j = \partial D_i \cap \partial D_j$

e.g.

<u>Fact</u>: if D is elementary and $f : D \to \mathbb{R}$ is cont^s, then f is integrable.

If $D = D_1 \cup \ldots \cup D_n$ almost disjoint elementary, then $\iint_D f \, dA = \iint_{D_1} f \, dA + \ldots + \iint_{D_n} f \, dA$

If D is x-simple, say $D = \{(x,y) \mid x \in [a,b], \phi(x) \le y \le \psi(x)\}$

then $\iint_D f \, dA = \int_a^b \int_{\phi(x)}^{\psi(x)} f(x,y) \, dy \, dx$

If D is y-simple, say $D = \{(x,y) \mid y \in [c,d], \phi(y) \le x \le \psi(y)\}$

then $\iint_D f \, dA = \int_c^d \int_{\phi(y)}^{\psi(y)} f(x,y) \, dx \, dy$

<u>Fact</u>: if D is p.e. and $f : D \to \mathbb{R}$ cont^s, then f int^{ble} and

<u>Example</u>: Let D be the semicircle

$$\iint_D xy^2 \, dA = \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy^2 \, dy \, dx \quad \text{(viewing it as a single } x\text{-simple region)}$$

$$= \frac{1}{3} \int_0^1 x \left((1-x^2)^{3/2} + (1-x^2)^{3/2} \right) dx$$

$$= \frac{-1}{3} \int_1^0 u^{3/2} \, du \qquad \begin{pmatrix} u = 1 - x^2 \\ \frac{du}{dx} = -2x \end{pmatrix}$$

$$= \frac{-2}{15}(0 - 1) = \underline{2/15}$$

So also $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} xy^2 \, dx \, dy = 2/15$, and so on.

2A03 Lec 22 cont^d

Example (6-20)

$D :=$ region between $y=x$ and $y=\sqrt[3]{x}$ in first quadrant $(x \geq 0, y \geq 0)$

$\iint_D e^{x/y}\, dA$

D is x-simple

$D = \{(x,y) \mid 0 \leq x \leq 1,\ x \leq y \leq \sqrt[3]{x}\}$

so $\iint_D e^{x/y}\, dA = \int_0^1 \int_x^{\sqrt[3]{x}} e^{x/y}\, dy\, dx$

But $\int e^{x/y} dy$ has no elementary solution!

Luckily...

D is also y-simple

$D = \{(x,y) \mid 0 \leq y \leq 1,\ y^3 \leq x \leq y\}$

so $\iint_D e^{x/y}\, dA = \int_0^1 \int_{y^3}^{y} e^{x/y}\, dx\, dy = \int_0^1 y\left(e^{y/y} - e^{y^3/y}\right) dy$

$= \int_0^1 \left(ye - ye^{y^2}\right) dy$

$= \frac{1}{2}e - \frac{1}{2}\int_0^1 e^u\, du \quad (u = y^2)$

$= \frac{1}{2}e - \frac{1}{2}(e-1) = \underline{1/2}$

Example: Evaluate $\int_0^2 \int_y^2 e^{x^2}\, dx\, dy$

Again, $\int e^{x^2} dx$ is not elementary, so can't do directly.

But $\int_0^2 \int_y^2 e^{x^2}\, dx\, dy = \iint_D e^{x^2}\, dA$ $\quad D =$ 

$= \int_0^2 \int_0^x e^{x^2}\, dy\, dx$

$= \int_0^2 x e^{x^2}\, dx = \frac{1}{2}\left[e^{x^2}\right]_0^2 = \frac{1}{2}(e^4 - 1)$

(α, β)

Lec 23

Triple integrals $\iiint_W f\, dV$

Def^n: $W \subseteq \mathbb{R}^3$ is xy-simple if $W = \{(x,y) \in D,\ K_1(x,y) \leq z \leq K_2(x,y)\}$ where D is elementary, $K_i: D \to \mathbb{R}$ are cont^s and $K_1 \leq K_2$ on D.

Then for f cont^s $: D \to \mathbb{R}$,

$$\iiint_W f\, dV = \iint_D \int_{K_1(x,y)}^{K_2(x,y)} f(x,y,z)\, dz\, dA$$

so eg if D is x-simple, $D = \{a \leq x \leq b,\ \phi(x) \leq y \leq \psi(x)\}$

$= \int_a^b \int_{\phi(x)}^{\psi(x)} \int_{K_1(x,y)}^{K_2(x,y)} f(x,y,z)\, dz\, dy\, dx$

- Analogously for yz-simple, xz-simple.
- W is elementary if it's xy-simple or yz-simple or xz-simple
- W is p.e. if it's the almost-disjoint union of finitely many elementary W_i, as in 2d case.

