2A03 Lec 1

$f : \mathbb{R}^2 \to \mathbb{R}$

e.g. $f(\underline{x}) = |\underline{x}|$

i.e. $f(x,y) = \sqrt{x^2+y^2}$

$U \subseteq \mathbb{R}^2$

$f : U \to \mathbb{R}$

e.g. temp at a point in this room

$U \subseteq \mathbb{R}^3$

$f : U \to \mathbb{R}^2$

e.g. temp, humidity

$f(x,y,z) = (x+y+z,\ xyz)$

$f(x,y,z) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x+2y+3z,\ 2x+3y+4z)$

$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^3 \to \mathbb{R}^3$

vector fields

$f(x,y) = (-y, x)$ Fig 2.3

$f(x,y,z) = \frac{(x,y,z)}{\|(x,y,z)\|} \in$ unit sphere

i.e. $f(\underline{r}) = \frac{\underline{r}}{\|\underline{r}\|}$

vectors pointing directly away from $\underline{0}$

all of same length

draw 2d version

Example: gravitational force field

~~planet of mass~~

mass of $M$ at $\underline{0}$

test mass of 1kg at dist $d$ metres feels force of $\frac{GM}{d^2}$ Newtons in direction of the mass

i.e. $F(\underline{r}) := -\frac{GM}{\|\underline{r}\|^2}\frac{\underline{r}}{\|\underline{r}\|}$ is force felt by test mass at $\underline{r}$

Suppose a planet of mass $M_p$ orbits a star of mass $M_s$. Describe the resulting force field when the planet is $d_p$ metres from the star.

$M_p = 10^{24}$
$M_s = 10^{30}$
$d_p = 10^{12}$

Sol$^n$: Establish co-ordinates s.t. star is at $\underline{0}$ and planet is at $\underline{r}_p := (d_p, 0, 0)$

Force at $\underline{r}$ due to star on test mass of 1kg

is $F_s(\underline{r}) = -\frac{GM_s}{\|\underline{r}\|^2}\frac{\underline{r}}{\|\underline{r}\|}$

— — — — planet — — —

is $F_p(\underline{r}) = -\frac{GM_p}{\|\underline{r}-\underline{r}_p\|^2}\frac{(\underline{r}-\underline{r}_p)}{\|\underline{r}-\underline{r}_p\|}$

total force is sum

$F(\underline{r}) = F_s(\underline{r}) + F_p(\underline{r})$ (no simplification!)

Graphs, & level sets

$f : U^{\subseteq \mathbb{R}^2} \to \mathbb{R}$

Graph of $f$ := $\{(x, y, f(x,y)) \mid (x,y) \in U\}$

sqdist.ogv

Level set of $f$ at value $c$ = $\{(x,y) \mid f(x,y) = c\} \subseteq U \subseteq \mathbb{R}^2$

limits & cont^y   2A03   Lec 2

- Balls
- $\varepsilon$-$\delta$ limits for $\underline{F}: \mathbb{R}^m \to \mathbb{R}^n$, cont^y
- specialisation to 1-var, $\mathbb{R}$-valued multivar
- $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{b}$ iff $\lim_{\underline{x}\to\underline{a}} F_i(\underline{x}) = b_i$
- $+, \cdot$ cont^s; hence limit laws
- cont^y of $\circ$s; hence of $\underline{F}+\underline{G}$, $f\cdot\underline{G}$
-

Notation: $f, g, h$ are real-valued f^ns of 1-or-more var^s
$\underline{F}, \underline{G}, \underline{H}$ are $\mathbb{R}^n$-valued f^ns of ...
$n \geq 1$

If $\underline{F}: U \to \mathbb{R}^n$   $U \subseteq \mathbb{R}^m$
write $F_i: U \to \mathbb{R}$ $i = 1, \dots, n$ for the comp^s
so $\underline{F}(\underline{x}) = (F_1(\underline{x}), \dots, F_n(\underline{x}))$
$= (F_1(x_1, \dots, x_m), \dots, F_n(x_1, \dots, x_m))$

e.g. $\underline{A} \in \mathbb{R}_{n\times m}$ : $\mathbb{R}^m \to \mathbb{R}^n$
$(x_1, \dots, x_m) \mapsto \underline{A}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$   just do $2\times 3$
$A_i$ — $i$th row
$A_i(x_1, \dots, x_m) = a_{i1}x_1 + \dots + a_{im}x_m$

~~$\underline{a} \in \mathbb{R}^m$, $\varepsilon > 0$ ball of rad $\varepsilon$ around $\underline{a}$~~
~~$B_{\underline{a}}(\varepsilon) \subseteq \mathbb{R}^m$   $B_{\underline{a}}(\varepsilon) = \{\underline{x} \mid \|\underline{x}-\underline{a}\| < \varepsilon\}$~~

$F: U \to \mathbb{R}^n$
- $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{b}$ iff $\forall \varepsilon > 0.\ \exists \delta > 0.$ for every $\underline{x} \in U$ s.t. $0 < \|\underline{x}-\underline{a}\| < \delta$, we have $\|\underline{F}(\underline{x}) - \underline{b}\| < \varepsilon$

In other words:
let $B_\varepsilon(\underline{b})$ := ball of radius $\varepsilon$ around $\underline{b}$
$= \{\underline{y} \mid \|\underline{y}-\underline{b}\| < \varepsilon\}$

$\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{b}$ iff $\forall \varepsilon > 0.$ all $\underline{x} \in U$ sufficiently close to, but not equal to, $\underline{a}$, map under $F$ into $B_\varepsilon(\underline{b})$

, $f: (0,\infty) \to \mathbb{R}$, $f(x) = e^{-\frac{1}{x}}$

1d, 2d vector examples
- ~~Agrees with usual notion for real-valued f^ns~~
- $\underline{F}$ is cont^s at $\underline{a}$ iff $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{F}(\underline{a})$
(in particular, both exist)

so $\underline{F}$ is cont^s at $\underline{a}$ iff $\underline{x} \in \mathrm{dom}(\underline{F})$ and $\forall \varepsilon > 0.$ $\exists \delta > 0.$ $\underline{x} \in B_\delta(\underline{a}) \cap U$
$\Rightarrow \underline{F}(\underline{x}) \in B_\varepsilon(\underline{F}(\underline{a}))$

**Theorem (2.3):** if $\underline{F}: U \to \mathbb{R}^n$ ($U \subseteq \mathbb{R}^m$), $\underline{G}: V \to \mathbb{R}^k$
$\mathrm{ran}\,\underline{F} \subseteq V = \mathrm{dom}\,\underline{G}$   $\underline{F}$ cont^s at $\underline{a}$, $\underline{G}$ cont^s at $\underline{F}(\underline{a})$
then $\underline{G}\circ\underline{F}$ is cont^s at $\underline{a}$

$$\begin{tikzcd} \underset{\underline{a}}{U} \arrow[r, "\underline{F}"] \arrow[rr, bend right, "\underline{G}\circ\underline{F}"'] & \underset{\underline{F}(\underline{a})}{V} \arrow[r, "\underline{G}"] & \underset{\underline{G}(\underline{F}(\underline{a}))}{\mathbb{R}^k} \end{tikzcd}$$

not on syllabus
pf^n: Let $\underline{a} \in U$, $\varepsilon > 0$
Since $\underline{G}$ is cont^s exists $\delta$ s.t.
$\underline{G}$ maps $B_\delta(\underline{F}(\underline{a})) \cap V$ into $B_\varepsilon(\underline{G}(\underline{F}(\underline{a})))$
but then since $\underline{F}$ is cont^s (using $\delta$ as the $\varepsilon$!)
exists $\eta > 0$ s.t. $F$ maps $B_\eta(\underline{a}) \cap U$ into $B_\delta(\underline{F}(\underline{a}))$

$$B_\eta(\underline{a}) \xrightarrow{\underline{F}} B_\delta(\underline{F}(\underline{a})) \xrightarrow{\underline{G}} B_\varepsilon(\underline{G}(\underline{F}(\underline{a})))$$

so $\underline{G}\circ\underline{F}$ is cont^s! $\square$

Remark: Can also (book does) define $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x})$ componentwise:
$\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = (\lim_{\underline{x}\to\underline{a}} F_1(\underline{x}), \dots, \lim_{\underline{x}\to\underline{a}} F_n(\underline{x}))$

when all exist,
and $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x})$ exists iff all $\lim_{\underline{x}\to\underline{a}} F_i(\underline{x})$ exist.

So $\underline{F}$ cont^s iff all $F_i$ cont^s

Examples: vector addition
$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$
$\quad \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$

$+ : (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (x_1+y_1, \dots, x_n+y_n)$
is cont^s, so is scalar mult^n

$\cdot : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$
$(c, \underline{x}) \mapsto c\underline{x}$
if $\underline{F}, \underline{G}$ are cont^s at $\underline{a} \in \mathbb{R}^m$, so is $\underline{F}+\underline{G}$

$$\left( \begin{array}{ccccc} \mathbb{R}^m & \xrightarrow{(\underline{F},\underline{G})} & \mathbb{R}^n\times\mathbb{R}^n & \xrightarrow{+} & \mathbb{R}^n \\ \underline{x} & \longmapsto & (\underline{F}(\underline{x}), \underline{G}(\underline{x})) & \longmapsto & \underline{F}(\underline{x})+\underline{G}(\underline{x}) \end{array} \right)$$

and $c\underline{F}$, $f\underline{G}$   (if cont^s at $\underline{a}$)

$f(x,y) = \dfrac{xy}{x^2+y^2}$ has no limit at $0$
indeed: $f(0,y) = 0$ any $y \neq 0$   so limit could only be $0$
$f(x,x) = \dfrac{x^2}{2x^2} = \frac{1}{2}$ any $x \neq 0$
so limit could only be $\frac{1}{2}$

~~$f(\underline{x})$~~

$\underline{F}(\underline{x}) = \dfrac{\underline{x}}{\|\underline{x}\|}$ has no limit at $0$

but $\dfrac{\underline{x}}{\|\underline{x}\|^{1/2}}$ does

and $f(x,y) = (-y, x)$ does

Note: $\lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) = \underline{b}$
means precisely that if we set
$\underline{F}(\underline{a}) := \underline{b}$
then $F$ becomes cont^s at $\underline{a}$

so we get limit laws -
$\lim_{\underline{x}\to\underline{a}} (\underline{F}+\underline{G})(\underline{x}) = \lim_{\underline{x}\to\underline{a}} \underline{F}(\underline{x}) + \lim_{\underline{x}\to\underline{a}} \underline{G}(\underline{x})$