Example: If W is the unit sphere and $f: W \to \mathbb{R}$ is cont^s, let's express $\iiint_W f\, dV$ in terms of integrals in 1 variable.

$W = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$

$= \{(x,y,z) \mid x^2+y^2 \leq 1,\ -\sqrt{1-(x^2+y^2)} \leq z \leq \sqrt{1-(x^2+y^2)}\}$

$= \{(x,y,z) \mid -1 \leq x \leq 1,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},\ -\sqrt{1-(x^2+y^2)} \leq z \leq \sqrt{1-(x^2+y^2)}\}$

so $\iiint_W f\, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-(x^2+y^2)}}^{\sqrt{1-(x^2+y^2)}} f(x,y,z)\, dz\, dy\, dx$.

Area and Jacobians

~~Change of variables~~

Remark: $\text{Area}(D) = \iint_D 1\, dA$, $\text{Vol}(W) = \iiint_W 1\, dV$

Let $D^* \subseteq \mathbb{R}^2$ elementary

let $T: D^* \to \mathbb{R}^2$ be 1-1

consider $D := T(D^*) = \text{im}\, T = \{T(u,v) \mid (u,v) \in D^*\}$

Question: what is $\text{Area}(D)$?

Example: $T(u,v) = (u+3, v-7)$

Clearly, $\text{Area}(T(D^*)) = \text{Area}(D^*) = \iint_{D^*} 1\, dA^*$

Lec 23 cont^d

Example: $T(u,v) = (2u, 3v)$

Area$(D) = 6\,$Area(D^*)

Example $T(u,v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$ (affine)

Recall: If A is an $n\times n$ matrix, $n=2$ or 3,

then the area/vol of the image of the unit box $[0,1]^n$ is $|\det A|$

(pf: e.g. $n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the matrix of T

$A\underline{i} = \begin{pmatrix} a \\ c \end{pmatrix}$ $A\underline{j} = \begin{pmatrix} b \\ d \end{pmatrix}$

want to see that the area of the parallelogram is $|\det A|$

clear for $A = I$

check preserved by elementary col ops:

- scalar mult of a col ✓
- swapping cols ✓
- adding a multiple of one col to another (shear) ✓)

So Area$(D) = \left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$ Area(D^*) $= \iint_{D^*} \left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| dA^*$

General case:

Assume T is C^1.

Fact: Area$(D) = \iint_{D^*} |\det(DT(u,v))|\, dA^*$

Jacobian determinant

why? Near any point (u_0,v_0)

T is approximately affine

$T(u_0+u, v_0+v) \approx DT(u_0,v_0)\begin{pmatrix} u \\ v \end{pmatrix} + T(u_0,v_0)$

so "stretching factor" near $(u_0,v_0) \approx |\det(DT(u_0,v_0))|$

Example – Area of unit disc

Jac. gp (1): $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ (2) $\left(\frac{x^2(y+1)}{2}, \frac{y^2(x+1)}{2}\right)$

Analogue in 1d:

Let $f: I^* = [a,b] \to [c,d] = I$ be C^1 and 1-1

Then

length$(I) = d-c = \int_c^d 1\,dx = \int_a^b |f'(u)|\,du$ $\quad$ ($x = f(u)$, $\frac{dx}{du} = f'(u)$)

(need $|f'(u)|$; if f is decreasing, $\int_c^d 1\,dx = \int_b^a f'(u)\,du = \int_a^b |f'(u)|\,du$)

We know more: for $g: [c,d] \to \mathbb{R}$ integrable

$\int_c^d g(x)\,dx = \int_a^b g(f(u))\,|f'(u)|\,du$

Lec 24

Fact [Change of Variables]:

Let D and D^* be piecewise C^1 regions in $\mathbb{R}^2$

Let $T: D^* \to D$ be C^1, onto, and 1-1 on the interior of D^*

Let $g: D \to \mathbb{R}$ be integrable

then $\iint_D g(x,y)\,dA = \iint_{D^*} g(T(u,v))\,\|DT(u,v)\|\,dA^*$

book's notation: $\iint_{D^*} g(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

The corresponding statement holds in 3d $\iiint_W h(x,y,z)\,dV = \iiint_{W^*} h(T(u,v,w))\,\|DT(u,v,w)\|\,dA^*$

(and in 1d, as above)

(generally: $\int_Y g(\underline{x})\,d\underline{x} = \int_X g(\underline{u})\,\|DT(\underline{u})\|\,d\underline{u}$ $\quad T: X \to Y$ C^1 bij^n)

Example: Find $\iint_D e^{-(x^2+y^2)/2}\,dA$ where D is the disc region

T polar $\to$ cartesian

$T(r,\theta) = (r\cos\theta, r\sin\theta)$

$|DT| = \left|\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| = |r\cos^2\theta + r\sin^2\theta| = |r| = r$

so $\iint_D e^{-(x^2+y^2)/2}\,dA = \iint_{D^*} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)/2}\, r\,dA^*$ $\quad D^* = [0,1]\times[0,2\pi]$

$= \int_0^b\int_0^{2\pi} e^{-r^2/2}\,d\theta\,dr = 2\pi\int_0^{b^2/2} e^{-u}\,du = 2\pi\,(1 - e^{-b^2/2})$

$\xrightarrow[b\to\infty]{} 2\pi$

(so $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dA = 2\pi$!)

13.5 ~~13.5~~ 11.5 → 13.5

1350244

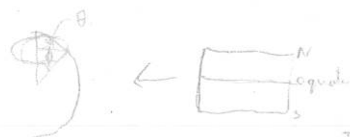
2A03 Lec 25, which actually is Lec 29
(numbering messed up somewhere(s)...)

Def^n: A parametrisation of a surface $S \subseteq \mathbb{R}^3$
is a map $\underline{r} : D \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}^3$
which is 1-1 on the interior of D,
and whose image is S, $\underline{r} : D \twoheadrightarrow S$

Examples:

- $f : \mathbb{R}^2 \to \mathbb{R}$
  $\underline{r}(x,y) := (x, y, f(x,y))$
  graph of f

- unit sphere:
  $\underline{r}(\phi, \theta) = (\cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi)$
  $r : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$



(ϕ = "inclination" / "zenithal angle")
(θ = "azimuth")

Suppose $\underline{r}$ is C^1.

$D\underline{r}(\underline{u}) : \mathbb{R}^2 \to \mathbb{R}^3$ linear $\quad (\underline{u} = (u, v))$

$D\underline{r}(\underline{u}) \begin{pmatrix} a \\ b \end{pmatrix}$ is a tangent vector to S at $\underline{r}(\underline{u})$

$\underline{T}_u(\underline{u}) := D\underline{r}(\underline{u}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial \underline{r}}{\partial u}(\underline{u})$

$\underline{T}_v(\underline{u}) := D\underline{r}(\underline{u}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial \underline{r}}{\partial v}(\underline{u})$

$\underline{N}(\underline{u}) := \underline{T}_u(\underline{u}) \times \underline{T}_v(\underline{u})$ is, if non-zero, normal to S at $\underline{r}(\underline{u})$

Note $\underline{N}(\underline{u}) = \underline{0}$ iff $\dim(\mathrm{Im}\, D\underline{r}(\underline{u})) < 2$

Def^n: A param^n $\underline{r} : D \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}^3$ is smooth
if it is C^1 and $\underline{N}(\underline{u}) \neq 0$ for all $\underline{u} \in D$.
A surface is smooth if it has a smooth param^n.

Spherical co-ordinates – leaving to Jamal

$\alpha : [0, \infty] \times [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$

$\alpha(\rho, \phi, \theta) := (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$

is onto $\mathbb{R}^3$, and 1-1 on the interior of dom α, and C^1

$$D\alpha = \begin{pmatrix} \cos\theta\sin\phi & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\theta & -\rho\sin\phi & 0 \end{pmatrix}$$

$$\|D\alpha\| = \rho^2 \sin\phi$$

Example: B := unit ball

$$\iiint_B x^2y^2z^2\, dV = \iiint_{[0,1]\times[0,\pi]\times[0,2\pi]} \rho^6 \cos^2\theta \sin^2\theta \sin^4\phi \cos^2\phi\, \rho^2 \sin\phi\, dV^*$$