$\lim_{\underline{x}\to\underline{a}} f(\underline{x})\underline{G}(\underline{x}) = (\lim_{\underline{x}\to\underline{a}} f(\underline{x}))(\lim_{\underline{x}\to\underline{a}} \underline{G}(\underline{x}))$

when RHS lims exist

2A03 Lec 3 Diff$^n$

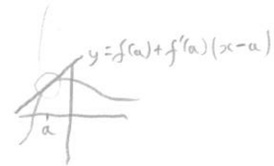
Plan: · Linear approximations of $f$

· " " " $F$

· Jacobian

Recall: for $f:\mathbb{R}\to\mathbb{R}$

$f'(a) = b$ means $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = b$




i.e. $\lim_{h\to 0} \frac{|f(a+h)-(f(a)+bh)|}{|h|} = 0$

i.e. error in the approximation $f(a)+bh$ ("tangent line") to $f(a+h)$ tends to 0 faster than $|h|$ does as $h\to 0$

call this a "good linear approximation" to $f$ at $a$

so $f$ diff$^{ble}$ at $a$ $\iff$ exists good linear approx

derivative $f'(a)=b$ is shorthand for the approx

Now consider $f:\mathbb{R}^2\to\mathbb{R}$

Again, say $f(\underline{a}) + b_1h_1 + b_2h_2$ is a good linear approx if

$$\lim_{\underline{h}\to 0} \frac{|f(\underline{a}+\underline{h}) - (f(\underline{a}) + b_1h_1 + b_2h_2)|}{\|\underline{h}\|} = 0$$

we say $f$ is diff$^{ble}$ at $\underline{a}$ if such an approx exists, and then $b_1 = \frac{\partial f}{\partial x}(\underline{a})$, $b_2 = \frac{\partial f}{\partial y}(\underline{a})$

e.g. $f(x,y) = xy$, $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$

$\underline{a} = (1,2)$

$(1+h_1)(2+h_2) = 2 + 2h_1 + h_2 + h_1h_2$ ($\to 0$ faster than $\|\underline{h}\|$ as $\underline{h}\to 0$)

$$f(1+h_1, 2+h_2) \approx (1)(2) + \frac{\partial f}{\partial x}(1,2)h_1 + \frac{\partial f}{\partial y}(1,2)h_2 = 2 + 2h_1 + h_2$$
$$= 2 + \nabla f(\underline{a})\cdot\underline{h}$$

Let $\nabla f(\underline{a}) := \left(\frac{\partial f}{\partial x}(\underline{a}), \frac{\partial f}{\partial y}(\underline{a})\right)$ "grad" $\quad \nabla f:\mathbb{R}^2\to\mathbb{R}^2$

$f$ diff$^{ble}$ at $\underline{a}$ $\iff$ $\frac{\partial f}{\partial x_i}(\underline{a})$ exist and $\lim_{\underline{h}\to 0} \frac{|f(\underline{a}+\underline{h}) - (f(\underline{a}) + \nabla f\cdot\underline{h})|}{\|\underline{h}\|} = 0$

$\underline{F}: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$

again, same, but with matrices

linear approx to $\underline{F}$ at $\underline{a}$ will be of form

$\underline{F}(\underline{a}) + M\underline{h}$ $\quad M\in \mathrm{Mat}_{n,m}$ $\quad M:\mathbb{R}^m\to\mathbb{R}^n$

treated as col vec

Again, can say what $M$ will be if it exists;

$$D\underline{F}(\underline{a}) := \left(\frac{\partial F_i}{\partial x_j}\right)_{i,j} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m}\end{pmatrix}$$

"Jacobian matrix"

So $\underline{F}$ is diff$^{ble}$ at $\underline{a}$ iff all $\frac{\partial F_i}{\partial x_j}$ exist and

$$\frac{\|\underline{F}(\underline{a}+\underline{h}) - (\underline{F}(\underline{a}) + D\underline{F}(\underline{a})\underline{h})\|}{\|\underline{h}\|} \xrightarrow[\underline{h}\to 0]{} 0$$

then call $D\underline{F}$ the derivative of $\underline{F}$!

Fact: if all $\frac{\partial F_i}{\partial x_j}$ cont$^s$ at $\underline{a}$, then $\underline{F}$ diff$^{ble}$ at $\underline{a}$

Fact: diff$^{ble}$ at $\underline{a}$ $\Rightarrow$ cont$^s$ at $\underline{a}$

e.g. $\underline{F}(x,y) = (-y, x)$ : already linear!

$D\underline{F}(\underline{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$ constant

~~$\underline{F}(\underline{x}) = \frac{\underline{x}}{\|\underline{x}\|}$ $D\underline{F}(\underline{x}) =$~~

Tutorial example: $\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$\underline{F}(x,y) = (xy, y)$ $\quad D\underline{F}(\underline{x}) = \begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}$

2A03 Lec 4,5 Derivatives cont'd

$\mathbb{R} \to \mathbb{R}^3$

Suppose an object moves in space and is at position $\underline{c}(t) \in \mathbb{R}^3$ at time $t$

e.g. if we fire a cannon ball from $(0,0,10)$ at $t=0$ such that it has initial velocity $\frac{d\underline{c}}{dt}$ $(1,1,0)$, then its position is given by

$\underline{c}(t) = (t, t, 10 - \frac{gt^2}{2})$


$\underline{c}$ traces out a curve

$\underline{c}$ gives a parametrisation of the curve (more on which later).

$\underline{c} : U^{\subseteq \mathbb{R}} \to \mathbb{R}^3$

$\frac{d\underline{c}}{dt} = \underline{c}' = D\underline{c} \in Mat_{3,1}$ column vector

$D\underline{c}(t)$ is velocity at time $t$

e.g. $D\underline{c}_{cb} = (1, 1, -gt)$.

$\underline{c}'' = D(D\underline{c})(t)$ is accel

$D^2 \underline{c}_{cb} = (0, 0, -g)$

Notation: $\underline{c}, \underline{v}, \underline{w}, \ldots : \mathbb{R} \to \mathbb{R}^n$

Chain rule;

$B(A\underline{x}) = BA\underline{x}$

Recall if $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^m \xrightarrow{T} \mathbb{R}^k$, $T \circ S : \mathbb{R}^n \to \mathbb{R}^k$

$S, T$ linear transformations

$S(\underline{x}) = A\underline{x}$ $A \in Mat_{m,n}$

$T(\underline{y}) = B\underline{y}$ $B \in Mat_{k,m}$

then $T \circ S$ is the linear transformation $(BA)\underline{x}$ $B \cdot A \in Mat_{k,n}$

i.e. composition $\leadsto$ matrix mult$^n$ for linear $f^{ns}$

sim for affine linear - $S(\underline{x}) = A\underline{x} + \underline{a}$ $T(\underline{y}) = B\underline{y} + \underline{b}$

$T \circ S = B(A\underline{x} + \underline{a}) + \underline{b} = (BA)\underline{x} + B\underline{a} + \underline{b}$!

chain rule says: composition of good linear approx$^{ns}$ is a good linear approx$^n$

i.e. **Fact [chain rule]**: if $U \xrightarrow{F} V \xrightarrow{G} \mathbb{R}^k$, $G \circ F : U \to \mathbb{R}^k$

$F$ diff$^{ble}$ at $\underline{a}$, $G$ diff$^{ble}$ at $F(\underline{a})$

then $G \circ F$ diff$^{ble}$ at $\underline{a}$

and $D(G \circ F)(\underline{a}) = DG(F(\underline{a})) \cdot DF(\underline{a})$ $\boxed{D(G \circ F) = (DG \circ F)(DF)}$

Formulae for derivatives

- linearity: $D(\underline{F} + \underline{G}) = D\underline{F} + D\underline{G}$ $f, g, \underline{F}, \underline{G}$ diff$^{ble}$, $c \in \mathbb{R}$

  $D(c\underline{F}) = cD\underline{F}$

- product rules: $\nabla(fg) = f\nabla g + g \nabla f$ $f, g : \mathbb{R}^n \to \mathbb{R}$

  $(\underline{v} \cdot \underline{w})' = \underline{v} \cdot \underline{w}' + (\underline{v}') \cdot \underline{w}$ $\underline{v}, \underline{w} : \mathbb{R} \to \mathbb{R}^n$

  $(\underline{v} \times \underline{w})' = \underline{v} \times \underline{w}' + (\underline{v}') \times \underline{w}$

(all hold pointwise

e.g. $\underline{F}, \underline{G}$ diff$^{ble}$ at $\underline{a}$

$\Rightarrow D(\underline{F} + \underline{G})(\underline{a}) = D\underline{F}(\underline{a}) + D\underline{G}(\underline{a})$ )

**Example;**

$\underline{r}(t)$ = position of object, assume diff$^{ble}$.

$$\frac{d}{dt} \|\underline{r}(t)\|^2 = \frac{d}{dt}(\underline{r}(t) \cdot \underline{r}(t)) = \underline{r}'(t) \cdot \underline{r}(t) + \underline{r}(t) \cdot \underline{r}'(t) = 2\underline{r}(t) \cdot \underline{v}(t)$$

so velocity is orthogonal to position vector

iff $\underline{r}(t) \cdot \underline{v}(t) = 0$

iff $\frac{d}{dt} \|\underline{r}(t)\|^2 = 0$

iff $\|\underline{r}(t)\|^2$ is constant

iff $\|\underline{r}(t)\|$ is constant

iff object moves on surface of a sphere

**Example 2:** let's prove $(\underline{v} \cdot \underline{w})' = \underline{v} \cdot \underline{w}' + \underline{v}' \cdot \underline{w}$.