Example: $f : \mathbb{R}^2 \to \mathbb{R}$

$\underline{r}(u, v) = (u, v, f(u,v))$
parametrises the graph of f
and is smooth: $\underline{N}(u,v) = \underline{T}_u(u,v) \times \underline{T}_v(u,v)$
$= (1, 0, f_u) \times (0, 1, f_v)$
$= (-f_u, -f_v, 1) \neq \underline{0}$

$$D\underline{r} = \begin{array}{c} \begin{array}{cc} T_\phi & T_\theta \end{array} \\ \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\cos\phi & \cos\theta\sin\phi \\ -\sin\phi & 0 \end{pmatrix} \end{array}$$

$\underline{N}(\phi, \theta) = T_\phi \times T_\theta = (\cos\theta\sin^2\phi,\ \sin\theta\sin^2\phi,\ \underbrace{\cos^2\theta\cos\phi\sin\phi + \sin^2\theta\cos\phi\sin\phi}_{= \cos\phi\sin\phi})$

$= \underline{0}$ iff $\sin\phi = 0$ iff $\phi = 0$ or π
not smooth $\quad$ (iff $T_\theta = 0$)
(hairy ball)

Orientation

Def^n: An orientation of a surface S
is a continuous choice of a unit normal vector at each point of S, i.e. $\underline{n} : S \to \mathbb{R}^3$ cont^s s.t. $\|\underline{n}(s)\| = 1$ and normal.
S is orientable (or two-sided) if it has an orientati[on]
Note it then has two.

Remark: Non-orientable (one-sided) surfaces exist,
e.g. Möbius strip

Def^n: A param^n $\underline{r} : D \twoheadrightarrow S$
preserves a chosen orientation $\underline{n}$ on S
if $\underline{N}(\underline{u}) = \underline{n}(\underline{r}(\underline{u}))\ \forall \underline{u}$

reverses

2 May 5

Lec 30 $\iint_S f\,dS$

e.g. $\rho(s)$ = surface density at s

$\iint_S \rho\,dS$ = mass of S

· $\iint_S 1\,dS$ = area of S

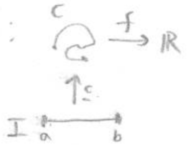
· $\dfrac{\iint_S f\,dS}{\iint_S 1\,dS}$ = av. value of f on S

So $\iint_S f\,dS$ = Riemann sum

"limit of sum of product of area of little pieces of S and value of f at (a point on) that piece as we let the area of a piece tend to 0"

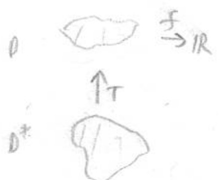
when this gives a well-defined number.

Recall:



$\int_C f\,ds = \int_I (f\circ c)\cdot|c'|\,dt_I$

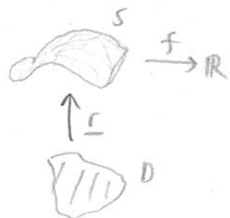
$= \int_a^b f(c(t))\cdot\|c'(t)\|\,dt$



$\iint_D f\,dA = \iint_{D^*} (f\circ T)\cdot\|DT\|\,dA^*$

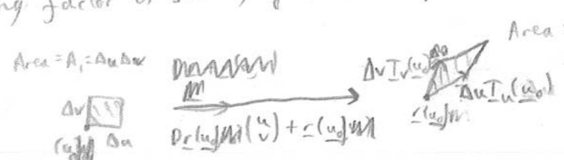
$= \iint_{D^*} f(T(u,v))\cdot\|DT(u,v)\|\,dA^*$

Now:



$\iint_S f\,dS = \iint_D (f\circ \underline{r})\cdot\boxed{?}\,dA$

Missing factor is scaling of area at a point



Recall: Area of parallelogram in $\mathbb{R}^3$ spanned by $\underline{a}, \underline{b}$

is $(\|\underline{a}\|\cdot\|\underline{b}\|\cdot\sin\theta)$

$= \|\underline{a}\times\underline{b}\|$

So $A_2 = \Delta u\Delta v\,\|\underline{T}_u(\underline{u}_0)\times\underline{T}_v(\underline{u}_0)\|$

so scaling factor is $\|\underline{T}_u(\underline{u}_0)\times\underline{T}_v(\underline{u}_0)\| = \|\underline{N}(\underline{u}_0)\|$

Fact:

If S is smoothly parametrised by $\underline{r}: D\to\mathbb{R}^3$, and $f: S\to\mathbb{R}$ is cont^s,

then $\iint_S f\,dS = \iint_D (f\circ\underline{r})\,\|\underline{N}\|\,dA$

$= \iint_D f(\underline{r}(u,v))\cdot\|\underline{N}(u,v)\|\,dA$

when this integral exists, which it does if D is piecewise cl^y.

Def^n: A piecewise smooth para^n of a surface S is a collection of $\underline{r}_1,\dots,\underline{r}_n$, $\underline{r}_i: D_i\to\mathbb{R}^3$, each smooth except maybe at finitely many points of S_i ($\underline{N}_i(\underline{u})=0$ for finitely many $\underline{r}_i(\underline{u})$)

$\underline{r}_i$ parametrising $S_i\subseteq S$

such that the S_i cover S $\quad(S = S_1\cup\dots\cup S_n)$

and the S_i are essentially disjoint (no interior points in common)

Fact: then $\iint_S f\,dS = \sum_i \iint_{S_i} f\,dS$

$= \sum_i \iint_{D_i} f(\underline{r}_i(\underline{u}))\,\|\underline{N}_i(\underline{u})\|\,dA$

Example

"Ice cream ball & cone"

$S := \{(x,y,z) \mid (z\geq 0 \text{ and } x^2+y^2+z^2=1)$

$\text{or } (-1\leq z\leq 0 \text{ and } x^2+y^2=z+1)\}$

surface area

$= \iint_S 1\,dS = \iint_{\text{hemisphere}} 1\,dS + \iint_{\text{cone}} 1\,dS$

$= \iint_{D_1} \|\underline{N}_1\|\,dA + \iint_{D_2} \|\underline{N}_2\|\,dA$

$= \int_0^{2\pi}\int_0^{\pi/2} \sin\phi\,d\phi\,d\theta + \int_{-1}^{0}\int_0^{2\pi} \sqrt{2}(z+1)\,d\theta\,dz$

$= \dots$

$\underline{r}_1(\phi,\theta) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$D_1 = [0,\pi/2]\times[0,2\pi]$

$\underline{N}_1 = (\cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \cos\phi\sin\phi)$

$\|\underline{N}_1\| = \sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \cos^2\phi\sin^2\phi}$

$= \sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}$

$= \sqrt{\sin^2\phi}$

$= |\sin\phi|$

$= \sin\phi$

$\underline{r}_2(\theta,z) = ((z+1)\cos\theta, (z+1)\sin\theta, z)$

$D_2 = [0,2\pi]\times[-1,0]$

$\|\underline{N}_2\| = \|(-(z+1)\sin\theta, (z+1)\cos\theta, 0)\times(\cos\theta,\sin\theta,1)\|$

$= \sqrt{(z+1)^2\cos^2\theta + (z+1)^2\sin^2\theta + (-(z+1)(\sin^2\theta+\cos^2\theta))^2}$

$= \sqrt{2(z+1)^2}$

$= \sqrt{2}(z+1)$

2A03 Lec 31 Flux

Def^n: An <u>oriented surface</u> is a surface S equipped with a choice of orientation $\underline{n}: S \to \mathbb{R}^3$

A param^n of an oriented surface is a param^n such that whenever $\underline{N}(\underline{u}) \neq 0$, $\hat{N}(\underline{u}) := \frac{\underline{N}(\underline{u})}{\|\underline{N}(\underline{u})\|} = \underline{n}(\underline{r}(\underline{u}))$

Def^n: Given S an oriented$^\wedge$ piecewise smooth surface, $\underline{n}: S \to \mathbb{R}^3$ its orientation, and $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a cts vector field the <u>surface integral of $\underline{F}$ on S</u> also known as the <u>flux of $\underline{F}$ through S</u> is $\iint_S \underline{F} \cdot d\underline{S} := \iint_S (\underline{F} \cdot \underline{n})\, dS$

If $\underline{r}$ is a smooth para^n of S, then

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_S (\underline{F} \cdot \underline{n})\, dS$$

$$= \iint_D (\underline{F} \cdot \underline{n})(\underline{u}) \|\underline{N}(\underline{u})\|\, dA$$

$$= \iint_D \underline{F}(\underline{u}) \cdot \frac{\underline{N}(\underline{u})}{\|\underline{N}(\underline{u})\|} \|\underline{N}(\underline{u})\|\, dA$$

$$\boxed{\iint_S \underline{F} \cdot d\underline{S} = \iint_D \underline{F}(\underline{u}) \cdot \underline{N}(\underline{u})\, dA}$$