$(\underline{v} \cdot \underline{w})(t) = (\underline{G} \circ \underline{F})(t)$

$\underline{F}(t) = (v_1(t), v_2(t), v_3(t), w_1(t), w_2(t), w_3(t))$

$\underline{G}(x_1, y_1, z_1, x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$

$t \xmapsto{F} (\underline{v}(t), \underline{w}(t)) \xmapsto{G} \underline{v}(t) \cdot \underline{w}(t)$

$\mathbb{R} \xrightarrow{F} \mathbb{R}^6 \xrightarrow{G} \mathbb{R}$

$D\underline{F} = (v_1', \ldots, w_3')^T = (\underline{v}', \underline{w}')^T$

$D\underline{G} = (x_2, y_2, z_2, x_1, y_1, z_1)$

so $D(G \circ F) = (DG \circ F)(DF) = (\underline{w}, \underline{v})(\underline{v}', \underline{w}')^T = \underline{w} \cdot \underline{v}' + \underline{v} \cdot \underline{w}'$ ✓

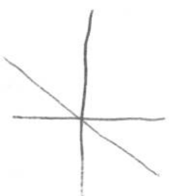
**Example 1:** $\mathbb{R}^2 \xrightarrow{f} \mathbb{R} \xrightarrow{\underline{v}} \mathbb{R}^2$, $\underline{v} \circ f$ $f(\underline{x}) = \|\underline{x}\|$

$\underline{v}(t) = (t, t^2)$

$D(\underline{v} \circ f) = (D\underline{v} \circ f) Df$

$$= \begin{pmatrix} 1 \\ 2\|\underline{x}\| \end{pmatrix} \left( \frac{x}{\|\underline{x}\|}, \frac{y}{\|\underline{x}\|} \right) = \begin{pmatrix} \frac{x}{\|\underline{x}\|} & \frac{y}{\|\underline{x}\|} \\ 2x & 2y \end{pmatrix}$$

Sol'n: (a) parabola (b) slope



2A03 Lec6

change of variables:

$x = r\cos\theta$
$y = r\sin\theta$

$P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(r,\theta) \longmapsto (r\cos\theta, r\sin\theta)$

Let $f$ a function on the plane expressed in cartesian co-ords
e.g. $f(x,y) = x^2+xy+y^2$

so in terms of polar co-ords, it's the function
$(r,\theta) \mapsto f(r\cos\theta, r\sin\theta)$
$= r^2\cos^2\theta + r^2\cos\theta\sin\theta + r^2\sin^2\theta$
$(= r^2(1+\frac{\sin 2\theta}{2}))$

call it $g(r,\theta)$

so $g = f\circ P$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "P"] \arrow[dr, "g"'] & \mathbb{R}^2 \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

Warning: often, people will write $f(r,\theta)$ to mean $g = f\circ P$

Confusing!

Now, what are $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial \theta}$? ("$\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial \theta}$")

$\nabla g = (\nabla f \circ P)\, DP$

$$\left(\frac{\partial g}{\partial r}, \frac{\partial g}{\partial \theta}\right) = \nabla f(r\cos\theta, r\sin\theta)\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}$$

if $f(x,y) = x^2+xy+y^2$
then $\nabla f(x,y) = (2x+y, x+2y)$
so $\nabla g(r,\theta) = (2r\cos\theta + r\sin\theta, r\cos\theta + 2r\sin\theta)\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}$

$= (2r\cos^2\theta + r\sin\theta\cos\theta + r\cos\theta\sin\theta + 2r\sin^2\theta,\ \ldots) = \begin{pmatrix}2r(1+\sin\theta\cos\theta), \\ r^2(\cos^2\theta - \sin^2\theta)\end{pmatrix} \left[= \begin{pmatrix} r(2+\sin(2\theta)), \\ r^2\cos(2\theta)\end{pmatrix}\right]$

Example 2 simplified

$\underline{v}, \underline{w} : \mathbb{R} \longrightarrow \mathbb{R}^2$

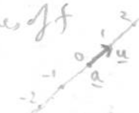
$\underline{v}\cdot\underline{w} = f\circ\underline{u}$
where $\underline{u}(t) = (v_1(t), v_2(t), w_1(t), w_2(t))$
$f(x,y,z,w) = xz + yw$

$\underline{u}' = (v_1', v_2', w_1', w_2')$
$\nabla f(\underline{x}) = (z, w, x, y)$
$(\underline{v}\cdot\underline{w})' = (f\circ\underline{u})' = (\nabla f\circ\underline{u})\cdot\underline{u}'$
$= (w_1, w_2, v_1, v_2)\begin{pmatrix}v_1' \\ v_2' \\ w_1' \\ w_2'\end{pmatrix}$

$= w_1v_1' + w_2v_2' + v_1w_1' + v_2w_2'$
$= \underline{w}\cdot\underline{v}' + \underline{v}\cdot\underline{w}'$

Directional derivatives:

$f : U \subseteq \mathbb{R}^n \to \mathbb{R}$
$\underline{a} \in U$
$\underline{u} \in \mathbb{R}^n$ a unit vector ($\|\underline{u}\| = 1$)
let $\underline{c}(t) := \underline{a} + t\underline{u}$
so $f(\underline{c}(t)) = f\circ\underline{c}(t)$ gives values of $f$ as we "walk" at constant speed! in direction of $\underline{u}$ from $\underline{a}$.




Defn: $D_{\underline{u}} f(\underline{a}) := \frac{d}{dt}(f\circ\underline{c})(t)|_0$

so by chain rule if $f$ is diff$^{ble}$ at $\underline{a}$ then $D_{\underline{u}} f(\underline{a})$ exists and
$D_{\underline{u}} f(\underline{a}) = \nabla f(\underline{c}(0))\cdot\underline{c}'(0)$
$= \nabla f(\underline{a})\cdot\underline{u}$ ← dot product

Recall: $\nabla f(\underline{a})\cdot\underline{u} = \|\nabla f(\underline{a})\|\cdot\|\underline{u}\|\cdot\cos\theta$
so $D_{\underline{u}} f(\underline{a}) = \|\nabla f(\underline{a})\|\cos\theta$
where $\theta$ is angle between $\nabla f(\underline{a})$ and $\underline{u}$

so $\nabla f(\underline{a})$ is in direction of maximal rate of change of $f$ from $\underline{a}$
and $\|\nabla f(\underline{a})\|$ is that maximum.

if $\underline{u}\cdot\nabla f(\underline{a}) = 0$
i.e. $\underline{u} \perp \nabla f(\underline{a})$
then $D_{\underline{u}} f(\underline{a}) = 0$
i.e. $f$ is (instantaneously) constant in dir $\underline{u}$

so for diff$^{ble}$ $f$
$\nabla f(\underline{a})$ is perpendicular to the level set
$\{\underline{x} \mid f(\underline{x}) = f(\underline{a})\}$

e.g. $f(\underline{x}) = \|\underline{x}\|$ $\quad \underline{x} = (x,y,z)$
level sets are spheres
$\nabla f(\underline{x})$ is normal
indeed, $\nabla f(\underline{x}) = \left(\frac{x}{\|\underline{x}\|}, \frac{y}{\|y\|}, \frac{z}{\|z\|}\right) = \frac{\underline{x}}{\|\underline{x}\|}$
("steepest" direction is straight out from $\underline{0}$
constant "steepness")

## 2A03 Lec 7 Higher Order Derivatives

$f : U^{\subseteq \mathbb{R}^n} \to \mathbb{R}^m$

$\nabla f : U^{\subseteq \mathbb{R}^n} \to \mathbb{R}^n$

Recall: if $f$ is diff^ble at $\underline{a}$ and $\|\underline{u}\|=1$,

$\underline{u} \cdot \nabla f(\underline{a}) = D_{\underline{u}} f(\underline{a})$ = directional derivative at $\bar{a}$ in dir$^n$ $\underline{u}$
= rate of change of $f$ at $\bar{a}$ as move from $\underline{a}$ in dir$^n$ $\underline{u}$
$= \frac{d}{dt} f(\underline{a} + t\underline{u})$

Remark: if $f$ is diff^ble and $\underline{u}$ "points along" (is tangent to) a level set $f(\underline{x}) = c$
then $\underline{u} \cdot \nabla f(\underline{a}) = 0$
so $\nabla f(\underline{a})$ is $\perp$ $\underline{u}$
so $\nabla f(\underline{a})$ is perpendicular (normal) to the level set at $\underline{a}$ $\{f(\underline{x}) = f(\underline{a})\}$

e.g. $f(\underline{x}) = \|\underline{x}\|$ level sets are spheres
$\underline{x}$ is indeed normal to the sphere of radius $\|\underline{x}\|$

Now $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$
each $\frac{\partial f}{\partial x_i}$ is a f$^n$ $U \to \mathbb{R}$
so we can try to differentiate it

$$\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i} = \frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j}$$ "second derivatives"

e.g. $f(x,y) = xe^{xy}$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x} x^2 e^{xy} = 2xe^{xy}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(e^{xy} + xye^{xy}\right) = xe^{xy} + xe^{xy} = 2xe^{xy}$$

$f_{xy} = f_{yx}$ !

$$f_{xx} = ye^{xy} + ye^{xy} + xy^2e^{xy} = (y + xy^2)e^{xy}$$

$$f_{yy} = x^3 e^{xy}$$

Note. $f : \mathbb{R}^3 \to \mathbb{R}$ has $\underline{9}$ second derivatives
$f : \mathbb{R}^m \to \mathbb{R}$ " " $\underline{m^2}$ " "

Jacobian of $\nabla f$ = matrix of 2$^{nd}$ deriv$^s$ of $f$

Def$^n$: $f : U^{\subseteq \mathbb{R}^m} \to \mathbb{R}$ is $\underline{C^1}$ if all 1$^{st}$ partial derivatives exist and are cont$^s$ on $U$

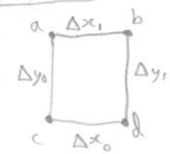
$f$ is $\underline{C^2}$ " " " 2$^{nd}$ " " "

$C^2 \Rightarrow C^1 \Rightarrow$ diff^ble

Fact: if $f$ is $C^2$
then $$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad (\text{all } i, j)$$

(as we saw with $xe^{xy}$ above)

Rough idea:



$\Delta x_1 := f(b) - f(a)$
$\Delta x_0 := f(d) - f(c)$
$\Delta y_1 := f(b) - f(d)$
$\Delta y_0 := f(a) - f(c)$
$\Delta_x \Delta y := \Delta y_1 - \Delta y_0$
$\Delta_y \Delta x := \Delta x_1 - \Delta x_0$

then $\Delta_x \Delta y = \Delta_y \Delta x$
now shrink and take limits..
if all cont$^s$, works nicely.

### Heat Equation:

Rod of conductive metal
$T(x,t)$ := heat at $x$ along rod at time $t$
no external heating or cooling

Fact (1d heat equation)
$$\frac{\partial T}{\partial t} = \sigma \frac{\partial^2 T}{\partial x^2}$$

Rough idea why: discrete model: divide rod into cells and time into steps, and say at each time step each cell gives $\frac{1}{3}$ its heat to its left neighbour, $\frac{1}{3}$ to right, keeps $\frac{1}{3}$
so the heat gained by a cell is $\frac{1}{3}$ the difference between the differences

$$t_{x,t+1} = \tfrac{1}{3}(t_{x-1,t} + t_{x,t} + t_{x+1,t})$$

$$t_{x,t+1} - t_{x,t} = \tfrac{1}{3}(t_{x+1,t} - t_{x,t}) - \tfrac{1}{3}(t_{x,t} - t_{x-1,t})$$

e.g. $T(x,t) = 7x$
is a "steady state sol$^n$"
$$\frac{\partial T}{\partial t} = 0 = \frac{\partial^2 T}{\partial x^2}$$
dynamic sol$^n$: $T(x,t) = \sin x \, e^{-\sigma t}$

### 2d heat eq$^n$.

$T(x,y,t)$ = temp in a sheet of metal at $(x,y)$ at time $t$

$$\frac{\partial T}{\partial t} = \sigma\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

steady state sol$^{ns}$:
$xy$
$x^2 - y^2$

~~More conventional explanation of heat eq$^n$:~~

## 2A03 Lec 8 Taylor's Th$^m$

Def$^n$: Hessian of $f$
$Hf = D\nabla f = (f_{x_i x_j})_{ij}$

Let $f : \mathbb{R} \to \mathbb{R}$ be $C^2$

$$\int_0^h (h-t) f''(t)\,dt$$
$$= \int_0^h \frac{d}{dt}\big((h-t)f'(t) + f(t)\big)\,dt$$
$$= \Big[(h-t)f'(t) + f(t)\Big]_0^h$$
$$= f(h) - \big(hf'(0) + f(0)\big)$$

$$f(h) = f(0) + hf'(0) + \int_0^h (h-t) f''(t)\,dt$$

$$f(x_0+h) = \underbrace{f(x_0) + hf'(x_0)}_{\substack{\text{linear approx} \\ T_1(x_0,h)}} + \underbrace{\int_{x_0}^{x_0+h} (x_0+h-t) f''(t)\,dt}_{\substack{\text{remainder} \\ R_1(x_0,h)}}$$

$f$ is $C^2$, so $f''(t)$ is cont$^s$ on $[x_0-c, x_0+c]$ hence bdd

so $|f''(t)| \le M$

then $|R_1(x_0,h)| = \left|\int_{x_0}^{x_0+h} (x_0+h-t) f''(t)\,dt\right|$

$\le \left|\int_{x_0}^{x_0+h} |h| M\,dt\right| = M|h|^2$  for any $-c \le h \le c$

$f$ $C^3$

$$R_2(0,h) = \int_0^h \frac{(h-t)^2}{2} f'''(t)\,dt = \int_0^h \frac{d}{dt}\left(\frac{(h-t)^2}{2} f''(t) + (h-t)f'(t) + f(t)\right)dt$$
$$= f(h) - \underbrace{\left(\frac{h^2}{2} f''(0) + hf'(0) + f(0)\right)}_{T_2(0,h)}$$

$f(x_0+h) = \ldots$

$R_2(x_0,h) \le M|h|^3$

$\Rightarrow \dfrac{f(x_0+h) - T_2(x_0,h)}{|h|^2} \xrightarrow[h\to 0]{} 0$

$T_2$ = quad$^c$ approx

---

$f : \mathbb{R}^m \to \mathbb{R}$  $C^3$

want linear & quad$^c$ approx$^{ns}$ with error estimates

$f(\underline{h}) = F_{\underline{h}}(1)$  $F_{\underline{h}}(t) := f(t\underline{h})$  $F_{\underline{h}} : \mathbb{R} \to \mathbb{R}$  $F_{\underline{h}}'(t) = \nabla f(t\underline{h}) \cdot \underline{h}$

$= F_{\underline{h}}(0) + F_{\underline{h}}'(0) + [\text{remainder}]$

$= f(\underline{0}) + \underline{h}\cdot\nabla f(\underline{0}) + R_1(\underline{h})$

again $C^2 \Rightarrow$ 2$^{nd}$ derivs bdd on disc of radius $c$
$\Rightarrow R_1(\underline{h}) \le \|\underline{h}\|^2 M_c$

so $\dfrac{R_1(\underline{h})}{\|\underline{h}\|} \xrightarrow[\underline{h}\to 0]{} 0$

we knew this.

$f(\underline{h}) = F_{\underline{h}}(1)$

$= F_{\underline{h}}(0) + F_{\underline{h}}'(0) + \frac{1}{2} F_{\underline{h}}''(0) + R_2(\underline{h})$

$= f(\underline{0}) + \nabla f(\underline{0})\cdot\underline{h} + \frac{1}{2}\underline{h}^t D\nabla f(\underline{0})\underline{h} + R_2(\underline{h})$

$F_{\underline{h}}''(t) = \dfrac{d}{dt}\big(\nabla f(t\underline{h})\cdot\underline{h}\big)$

$= \left(\dfrac{d}{dt}\nabla f(t\underline{h})\right)\cdot\underline{h}$

$= \big(D\nabla f(t\underline{h})\cdot\underline{h}\big)\cdot\underline{h}$

$= \big(Hf(t\underline{h})\cdot\underline{h}\big)\cdot\underline{h}$

$= \underline{h}^t Hf(t\underline{h})\,\underline{h}$

$\dfrac{d}{dt}\left(\begin{pmatrix} f_x \\ f_y \end{pmatrix}(t\underline{h}) \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right) = \left(\dfrac{d}{dt}\begin{pmatrix} f_x \\ f_y \end{pmatrix}(t\underline{h})\right)\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \left(\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}(t\underline{h})\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right)\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = (h_1\ h_2)\left(\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}(t\underline{h})\right)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

eg $m=2$: $\big(D\nabla f(\underline{0})\cdot\underline{h}\big)\cdot\underline{h}$

$= \big(D(f_x, f_y)\cdot\underline{h}\big)\cdot\underline{h}$

$= \begin{pmatrix} f_{xx}^{(0)} & f_{xy}^{(0)} \\ f_{yx}^{(0)} & f_{yy}^{(0)} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$  $\underline{h} = (x,y)$

$=$

$= x^2 f_{xx}(0) + 2xy f_{xy}(0) + y^2 f_{yy}(0)$

So 2$^{nd}$ order Taylor's formula in two vars

$f : \mathbb{R}^2 \to \mathbb{R}$  $C^3$

$$f(\underline{x}_0+\underline{h}) = f(x_0+x, y_0+y) = f(\underline{x}_0) + x f_x(\underline{x}_0) + y f_y(\underline{x}_0)$$
$$+ x^2 f_{xx}(\underline{x}_0) + 2xy f_{xy}(\underline{x}_0) + y^2 f_{yy}(\underline{x}_0)$$
$$+ R_2(\underline{x}_0, \underline{h})$$

where $\dfrac{R_2(\underline{x}_0,\underline{h})}{\|\underline{h}\|^2} \xrightarrow[\underline{h}\to 0]{} 0$

Notation: $Hf(\underline{x}) := D\nabla f(\underline{x})$ "Hessian"

2$^{nd}$ order in $m$ vars: $\boxed{f(\underline{x}_0+\underline{h}) = f(\underline{x}_0) + \underline{h}\,\nabla f(\underline{x}_0) + \underline{h}^t Hf(\underline{x}_0)\underline{h} + R_2(\underline{x}_0,\underline{h})}$

## 2A03 Lec 9 Examples of Taylor

$f(x,y) = e^{xy}$

$\nabla f = (f_x, f_y) = (y e^{xy}, x e^{xy})$

$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} y^2 e^{xy} & (1+xy)e^{xy} \\ (1+xy)e^{xy} & x^2 e^{xy} \end{pmatrix}$

So 2nd order Taylor approx^n at $\underline{x}_0 = (x_0, y_0)$ via

$\cancel{f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + \nabla f \cdot \underline{h} + \frac{1}{2} \underline{h}^t Hf \underline{h} + R_2}$

$\cancel{= f(\underline{x}_0) + h_1 y_0 e^{x_0 y_0} + h_2 x_0 e^{x_0 y_0}}$

at $\underline{x}_0 := (0,1)$:

$f(x, 1+y) = f(0,1) + \nabla f(0,1) \cdot \underline{h} + \frac{1}{2} \underline{h}^t Hf(0,1) \underline{h} + R_2(\underline{h})$

$= 1 + (1, 0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2} (x, y) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + R_2(\underline{h})$

$= 1 + x + \frac{1}{2} (x\ y) \begin{pmatrix} x+y \\ x \end{pmatrix} + R_2(\underline{h})$

$= 1 + x + \frac{1}{2}(x^2 + 2xy) + R_2(\underline{h})$

---

$f(x,y,z) = x^3 + y^3 + z^3$

$\nabla f(\underline{x}) = (3x^2\ \ 3y^2\ \ 3z^2)$

$Hf = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{pmatrix}$

$f(\underline{x}_0 + \underline{h}) = f(x_0) + 3x_0^2 x + 3y_0^2 y + 3z_0^2 z$
$\quad + \frac{1}{2}(6x_0 x^2 + 6y_0 y^2 + 6z_0 z^2)$
$\quad + R_2$

2A03 Lec 10, 11 Optimisation & Extreme values

$f : X^{\subseteq \mathbb{R}^2} \to \mathbb{R}^n$

$f$ has a local minimum at $x_0$

if for some open ball $B$ at $x_0$

$f(x_0) \le f(x)$ for all $x \in B$

i.e. $f(x_0)$ is a local min val of $f$.

$x_0$ is a global/absolute minimum of $f$

if $f(x_0) \le f(x)$ for all $x \in X$

Obviously, global min $\Rightarrow$ local min

"extreme" means "min or max"

~~Th$^m$: If $f$ has a local extremum at $x_0$ and $f$ is diff$^{ble}$ in an open ball around $x_0$ then $\nabla f(x_0) = 0$~~

Def$^n$: Let $X \subseteq \mathbb{R}^2$.

$x \in X$ is interior to $X$

if some open ball around $x$ is contained in $X$.