<u>Interpretation:</u>

If $\underline{F}(\underline{x})$ represents some sort of rate of flow $\iint_S \underline{F} \cdot d\underline{S}$ is total rate of flow through S

e.g. if $\underline{v}(\underline{x})$ is velocity at $\underline{x}$ of a fluid then the flux $\iint_S \underline{v} \cdot d\underline{S}$ is the rate of flow of the fluid through S.

e.g. $\underline{v}(\underline{x}) = (0,0,1)$

flow rate = area if head-on

$\iint_S (0,0,1) \cdot \underline{n}$
$= \cos\theta \cdot \text{Area}$
$=$ area of projection
$=$ flow rate

Let S be the graph of a function $f: D \to \mathbb{R}$ where D is the unit disc, oriented upwards

Then $\iint_S \underline{v} \cdot d\underline{S} = \iint_D \underline{v} \cdot (-f_y, -f_x, 1) = \iint_D 1 = 2\pi$

Example: $\underline{F}(\underline{x}) := \frac{\underline{x}}{\|\underline{x}\|^3}$ $S :=$ sphere at 0 of radius r, oriented outwards

Don't parametrise!

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_S \underline{F} \cdot \underline{n}\, dS = \iint_S \underline{F}(\underline{x}) \cdot \frac{\underline{x}}{\|\underline{x}\|}\, dS$$

$$= \iint_S \frac{\underline{x} \cdot \underline{x}}{\|\underline{x}\|^4}\, dS$$

$$= \iint_S \frac{1}{\|\underline{x}\|^2}\, dS$$

$$= \iint_S \frac{1}{r^2}\, dS \quad \text{since } \|\underline{x}\| = r \text{ on } S!$$

$$= \frac{1}{r^2} \iint_S 1\, dS = \frac{4\pi r^2}{r^2} = \underline{4\pi}$$

Example $\underline{v}(\underline{x}) = (0,0,1)$, $S =$ unit sphere, oriented outwards

split into hemispheres $z \geq 0$ and $z \leq 0$ S_1, S_2

each is graph of a f^n as above, oriented up resp. down

so $\iint_S \underline{v} \cdot d\underline{S} = \iint_{S_1} \underline{v} \cdot d\underline{S} + \iint_{S_2} \underline{v} \cdot d\underline{S} = 2\pi - 2\pi = \underline{0}$

Why?

2A03 Lee Penult

Special case of Stokes':
$D \subseteq \mathbb{R}^2$ p.c., ∂D p.s.
S_v surface in $z=0$ plane corresponding, oriented up

$\underline{F}: \mathbb{R}^2 \to \mathbb{R}^2$ vector field

$\underline{G}: \mathbb{R}^3 \to \mathbb{R}^3$, $\underline{G}(x,y,z) = (F_1(x,y), F_2(x,y), 0)$

so $\operatorname{curl} \underline{G} = (0, 0, \operatorname{scurl} \underline{F})$
$= \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

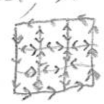
so by Stokes', $\int_{\partial S} \underline{G} \cdot d\underline{s} = \iint_S \operatorname{curl} \underline{G} \cdot d\underline{S}$

so Th^n [Green]: $\int_{\partial D} \underline{F} \cdot d\underline{s} = \iint_D \operatorname{scurl} \underline{F} \, dA$

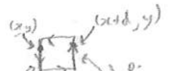
Sketch pf

Suppose $D = [0,1]^2$
split into n^2 squares of side length $d = 1/n$, D_i, $x_i \in D_i$
contributions on common sides cancel, so

$\sum_i \int_{\partial D_i} \underline{F} \cdot d\underline{s} = \int_{\partial D} \underline{F} \cdot d\underline{s}$



$\lim_{d \to 0} \int_{\partial D_i} \underline{F} \cdot d\underline{s} = d^2 \operatorname{scurl} \underline{F}(x_i)$



$F_2(x+d,y) - F_2(x,y) \approx d \frac{\partial F_2}{\partial x}$

so $\int_{\partial_R D_i} \underline{F} \cdot d\underline{s} + \int_{\partial_L D_i} \underline{F} \cdot d\underline{s}$
$= \int_{\partial_R D_i} F_2 \, ds - \int_{\partial_L D_i} F_2 \, ds$
$\approx d \left(d \frac{\partial F_2}{\partial x}\right)$

and $\iint_D \operatorname{scurl} \underline{F} \, dA$
$= \lim_{n \to \infty} \sum_i d^2 \operatorname{scurl} \underline{F}(x_i)$
$= \lim_{n \to \infty} \sum_i \int_{\partial D_i} \underline{F} \cdot d\underline{s}$
$= \int_{\partial D} \underline{F} \cdot d\underline{s}$

Full Stokes can be deduced from Green
(see book for partial argument)

Consequences of Stokes

- by Gauss, $\operatorname{div} \underline{F}(\underline{x}) = \lim_{\operatorname{vol} V \to 0} \frac{1}{\operatorname{vol} V} \iint_{\partial V} \underline{F} \cdot d\underline{S}$

  where V is a sphere (say) centred on $\underline{x}$

  so $\operatorname{div} F$ = "outward flux per unit volume"

- by Stokes, $\operatorname{curl} \underline{F}(\underline{x}) \cdot \underline{n}$, $\underline{n}$ unit vector

  $= \lim_{\operatorname{area}(D) \to 0} \frac{1}{\operatorname{area}(D)} \int_{\partial D} \underline{F} \cdot d\underline{s}$

  D = disc (say) centred at $\underline{x}$ with normal $\underline{n}$

  so $\operatorname{curl} \underline{F}(\underline{x}) \cdot \underline{n}$ = "circulation anticlockwise around $\underline{n}$ per unit area"

Example:
Fluid Dynamics

$\underline{v}$ = flow = velocity field

$\operatorname{curl} \underline{v}$ =: "vorticity"

Interpretation of circulation:
C smooth closed curve
of a thin tube suddenly appears along C

$\frac{\int_C \underline{v} \cdot d\underline{s}}{\operatorname{len}(C)}$ = average velocity around curve of fluid

vortex tubes (Kelvin, Helmholtz ~1870)

C_1 smooth closed curve, $\operatorname{curl} \underline{v} \neq \underline{0}$ at each point of C_1.
extend each point along $\operatorname{curl} \underline{v}$
i.e. to a curve everywhere tangent to $\operatorname{curl} \underline{v}$
to get a vortex tube

Let C_2 be any other curve on the boundary of the tube going round once

Say $C_i = \partial S_i$ ("cross-sections")

Then $\int_{C_1} \underline{v} \cdot d\underline{s} = \int_{C_2} \underline{v} \cdot d\underline{s}$

$\iint_{S_1} \operatorname{curl} \underline{v} \cdot d\underline{S} = \iint_{S_2} \operatorname{curl} \underline{v} \cdot d\underline{S}$

this constant is the strength of the vortex tube.

pf: $C_1 = \partial(S_0 \cup S_2)$

so $\int_{C_1} \underline{v} \cdot d\underline{s} = \iint_{S_0} \operatorname{curl} \underline{v} \cdot d\underline{S} + \iint_{S_2} \operatorname{curl} \underline{v} \cdot d\underline{S}$
$= 0 + \int_{C_2} \underline{v} \cdot d\underline{s}$

so vortex tubes don't "die"

so either: ends are at surface of fluid
e.g. whirlpools, hurricanes...

or ends are joined up
i.e. tube is a closed surface

e.g. smoke rings, vortex knots??

2A03 Lec VIt: Classical Electromagnetism

$\operatorname{div}\operatorname{curl}\underline{E} = 0$

$\underline{E}$ electric field
$\underline{B}$ magnetic field
} vector fields, time dependent

$\boxed{\underline{F}}$ = force on particle of charge 1 at $\underline{x}$ with velocity $\underline{v}$
$= \underline{E}(\underline{x}) + \underline{v} \times \underline{B}(\underline{x})$

e.g. $\underline{B} = (0,0,-1)$ into page

charged particles in plane make circles

Maxwell's equations:

$$\boxed{\begin{array}{ll} \operatorname{div}\underline{E} = \frac{\rho}{\varepsilon_0} & \operatorname{curl}\underline{E} = \frac{\partial B}{\partial t} \\ \operatorname{div}\underline{B} = 0 & \operatorname{curl}\underline{B} = \frac{1}{c^2}\left(\frac{\partial E}{\partial t} + \frac{j}{\varepsilon_0}\right) \end{array}}$$

$\rho(\underline{x})$ = charge density at $\underline{x}$
$j(x)$ = current at $\underline{x}$ (vector)