Th$^m$: Suppose $x_0$ is interior to $X = \operatorname{dom} f$

and $f$ is diff$^{ble}$ at $x_0$

and $f$ has a local extremum at $x_0$.

Then $\nabla f(x_0) = 0$

Pf: By the 1 var version, the directional derivs are 0 $\square$

e.g. $x^2 - y^2$

Def$^n$: $x$ interior to $X$ is critical

if $\nabla f(x) = 0$

or $f$ is not diff$^{ble}$ at $x$.

So for interior points,

extreme $\Rightarrow$ critical

but critical $\not\Rightarrow$ extreme

e.g. $x^2 - y^2$ "saddle point"

Suppose $f$ $C^2$ at $x_0$ and $x_0$ critical ($\nabla f(x_0) = 0$)

Then by Taylor, $f$ has good quad$^c$ approx near $x_0$:

$f(x_0 + x) \approx f(x_0) + \frac{1}{2}\left(f_{xx}(x_0)x^2 + 2f_{xy}(x_0)xy + f_{yy}(x_0)y^2\right)$

$= f(x_0) + \frac{1}{2} x^t Hf(x_0) x$

$H := Hf(x_0) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $\quad |H| = ac - b^2$

~~eigenvalues: $\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = (a-\lambda)(c-\lambda) - b^2 = 0$~~

~~$\Leftrightarrow \lambda^2 - (a+c)\lambda + ac - b^2 = 0$~~

~~$\Leftrightarrow \lambda = \frac{a+c \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2}$~~

~~$D := ac - b^2 = f_{xx}(x_0) f_{yy}(x_0) - f_{xy}(x_0)^2$~~

~~Exercise: e-values are of same sign if $D > 0$, diff $D < 0$~~

~~and are distinct in either case.~~

~~Fact: after rotating co-ordinates, $H$ becomes of form $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and it follows that $f(x,y) = f(x_0) + g(u,v)$ where $(u,v) \leftrightarrow (x,y)$ is a rotation centred at $x_0$~~

~~2nd deriv test: Fact: If $|H| > 0$, e.g. $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ with $a, c > 0$ or $a, c < 0$ then~~

Fact [2$^{nd}$ deriv test]: Suppose $f$ is $C^2$ on a ball around $x_0$, and $\nabla f(x_0) = 0$.

Let $D := \det(Hf(x_0)) = f_{xx}(x_0) f_{yy}(x_0) - f_{xy}(x_0)^2$

Then if $D > 0$

then $f$ has an extreme value at $x_0$,

min if $f_{xx}(x_0) < 0$

max ——— $> 0$

and if $D < 0$,

then $f$ does not have an extreme value at $x_0$.

Very Sketchy Proof: Easily correct for $f(x,y) = ax^2 + cy^2$;

after rotation of co-ords every $h^t Hf h$ is of that form (not obvious!)

by Taylor, behaviour of $f$ near a crit point ... $h^t Hf h$

Remark: In higher dimensions, similar ideas work, but no longer just about $\det Hf$

Def$^n$: $x$ is in the closure of $X \subseteq \mathbb{R}^m$

if every open ball around $x$ contains some point of $X$

$X$ is closed if $X = \text{closure}(X)$

i.e. if every point in the closure of $X$ is in $X$.

$X$ is open if every point of $X$ is interior to $X$

Exercise: $X \subseteq \mathbb{R}^m$ is closed iff $\mathbb{R}^3 \setminus X$ is open

e.g. an open ball $\{x \mid \|x - a\| < r\}$ is open

a closed ball $\{x \mid \|x - a\| \le r\}$ is closed.

Def$^n$: $X \subseteq \mathbb{R}^m$ is bounded if it is contained in some ball

Th$^m$: Suppose $f$ is continuous on its domain $X$, and $X$ is closed and bounded.

Then $f$ has a global max and a global min on $X$

Def$^n$: The boundary $\partial X$ of a closed set $X$ is the set of points of $X$ which are not interior to $X$.

e.g. $C :=$ closed ball $= \{\|x - a\| \le r\}$

$\partial C =$ bounding circle $= \{\|x - a\| = r\}$

Finding global extrema

$f : X^{\subseteq \mathbb{R}^2} \to \mathbb{R}$

If $X$ is not closed or if it is not bounded, $f$ might not have a min on $X$ and/or max

Interior points: check critical points, if $C^2$, can use 2$^{nd}$ deriv test for this.

The rest: can often parametrise the rest by curves, giving 1-variable f$^{ns}$ to find extrema of.

Always be aware that if $X$ is not closed or not bdd $f$ might not have a global min and/or max.

Example: $f(x,y) = x^3 - x^2 - y^2$ on $D$.

2A03 Lec 12 Lagrange Multipliers

Def$^n$: If $X \subseteq \mathrm{dom}\, f$, the restriction of $f$ to $X$ is the function $f|_X$ whose domain is $X$ and whose values are those of $f$

$(f|_X)(\underline{x}) = f(\underline{x})$ for $x \in X$
undefined elsewhere,

Let $f: X^{\subseteq \mathbb{R}^2} \to \mathbb{R}$ $C^1$
let $C$ be a curve in $X$

consider the problem of finding extreme values of $f|_C$ "optimisation of $f$ with constraint $C$"

e.g. if $X$ is closed,
let $\partial X$ be the boundary of $X$ being the points of $X$ not interior to $X$.
For "nice" $X$, $\partial X$ is a curve.
$f$ may have extrema on $\partial X$ and these will be extrema of $f|_{\partial X}$

e.g. Suppose we want to 3d-print a cylindrical cup, but we only have so much plastic to extrude.
what radius and height should we choose to maximize the volume?
constraint on materials $\Rightarrow$
$(r,h)$ lies on a curve, say

$$\pi r^2 + 2\pi r h = 10$$

want to maximise the restriction of $V(r,h) = \pi r^2 h$ to this curve.

Approach 1:
parametrise the curve as $\underline{c}(t)$ $\quad \underline{c}: I^{\subseteq \mathbb{R}} \to \mathbb{R}^2$
so every point of $C$ is $\underline{c}(t)$ for some $t \in I$

e.g. parametrise semicircle as

$\underline{c}(t) = (\sqrt{1-t^2}, t) \quad t \in [-1,1]$

then $f \circ \underline{c}: \mathbb{R} \to \mathbb{R}$
use 1-var techniques to find extrema

Approach 2:
If $C$ is a level curve of a $C^1$ function $g(x,y)$
i.e. $C = \{(x,y) \mid g(x,y) = k\}$
can use Lagrange multipliers:

Th$^m$ [Lagrange multipliers]:
$f, g, C$ as above
suppose $\underline{x} \in C$ is a local extremum of $f|_C$, and $\nabla g(\underline{x}) \neq \underline{0}$
then $\nabla f(\underline{x})$ is collinear with $\nabla g(\underline{x})$
i.e. for some $\lambda \in \mathbb{R}$, $\nabla f(\underline{x}) = \lambda \nabla g(\underline{x})$

e.g. $C = \{\|\underline{x}\| = 1\}$, $f\binom{x}{y} = x+y$
$g(x) = \|\underline{x}\|$
$\nabla g(x) = \hat{\underline{x}} = \frac{\underline{x}}{\|\underline{x}\|}$ $\quad \nabla f(\underline{x}) = (1,1)$

max $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
min

Idea of Pf
Recall: a level curve of $g$ is everywhere $\perp$ to $\nabla g$
so STS that at a local extremum of $f|_C$, $\nabla f$ is $\perp$ to the curve.
Fact: we can find a $C^1$ parametrisation of part of the curve near $\underline{x}$
$\underline{c}: (-1,1) \to C$ s.t. $\underline{c}(0) = \underline{x}$, $\underline{c}'(0) \neq \underline{0}$
$x$ local extremum
$\Rightarrow \frac{d}{dt} f(c(0)) = 0$
$\Rightarrow \nabla f(c(0)) \cdot c'(0) = 0$

so $\nabla f(\underline{x})$ is $\perp$ to $\underline{c}'(0)$ which points along the curve at $\underline{x}$ $\square$

Example:
maximise $V(r,h) = \pi r^2 h$
subject to the constraint
$A(r,h) = \pi r^2 + 2\pi r h = 10$, $r \geq 0, h \geq 0$

~~Note the constraint region is closed and bdd so has a global max~~

Let $(r,h)$ be a local max
$\nabla V = (2\pi r h, \pi r^2)$
$\nabla A(r,h) = (2\pi r + 2\pi h, 2\pi r)$
$r = 0$ blatantly not max,
so $\nabla A(r,h) \neq 0$
$A, V$ $C^1$
so by Lagrange multipliers
$\nabla V(r,h) = \lambda \nabla A(r,h)$ some $\lambda$
i.e. ① $2\pi r h = \lambda(2\pi r + 2\pi h)$
and ② $\pi r^2 = \lambda 2\pi r \Rightarrow r = 2\lambda$
meanwhile, ③ $A(r,h) = \pi r^2 + 2\pi r h = 10$

~~$\pi(2h)^2 + 2\pi(2h)h = 10$~~
~~$h^2(4\pi + 4\pi) = 10$~~
~~$h = \sqrt{10/8\pi}$~~
~~$r = 2\sqrt{10/8\pi}$~~
~~only local max and closed & bdd so must be global max~~

Solve: ② $\Rightarrow$ $r = 2\lambda$
so ① $\Rightarrow 2\pi r h = \pi(r^2 + rh)$
$\Rightarrow rh = r^2$
$\Rightarrow r = h$
then ③ $\Rightarrow \pi r^2 + 2\pi r^2 = 10$
$\Rightarrow r^2 = \frac{10}{3\pi}$
$\Rightarrow r = \sqrt{\frac{10}{3\pi}}$ (+ve square root, since we know $r \geq 0$!)

so $\boxed{(r,h) = \left(\sqrt{\tfrac{10}{3\pi}}, \sqrt{\tfrac{10}{3\pi}}\right)}$
is a local extremum, and the only one with $r > 0$
so this must be the global max,
since $V(r,h) \to 0$ as $r \to 0$

## 2A03 Lecs 13/14

<u>Def^n</u>: a path in $\mathbb{R}^n$ is a function
$\underline{c} : I^{\subseteq \mathbb{R}} \to \mathbb{R}^n$ where $I \subseteq \mathbb{R}$ is an interval

e.g. $\underline{c}(t) = (t, \sin t, \cos t)$ $\overset{t=0}{}$ $\xrightarrow[t \text{ increasing}]{}$
dom $\underline{c} = [0, \infty)$

a path $\underline{c}$ <u>parametises</u> the curve it traces out

Not all curves come parametrised!
e.g. level curves
(book graphs)