ε_0 constant (vacuum permittivity $\approx 9 \times 10^{-12}$)
c constant (speed of light) (3×10^8)

- By def^n, $\iiint_V \rho \, dV$ = amount of charge in V

so e.g. if we have some spherically symmetric charge Q centred at 0, none elsewhere
and V is a ball centred at 0 of radius r, containing the charge
then symmetry $\Rightarrow$

$$\iint_{\partial V} \underline{E} \cdot d\underline{S} = 4\pi r^2 f(r)$$
$\|$ Gauss
$$\iiint_V \operatorname{div}\underline{E} \, dV = \frac{1}{\varepsilon_0}\iiint_V \rho \, dV = \frac{Q}{\varepsilon_0}$$

so $f(r)$ = strength of electric field at dist r
$$= \frac{Q}{4\pi\varepsilon_0 r^2}$$

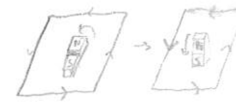
inverse square law

- $\operatorname{curl}\underline{E} = \frac{\partial B}{\partial t}$

so if C is a smooth closed curve $C = \partial S$, S a surface

then $\int_C \underline{E} \cdot d\underline{s} = \iint_S \frac{\partial B}{\partial t} \cdot d\underline{S} = \frac{\partial}{\partial t}\iint_S \underline{B} \cdot d\underline{S}$

$\|$
electromagnetic force
voltage



- $\operatorname{div}\underline{B} = 0$ so $\iint_{\partial V} \underline{B} \cdot d\underline{S} = 0$ any V

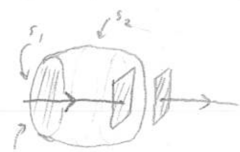
"no magnetic monopoles"

- $\operatorname{curl}\underline{B} = \frac{1}{c^2}\left(\frac{\partial E}{\partial t} + \frac{j}{\varepsilon_0}\right)$

($\underline{B}$ like incompressible fluid with electrical current and changing electrical fields as vortices)

steady current (so $\frac{\partial E}{\partial t} = 0$)
intensity drops linearly with distance

$\frac{\partial E}{\partial t}$ term needed!:



S_1, S_2, C

$\iint_{S_2}\left(\frac{\partial E}{\partial t} + \frac{j}{\varepsilon_0}\right) \cdot d\underline{S} \qquad c^2\int_C \underline{B} \cdot d\underline{s} = \iint_{S_1}\left(\frac{\partial E}{\partial t} + \frac{j}{\varepsilon_0}\right) \cdot d\underline{S}$

$= 0$ approx 0

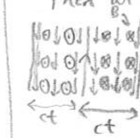
$\operatorname{div}\underline{E} = \rho/\varepsilon_0 \Rightarrow \frac{\partial}{\partial t}\iint_{S_2} \underline{E} \cdot d\underline{s} = \frac{\partial}{\partial t}\iiint_V \frac{\rho}{\varepsilon_0} dV = \frac{1}{\varepsilon_0} I$

rate of charge accumulation

$\iint_{S_1} \frac{j}{\varepsilon_0} \cdot d\underline{S} = \frac{I}{\varepsilon_0}$ ✓

Suppose a current starts flowing in a wire at time 0, and stays steady thereafter.

Then at time t:

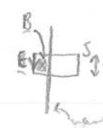


ct ct "wavefront"

intensity of B = $\frac{B}{r}$ for $r \le ct$
0 for $r > ct$

intensity of E = $\frac{E}{ct}$ at $r \le ct$, 0 at $r > ct$

consider fluxes:



wavefront at $r = ct$

$$\frac{\partial}{\partial t}\iint_S \frac{\partial E}{\partial t} \cdot d\underline{S} = \frac{cB\ell}{r} = \int_{\partial S} \underline{E} \cdot d\underline{s}$$

so $E = cB$

Sim, by wavefront

$$\frac{1}{c^2}\frac{\partial}{\partial t}\iint_S \frac{\partial E}{\partial t} \cdot d\underline{S} = \frac{cE\ell}{c^2 r} = \int_{\partial S} \underline{B} \cdot d\underline{s}$$

so $B = \frac{E}{c}$ ✓

if either E or B stopped spreading, one of these equations would fail.
And velocity must be c.
Hence, light.

My notes for Lecs 32 – 34 have gone missing!
Sorry about that. I'll add them if I find them.
They covered Gauss and the statement of Stokes.
Please see §8.2 – §8.3 of the book instead.