The graph of a 1-var $f^n$ 
$\{(x,y) \mid y = f(x)\}$ is a curve with an obv parametisation: $\underline{c}(t) = (t, f(t))$

### Implicit $f^n$ Th$^m$ for plane curves

If $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^1$, $f(a,b) = 0$, $f_y(a,b) \neq 0$
then $C := \{(x,y) \mid f(x,y) = 0\}$ "looks locally like the graph of a $C^1 f^n$"
i.e. there exists an open interval $U \ni a$
and $g : U \to \mathbb{R}$ $C^1$
s.t. $g(a) = b$
$(x, g(x)) \in C \quad \forall x \in U$
$$g'(x) = -\frac{f_x(x, g(x))}{f_y(x, g(x))}$$
for $x \in U$


e.g. $y^2 = x^3 + x^2$ 

$f(x,y) = y^2 - x^3 - x^2$
$\nabla f = (-3x^2 - 2x, 2y)$
at e.g. $(-\frac{1}{2}, -\frac{1}{2\sqrt{2}})$
$f_y \neq 0$ graph of y
$\nabla f = (\frac{3}{4} + 1, -\frac{1}{\sqrt{2}})$
$g'(x) = \frac{-7/4}{-1/\sqrt{2}} = \frac{7\sqrt{2}}{4}$

### Smooth Parametisations and arc-length

A curve can have many parametisations
e.g. $\underline{c}_1(\theta) = (\cos\theta, \sin\theta)$ $\theta \in [0, 2\pi]$
$\underline{c}_2(\theta) = (\cos\theta^2, \sin\theta^2)$ $\theta \in [0, \sqrt{2\pi}]$
$\underline{c}_3(\theta) = (\cos\theta, \sin\theta)$ $\theta \in [0, 3\pi]$
$$\underline{c}_4(x) = \begin{cases} (x, \sqrt{1-x^2}) & \text{if } x \leq 1 \\ (x-2, -\sqrt{1-(x-2)^2}) & \text{if } x > 1 \end{cases} \quad x \in [-1, 3]$$

A <u>smooth parametisation</u> of a curve $C$ is a path $\underline{c} : [a,b] \to \mathbb{R}$
- each point of $C$ appears as $\underline{c}(t)$ for a <u>unique</u> $t$ ("no doubling up") 
- $\underline{c}$ is $C^1$ 
- $\underline{c}'(t) \neq 0$ for any $t$

A <u>smooth curve</u> is a curve which has a smooth parametisation.

e.g.
- a circle is smoothly parametised by $\underline{c}_1$
- the graph of a $C^1$ $f^n$ $f : [a,b] \to \mathbb{R}$ is sm$^{th}$ param$^{ed}$ by $(x, f(x))$
- $(t, \sin t, \cos t)$ $t \in [0, 5]$

Recall: If we think of the parameter as time
$\underline{c}(t)$ = position on curve at time $t$,
then $\underline{c}'(t)$ = velocity
$\underline{c}''(t)$ = acceleration
$\|\underline{c}'(t)\|$ = speed

<u>Def^n</u>: The <u>length</u> of a smooth curve is $\int_a^b \|\underline{c}'(t)\| \, dt$
where $\underline{c} : [a,b] \to \mathbb{R}$ is any smooth param$^n$

### Example:

arc of a circle 

could parametise as
$\underline{c}(\theta) = (\cos\theta, \sin\theta)$ $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$
$$\|\underline{c}'(\theta)\| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = 1$$
$$\int_{-\pi/4}^{\pi/4} 1 \, d\theta = \frac{\pi}{2}$$

or as $\underline{c}(y) = (\sqrt{1-y^2}, y)$ $y \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$
$$\underline{c}'(y) = \left(\frac{-y}{\sqrt{1-y^2}}, 1\right)$$
$$\|\underline{c}'(y)\| = \sqrt{\frac{y^2}{1-y^2} + 1^2} = \sqrt{\frac{1}{1-y^2}}$$
$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{dy}{\sqrt{1-y^2}} = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{dy}{\sqrt{1-\sin^2\theta}} \qquad y = \sin\theta, \quad \frac{dy}{d\theta} = \cos\theta$$
$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{dy}{\cos\theta} = \int_{\arcsin(-1/\sqrt{2})}^{\arcsin(1/\sqrt{2})} d\theta = \int_{-\pi/4}^{\pi/4} d\theta = \frac{\pi}{2} \quad \text{phew!}$$

### pf of Lagrange:
If $\underline{a} = (a,b)$ local extr$^n$ of $f(x,y)$ on $g(x,y) = k$
and $\nabla g(a,b) \neq 0$
by implicit $f^n$ th$^m$ have local param$^n$
$\underline{c} : [-1,1] \to \mathbb{R}^2$ $g(\underline{c}(t)) = k$, $\underline{c}(0) = \underline{a}$, $\underline{c}'(0) \neq 0$

$\underline{a}$ local extremum $\Rightarrow \frac{d}{dt} f(\underline{c}(0)) = 0$
$\Rightarrow \nabla f(\underline{c}(0)) \cdot \underline{c}'(0) = 0$

$g(\underline{c}(t)) = 0 \Rightarrow \frac{d}{dt} g(\underline{c}(t)) = 0$
$\Rightarrow \nabla g(\underline{c}(0)) \cdot \underline{c}'(0) = 0$

$\underline{c}'(0) \neq 0$, in plane, so $\nabla f(\underline{a})$ collinear with $\nabla g(\underline{a})$ $\square$

2A03 Lec 15

Canonical way to parametrise a smooth curve;

Def$^n$: If $c$ is a smooth curve, the parametrisation by arc-length is $k(s) = c(\ell^{-1}(s))$

where $c : [a,b] \to \mathbb{R}^n$ is a smooth para$^n$ of $c$

and $\ell(t) = \int_a^t \|c'(x)\|\,dx$

(Note: not quite unique - need to pick an orientation and a start point)

Note: $c(t) = k(\ell(t))$

never 0 by smoothness

so $c'(t) = k'(\ell(t))\,\ell'(t) = k'(\ell(t))\,\|c'(t)\|$

so $k'(s) = \frac{c'(\ell^{-1}(s))}{\|c'(\ell^{-1}(s))\|} = \widehat{c'(\ell^{-1}(s))}$

so length from $k(0) = c(a)$ to $k(s) = \int_0^s \|k'(x)\|\,dx = \int_0^s dx = s$

Example: $a(t) = (\sqrt{1-t^2}, t)$, $t \in [-1/\sqrt{2}, 1/\sqrt{2}]$

$\ell(t) = \int_{-1/\sqrt{2}}^{t} \frac{dy}{\sqrt{1-y^2}} = \int_{\operatorname{asin}(-1/\sqrt{2})}^{\operatorname{asin} t} d\theta = \operatorname{asin} t - (-\pi/4)$

so $\ell^{-1}(s) = \sin(s - \pi/4)$ (solve $s = \ell(t) = \operatorname{asin} t + \pi/4$, $t = \sin(s-\pi/4)$)

so arc-length param$^n$ is

$c(s) = a(\ell^{-1}(s)) = (\sqrt{1-\sin^2(s-\pi/4)}, \sin(s-\pi/4))$

$= (\cos(s-\pi/4), \sin(s-\pi/4))$

Warning: $k(s)$ often written "$c(s)$"

Curvature

If $k$ is an arclength param$^n$ of $c$

$\|k'(s)\| = 1$ for all $s$

$k'(s)$ is unit tangent vector at $k(s)$

$\|k'(s)\| = 1 \Rightarrow 0 = \frac{d}{ds} k'(s)\cdot k'(s) = 2k'(s)\cdot k''(s)$

so $k''(s) \perp k'(s)$

$\kappa(s) := \|k''(s)\|$ is the curvature at $k(s)$ of $c$

Example;

Helix $c(t) = (t, \cos t, \sin t)$ $[0, 6\pi]$

$\ell(t) = \int_0^t \|c'(x)\|\,dx = \int_0^t \sqrt{2}\,dx = \sqrt{2}\,t$

so $k(s) = c(\ell^{-1}(s)) = c(s/\sqrt{2})$

$k'(s) = \widehat{c'(s/\sqrt{2})} = \frac{(1, -\sin(s/\sqrt{2}), \cos(s/\sqrt{2}))}{\sqrt{2}}$

$k''(s) = \frac{d}{ds} k'(s) = \left(0, \frac{-\cos(s/\sqrt{2})}{2}, \frac{-\sin(s/\sqrt{2})}{2}\right)$

$\kappa(s) = \|k''(s)\| = 1/2$

Often convenient to find curvature in terms of $c(t)$ without going via $k(s)$

$c(t) = k(\ell(t))$

$T_c(t) := \widehat{c'(t)} = k'(\ell(t))$ = unit tangent vector at $c(t)$

so $T_c'(t) = k''(\ell(t))\,\|c'(t)\|$

Let $\kappa_c(t) := \kappa(\ell(t))$ = curvature at $c(t)$

so $\kappa_c(t) = \frac{\|T_c'(t)\|}{\|c'(t)\|}$ (note need smoothness!)

Warning: $T_c(t)$ and $\kappa_c(t)$ often just written $T(t)$ and $\kappa(t)$

Example

what is the curvature of the graph of sin?

Parametrise by $x$: $c(x) = (x, \sin x)$

$c'(x) = (1, \cos x)$ $\|c'(x)\| = \sqrt{1+\cos^2 x}$

$T_c(x) = \frac{(1, \cos x)}{\sqrt{1+\cos^2 x}}$ $T_c'(x) = (-\sin x\cos x\,(1+\cos^2 x)^{-3/2}, -\sin x\,(1+\cos^2 x)^{-1/2} - \sin x\cos^2 x\,(1+\cos^2 x)^{-3/2})$

$\|T_c'(x)\| = \left(\sin^2 x\cos^2 x\,(1+\cos^2 x)^{-3} + \frac{\sin^2 x}{1+\cos^2 x} + \frac{\sin^2 x\cos^2 x}{(1+\cos^2 x)^2} + \frac{\sin^2 x\cos^4 x}{(1+\cos^2 x)^3}\right)^{1/2}$

$\frac{d}{dt}\|b(t)\| = \frac{d}{dt}\sqrt{b(t)\cdot b(t)} = \frac{b'(t)\cdot b(t)}{\|b(t)\|}$

$= \frac{\sin^2 x}{(1+\cos^2 x)^2}$

$= \frac{\sin x}{1+\cos^2 x}$

so $\kappa_c(x) = \frac{\|T_c'(t)\|}{\|c'(t)\|} = \frac{\sin x}{(1+\cos^2 x)^{3/2}}$

2A03 Lec 16 Reparametrisations, Orientation

Sean Fraser 11·5

<u>Def$^n$</u> If $\underline{c} : [a,b] \to \mathbb{R}^n$ is a $C^1$ path,
a <u>reparametrisation</u> of $\underline{c}$
is a path $\underline{\gamma} = \underline{c} \circ \phi : [c,d] \to \mathbb{R}^n$
where $\phi : [c,d] \to [a,b]$ is a $C^1$ bijection.

<u>Example</u>. $\underline{c} = \underline{k} \circ \underline{\ell}_c$

any smooth path is a reparametrisation of a parametrisation by arc length.

<u>Note</u>: <u>two</u> arc-length param$^{n}$s

If $\phi : [a,b] \to [c,d]$ is a bij$^n$, then
either (i) $\phi$ is strictly increasing, $\phi(a) = c$, $\phi(b) = d$
or (ii) " " decreasing, $\phi(a) = d$, $\phi(b) = c$

in case (i), we say the reparam$^n$ $\underline{\gamma} = \underline{c} \circ \phi$ is orientation-preserving
" " (ii) orientation-reversing

<u>Example</u> $\underline{c}(t) = (\cos t, \sin t)$ $(= \underline{k}(t))$

$\phi_1(t) = 2t$
$\phi_2(t) = -t$
$\phi_3(t) = (t - \pi)^3$

<u>Note</u>: $\phi_3$ is orientation-preserving

<u>Def$^n$</u>: a smooth curve is <u>closed</u>
if $\underline{c}(a) = \underline{c}(b)$ where
$\underline{c} : [a,b] \to \mathbb{R}^n$ is a (any) smooth param$^n$.

<u>Fact</u>: any two smooth param$^n$s of a smooth curve (assuming, in the case of a closed curve, that they have the same endpoints) are reparam$^n$s of each other.
(<u>pf</u>: go via arc-length params)

§5.2 4, 12, 28
§5.1 20
§3.4 24
§3.3 26
§3.1 20

## 2A03 Lec 17 Path integrals

**Example:**

Suppose a straight thin rod, placed in the plane along the x-axis, $0 \le x \le 10$ has (cross-sectional) density $\rho(x)$ kg m$^{-1}$. What is its mass?

Answer: $\int_0^{10} \rho(x)\,dx$.

Now suppose a curved thin rod, describing a parabolic arch $y = x^2$, $-5 \le x \le 5$, has (cross-sectional) density $\rho(x,y)$ at $(x,y)$ e.g. $\rho(x,y) = 1+y$. What is its mass?

Or: given a loop of wire and its temp at $(x,y)$, what is its average temp?

**Def$^n$:** If $\underline{c} : [a,b] \to \mathbb{R}^n$ is a $C^1$ path and if $f$ is a $\mathbb{R}$-val$^d$ fn s.t. $f \circ \underline{c} : [a,b] \to \mathbb{R}$ is cont$^s$ then the path integral of $f$ along $\underline{c}$ is

$$\int_{\underline{c}} f\,ds := \int_a^b f(\underline{c}(t))\|\underline{c}'(t)\|\,dt$$

**Def$^n$:** If $C \subseteq \mathbb{R}^n$ is a smooth curve and if $f : C \to \mathbb{R}$ is cont$^s$ then the integral of $f$ on $C$ is

$$\int_C f\,ds = \int_{\underline{c}} f\,ds$$

where $\underline{c}$ is any smooth para$^n$ of $C$.

e.g. • if $\rho(\underline{x})$ = density at $x$, $\int_C \rho\,ds$ is total mass

• $\int_C 1\,ds = \int_C ds = \int_{\underline{c}} \|\underline{c}'(t)\| = \text{length}$

• Average value of $f$ on $C$: $\dfrac{\int_C f\,ds}{\text{length}}$.

## Explanation

$f$ = density.

Say $[a,b] = [0,1]$. Then $\int_{\underline{c}} f\,ds$

$$= \int_0^1 f(\underline{c}(t))\|\underline{c}'(t)\|\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\underline{c}(\tfrac{i}{n}))\|\underline{c}'(\tfrac{i}{n})\|\tfrac{1}{n}$$

consider $[\frac{i}{n}, \frac{i+1}{n}]$ for $n$ large

corresponding to segment of curve between $\underline{c}(\frac{i}{n})$ and $\underline{c}(\frac{i+1}{n})$

For large $n$, approx straight, of length $\|\underline{c}'(\frac{i}{n})\| \cdot \frac{1}{n}$ ("speed", "time")

and value of $f$ is approx constant.

so $f(\underline{c}(\frac{i}{n}))\|\underline{c}'(\frac{i}{n})\|\frac{1}{n}$ $\approx$ mass of this segment.

**Example:** $C = \{y = x^2,\ x \in [-5,5]\}$, $\rho(x,y) = 1+y$

$\underline{c}(t) = (t, t^2)$ $t \in [-5,5]$

$\underline{c}'(t) = (1, 2t)$

$\|\underline{c}'(t)\| = \sqrt{1+4t^2}$

$$\int_C \rho\,ds = \int_{\underline{c}} \rho\,ds = \int_{-5}^{5} \rho(\underline{c}(t))\|\underline{c}'(t)\|\,dt = \int_{-5}^{5} (1+t^2)\sqrt{1+4t^2}\,dt = \ldots$$

**Th$^m$:** If $\underline{\gamma}$ is a reparametrisation of $\underline{c}$ then $\int_{\underline{\gamma}} f\,ds = \int_{\underline{c}} f\,ds$

(so $\int_C f\,ds$ is well-defined)

**Pf:** Say $\underline{c} : [a,b] \to \mathbb{R}^n$, $\phi : [c,d] \xrightarrow{\text{bij}^n} [a,b]$

$\underline{\gamma} = \underline{c} \circ \phi : [c,d] \to \mathbb{R}^n$

$$\int_{\underline{\gamma}} f\,ds = \int_c^d f(\underline{\gamma}(t))\|\underline{\gamma}'(t)\|\,dt = \int_c^d f(\underline{c}(\phi(t)))\cdot\|\underline{c}'(\phi(t))\||\phi'(t)|\,dt$$

Suppose $\phi$ increasing so $|\phi'(t)| = \phi'(t)$, $\phi(c) = a$, $\phi(d) = b$

$$= \int_a^b f(\underline{c}(u))\cdot\|\underline{c}'(u)\|\,du \quad \begin{pmatrix} u = \phi(t) \\ \frac{du}{dt} = \phi'(t) \end{pmatrix}$$

$$= \int_{\underline{c}} f\,ds$$

else $\phi$ is decreasing so $|\phi'(t)| = -\phi'(t)$, $\phi(c) = b$, $\phi(d) = a$

$$= \int_b^a f(\underline{c}(u))\|\underline{c}'(u)\|(-1)\,du = \int_a^b f(\underline{c}(u))\cdot\|\underline{c}'(u)\|\,du = \int_{\underline{c}} f\,ds \quad \square$$

## Lec 18 Path integrals of vector fields

**Def$^n$:** If $\underline{c} : [a,b] \to \mathbb{R}^n$ is a $C^1$ path and $\underline{F}$ is a vector field s.t. $\underline{F} \circ \underline{c} : [a,b] \to \mathbb{R}^n$ is cont$^s$ then the path integral (or line integral) of $\underline{F}$ along $\underline{c}$ is

$$\int_{\underline{c}} \underline{F}\cdot d\underline{s} := \int_a^b \underline{F}(\underline{c}(t))\cdot\underline{c}'(t)\,dt$$

**Example:**

Let $\underline{F}(\underline{x}) = \dfrac{-\underline{x}}{\|\underline{x}\|^3} k$

Let $\underline{c}(t) = (\cos t, \sin t)$ $t \in [0, \pi/2]$

$$\int_{\underline{c}} \underline{F}\cdot d\underline{s} = k\int_0^{\pi/2} \frac{-\underline{c}(t)}{\|\underline{c}(t)\|^3}\cdot\underline{c}'(t)\,dt$$

$$= k\int_0^{\pi/2} 0\,dt \quad \text{since } \underline{c}(t)\cdot\underline{c}'(t) = 0 \ \forall t$$

$$= 0$$

Let $\underline{\gamma}(t) = (2\cos t, \sin t)$ $t \in [0, \pi/2]$

$\underline{\gamma}'(t) = (-2\sin t, \cos t)$

$\underline{\gamma}(t)\cdot\underline{\gamma}'(t) = -3\cos t\sin t$

$\|\underline{\gamma}(t)\| = \sqrt{4\cos^2 t + \sin^2 t} = \sqrt{3\cos^2 t + 1}$

$$\int_{\underline{\gamma}} \underline{F}\cdot d\underline{s} = k\int_0^{\pi/2} \frac{-\underline{\gamma}(t)}{\|\underline{\gamma}(t)\|^3}\cdot\underline{\gamma}'(t)\,dt$$

$$= k\int_0^{\pi/2} \frac{3\cos t\sin t}{(3\cos^2 t + 1)^{3/2}}\,dt$$

$$= (6.2)\,k$$

**Physics terminology:**

If $\underline{F}(\underline{x})$ is a force field and $\underline{c}(t)$ is the trajectory of a particle

$\int_{\underline{c}} \underline{F}\cdot d\underline{s}$ is the "work done" by the force on the particle along the path.

= "change in energy"

If $\underline{F}$ is the only force acting on a rigid body, work done = change in kinetic energy ($= \frac{1}{2}m\|\underline{v}\|^2$)

Indeed $\int_{\underline{c}} \underline{F}\cdot d\underline{s} = \int_a^b \underline{F}(\underline{c}(t))\cdot\underline{c}'(t)\,dt$ $\quad \underline{c}(t)$ = pos$^n$ at time $t$

$= \int_a^b m\,\underline{a}(t)\cdot\underline{v}(t)\,dt$ (N2L)

$= \int_a^b \frac{d}{dt}(\ldots)\,dt = \frac{1}{2}m(\|\underline{v}(b)\|^2 - \|\underline{v}(a)\|^2)$

2A03 Lec 17 cont$^d$

**Th$^m$**: If $\underline{\gamma} = \underline{c} \circ \phi$ is a reparam$^n$ of $\underline{c}$

then $\int_{\underline{\gamma}} \underline{F} \cdot d\underline{s} = \begin{cases} \int_{\underline{c}} \underline{F} \cdot d\underline{s} & \text{if } \phi \text{ is orientation preserving} \\ -\int_{\underline{c}} \underline{F} \cdot d\underline{s} & \text{reversing} \end{cases}$

**Pf.**
$$\int_{\underline{\gamma}} \underline{F} \cdot d\underline{s} = \int_c^d \underline{F}(\underline{\gamma}(t)) \cdot \underline{\gamma}'(t)\, dt$$
$$= \int_c^d \underline{F}(\underline{c}(\phi(t))) \cdot \underline{c}'(\phi(t))\, \phi'(t)\, dt$$
$u = \phi(t)$
$$= \int_{\phi(c)}^{\phi(d)} \underline{F}(\underline{c}(u)) \cdot \underline{c}'(u)\, du$$
$$= \begin{cases} \int_a^b \underline{F}(\underline{c}(u)) \cdot \underline{c}'(u)\, du = \int_{\underline{c}} \underline{F} \cdot d\underline{s} & \text{if } \phi \text{ is orientation preserving} \\ \int_b^a \text{———} = -\int_{\underline{c}} \underline{F} \cdot d\underline{s} & \text{reversing} \end{cases}$$

$\square$

So e.g. in elliptic orbit example; would have got same answer if parametrised with time

so change in kinetic energy is 6.2 J

**Def$^n$**: An oriented smooth curve is a smooth curve with a choice of orientation. So $\int \underline{F} \cdot d\underline{s}$ makes sense for $\underline{F}$ a cont$^s$ vector field.

§4.6, 5.3

## Lec 18 curl and circulation

**Def$^n$**: The scalar curl of a diff$^{ble}$ vector field $\underline{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is the f$^n$
$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} : \mathbb{R}^2 \to \mathbb{R}$$

e.g. if $\underline{F}(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (ax+by, cx+dy)$

the scalar curl is constantly $c-b$

**Def$^n$** The curl of a diff$^{ble}$ vector field $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field
$$\operatorname{curl} \underline{F} = \nabla \times \underline{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

**Mnemonic**: "$\nabla \times \underline{F} = \det \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix}$" (c.f. "$\underline{a} \times \underline{b} = \det \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$")

**Example**: If $\underline{F}(x,y,z) = (G_1(x,y), G_2(x,y), 0)$

then $\operatorname{curl} \underline{F} = (0, 0, \text{scalarcurl } \underline{G})$

**Example**: If a rigid body is spinning around an axis through the origin then the velocity at a point is $\underline{v}(\underline{x}) = \underline{w} \times \underline{x}$

where $\underline{w}$ (the "angular velocity") points along the axis of rotation

then $\operatorname{curl} \underline{v}(\underline{x}) = \operatorname{curl}(w_2 x_3 - w_3 x_2, w_3 x_1 - w_1 x_3, w_1 x_2 - w_2 x_1)$
$$= (w_1 - (-w_1), w_2 - (-w_2), w_3 - (-w_3))$$
$$= 2\underline{w}$$

**Example**: If $\underline{F}$ is a force field it exerts a torque on a small object placed at $\underline{x}$ proportional to $\operatorname{curl} \underline{F}$

### Circulation

**Def$^n$**: A smooth curve is closed if it has a smooth param$^n$ $\underline{c} : [a,b] \to \mathbb{R}^n$ with $\underline{c}(a) = \underline{c}(b)$.

$\underline{c}(a) = \underline{c}(b)$

**Def$^n$**: Let $C$ be a smooth closed oriented curve. The integral $\int_C \underline{F} \cdot d\underline{s}$ of a cont$^s$ vector field is called the circulation of $\underline{F}$ around $C$.

**Fact**: "curl is infinitesimal circulation":

If $\hat{\underline{u}}$ is a unit vector, $\underline{x} \in \mathbb{R}^3$, $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ $C^1$

if we take smooth closed curves $C_i$ in the plane through $\underline{x}$ orthogonal to $\hat{\underline{u}}$, oriented right-handedly around $\underline{u}$, with the maximal distance from $\underline{x}$ on $C_i$ tending to 0 as $i \to \infty$

then $\lim_{i \to \infty} \left( \frac{1}{\operatorname{area}(C_i)} \int_{C_i} \underline{F} \cdot d\underline{s} \right) = \hat{\underline{u}} \cdot \operatorname{curl}(\underline{F})$

$\hat{u}$

**Fact**: For s.c. domain $\operatorname{curl}(\underline{F}) = \underline{0}$ everywhere iff all circulations are 0.

"irrotational"

These are consequences of Stoke's Th$^m$

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**Def$^n$**: A vector field $\underline{F}: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a gradient vector field if $\underline{F} = \nabla f$ for some $f: U \to \mathbb{R}^n$

**Example** A force field $\underline{F}$ is conservative iff it is a gradient vector field.

Then $\underline{F} = -\nabla\phi$ for some $\phi$; such a $\phi$ is called a potential for the force

e.g. $\underline{F} = -k\frac{\underline{x}}{\|\underline{x}\|^3} = -\nabla\left(\frac{-k}{\|\underline{x}\|}\right)$

$\frac{-k}{\|\underline{x}\|}$ is "gravitational potential energy"

**Example**: Let $\underline{v}(\underline{x},t)$ be the velocity of a fluid at $\underline{x}$ at time $t$.

Understanding such $\underline{v}$ is the subject of fluid mechanics.

A fluid is said to be under "potential flow" if $\underline{v}(\underline{x},t) = -\nabla\phi(t)$ for some $\phi(t)$

i.e. if for any $t$, $\underline{v}(\underline{x},t)$ is a gradient vector field

(so problem reduced to understanding evolution of $\phi$...)

**Lemma 2**:

If $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is a $C^1$ gradient vector field then $\operatorname{curl}\underline{F} = \underline{0}$

**Proof**: Say $\underline{F} = \nabla f$

Then e.g. $(\operatorname{curl}\underline{F})_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial f}{\partial y \partial z} - \frac{\partial f}{\partial z \partial y} = 0$

(since $f$ is $C^2$, since $\underline{F}$ is $C^1$) $\square$

Converse not quite true, e.g.

$\underline{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$; $\underline{F}: \mathbb{R}^3 \setminus \{x=0=y\} \to \mathbb{R}^3$

$$\operatorname{curl}\underline{F} = \left(0, 0, \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)$$

$$= \left(0, 0, \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right)$$

$= \underline{0}$ everywhere on $\operatorname{dom}\underline{F} = \mathbb{R}^3 \setminus \{x=0=y\}$

but let $\underline{c}(t) = (\cos t, \sin t, 0)$ $t \in [0, 2\pi]$

then $\int_{\underline{c}} \underline{F}\cdot d\underline{s} = \int_0^{2\pi} (-\sin t, \cos t, 0)\cdot(-\sin t, \cos t, 0)\,dt = 2\pi$

~~Lemma: The circulation of a gradient vector field around any closed curve is 0~~

~~pf: $\int_{\underline{c}} \nabla f \cdot d\underline{s} = \int_a^b \nabla f \cdot \underline{c}'(t)\,dt = \int_a^b$~~

**Lemma 1**: If $\underline{F} = \nabla f$ is a gradient vector field and $\underline{c}: [a,b] \to \mathbb{R}^n$ is a smooth path,

$\int_{\underline{c}} \underline{F}\cdot d\underline{s} = f(\underline{c}(b)) - f(\underline{c}(a))$.

In particular, all circulations are 0

**pf**: $\int_{\underline{c}} \underline{F}\cdot d\underline{s} = \int_{\underline{c}} \nabla f \cdot d\underline{s} = \int_a^b \nabla f \cdot \underline{c}'(t)\,dt$

$= \int_a^b \frac{d}{dt}(f(\underline{c}(t)))\,dt$

$= f(\underline{c}(b)) - f(\underline{c}(a))$ (FTC) $\square$

**Def$^n$**: $X \subseteq \mathbb{R}^n$ is simply connected (s.c.) if (i) any two points of $X$ are connected by a path in $X$ ("path-connected") and (ii) any closed path in $X$ can be "contracted" ("continuously") to a point in $X$ without leaving $X$

**Examples**: $\mathbb{R}^n$ is s.c.

$\mathbb{R}^2 \setminus \{0\}$ is not s.c.

$\mathbb{R}^3 \setminus \{x=0=y\}$ is not s.c.

$\mathbb{R}^3 \setminus \{0\}$ is s.c.

A torus is not s.c.

A sphere is s.c.

**Fact**: If $\underline{F}: U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ and $U = \operatorname{dom}\underline{F}$ is s.c. then if $\operatorname{curl}\underline{F} = \underline{0}$ on $U$ then any circulation $\int_{\underline{c}} \underline{F}\cdot d\underline{s}$, $\underline{c}$ a closed oriented smooth curve, is 0.

("irrotational on s.c. $\Rightarrow$ circulation-free")

This is a consequence of Stokes' theorem, which we'll hopefully eventually (mostly) prove.

**Example**

If $\underline{F}$ is a conservative force field, say $\underline{F} = -\nabla\phi = \nabla(-\phi)$

then if a particle is acted on by $\underline{F}$ as it moves along a trajectory $\underline{c}$ from $\underline{\alpha} = \underline{c}(a)$ to $\underline{\beta} = \underline{c}(b)$, then the work done is

$\int_{\underline{c}} \underline{F}\cdot d\underline{s} = -\phi(\underline{\beta}) - (-\phi(\underline{\alpha}))$

$= \phi(\underline{\alpha}) - \phi(\underline{\beta})$

(doesn't depend on the curve!)

Recall (ignoring other forces) this is the change in kinetic energy, so if we interpret $\phi$ as potential energy, we have

change in energy = change in kin. energy + change in pot energy

$= (\phi(\underline{\alpha}) - \phi(\underline{\beta})) + (\phi(\underline{\beta}) - \phi(\underline{\alpha}))$

$= 0$

So for conservative forces, (potential energy + kin energy) is conserved.