## 2A03 Lec 21

Outline:
- Stokes' Motivation
- Integrability of $f: R \to \mathbb{R}$ rectangle
- Fubini
- Integrability of $f: D \to \mathbb{R}$ D bounded
- Elementary regions, integration thereon
- piecewise elementary regions
- 3d analogues

Stokes' Th^m (rough version):

Let $\underline{F}: U^{\subseteq \mathbb{R}^3} \longrightarrow \mathbb{R}^3$ $C'$ vector field

Let $S \subseteq \mathbb{R}^3$ be a "nice" surface

with boundary $\partial S$ a closed p.s. curve

Then

$\int_{\partial S} \underline{F}\cdot d\underline{s} =$ "the integral of curl $\underline{F}$ on S"

To make sense of such integrals on surfaces, we'll do as we did for curves:
- parametrise the surface, putting it (piecewise) in correspondence with a region in the plane
- define the integral on the surface as an integral on that region

So first, we should understand

### Integrals on regions in the plane, and in $\mathbb{R}^3$

Rectangles:

Let $R = [a,b]\times[c,d] \subseteq \mathbb{R}^2$, a closed rectangle.

Let $f: R \to \mathbb{R}$

Def^n: f is integrable if the limit of Riemann sums

$$\lim_{n\to\infty} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(x_{i,n}, y_{i,n}) \frac{\Delta x \Delta y}{n^2}$$

has a finite value which does not depend on the choice of

$x_{i,n} \in [a + i\Delta x, a+(i+1)\Delta x]$

$y_{j,n} \in [c + j\Delta y, c+(j+1)\Delta y]$

We then write $\iint_R f\,dA$ for this value.

Fact [Fubini]:

For $f: R \to \mathbb{R}$ integrable

$$\iint_A f\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

e.g.
$$\iint_R xy\,dA = \int_c^d\int_a^b xy\,dx\,dy = \int_c^d y\left(\frac{b^2-a^2}{2}\right)dy = \frac{(d^2-c^2)}{2}\frac{(b^2-a^2)}{2}$$

Actually, recall: for f and g integrable,

$$\iint_R f(x)g(y)\,dA = \int_c^d\int_a^b f(x)g(y)\,dx\,dy = \int_c^d g(y)\left(\int_a^b f(x)\,dx\right)dy = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$$

Now suppose D is a bounded region in $\mathbb{R}^2$, $f: D \to \mathbb{R}$

Let $R = [a,b]\times[c,d]$ be a closed rectangle containing D

Extend f to $f_0: R \to \mathbb{R}^2$

$$f_0(\underline{x}) = \begin{cases} f(\underline{x}) & \text{if } \underline{x} \in D \\ 0 & \text{else} \end{cases}$$

Def^n: $f: D \to \mathbb{R}$ is integrable if $f_0: R \to \mathbb{R}$ is, and then $\iint_D f\,dA := \iint_R f_0\,dA$

(easily, this doesn't depend on R)

But this is hard to calculate directly.

Def^n: $D \subseteq \mathbb{R}^2$ of the form

$D = \{(x,y) \mid x \in [a,b]$
$\phi(x) \leq y \leq \psi(x)\}$

where $\phi, \psi: [a,b] \to \mathbb{R}$ cont^s

$\phi(x) \leq \psi(x)$ for $x \in [a,b]$

("region between graphs of two f^ns of x on a closed interval")

is called an x-simple region.

y-simple: same with vars swapped:

$\{(x,y) \mid y \in [c,d], \phi(y) \leq x \leq \psi(y)\}$

$\phi, \psi: [c,d] \to \mathbb{R}$ cont^s, $\phi \leq \psi$

Lec 22

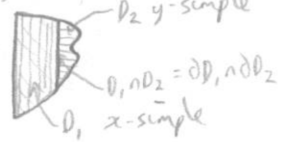
Def^n: An elementary region is a $D \subseteq \mathbb{R}^2$ which is either x-simple or y-simple.

- A piecewise elementary region is a $D \subseteq \mathbb{R}^2$ which is a finite union $D = D_1 \cup \ldots \cup D_n$

which are "almost disjoint" – intersect only on the boundaries (i.e. for any $i \neq j$, any point in the intersection $D_i \cap D_j$ is in the boundary of $D_i$ and in the boundary of $D_j$

i.e. $D_i \cap D_j = \partial D_i \cap \partial D_j$

e.g.



Fact: if D is elementary and $f: D \to \mathbb{R}$ is cont^s then f is integrable.

If $D = D_1 \cup \ldots \cup D_n$ almost disjoint elementary, then $\iint_D f\,dA = \iint_{D_1} f\,dA + \ldots + \iint_{D_n} f\,dA$

If D is x-simple, say $D = \{(x,y) \mid x \in [a,b], \phi(x) \leq y \leq \psi(x)\}$

then $$\iint_D f\,dA = \int_a^b \int_{\phi(x)}^{\psi(x)} f(x,y)\,dy\,dx$$

If D is y-simple, say $D = \{(x,y) \mid y \in [c,d], \phi(y) \leq x \leq \psi(y)\}$

then $$\iint_D f\,dA = \int_c^d \int_{\phi(y)}^{\psi(y)} f(x,y)\,dx\,dy$$

Fact: if D is p.e. and $f: D \to \mathbb{R}$ cont^s, then f int^ble and

Example: Let D be the semicircle

$$\iint_D xy^2\,dA = \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy^2\,dy\,dx$$ (viewing it as a single x-simple region)

$$= \frac{1}{3}\int_0^1 x\left((1-x^2)^{3/2} + (1-x^2)^{3/2}\right)dx$$

$$= \frac{-1}{3}\int_1^0 u^{3/2}\,du \quad \left(\begin{aligned} u &= 1-x^2 \\ \frac{du}{dx} &= -2x\end{aligned}\right)$$

$$= \frac{-2}{15}(0-1) = \underline{2/15}$$

So also $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} xy^2\,dx\,dy = 2/15$, and so on.

2A03 Lec 22 cont$^d$

Example (6-20)

D := region between $y=x$ and $y=\sqrt[3]{x}$
in first quadrant $(x \geq 0, y \geq 0)$

$\iint_D e^{x/y}\, dA$

D is x-simple

$D = \{(x,y) \mid 0 \leq x \leq 1, x \leq y \leq \sqrt[3]{x}\}$

so $\iint_D e^{x/y}\, dA = \int_0^1 \int_x^{\sqrt[3]{x}} e^{x/y}\, dy\, dx$

But $\int e^{x/y} dy$ has no elementary solution!

Luckily...
D is also y-simple

$D = \{(x,y) \mid 0 \leq y \leq 1, y^3 \leq x \leq y\}$

so $\iint_D e^{x/y}\, dA = \int_0^1 \int_{y^3}^{y} e^{x/y}\, dx\, dy = \int_0^1 y(e^{y/y} - e^{y^3/y})\, dy$

$= \int_0^1 (ye - ye^{y^2})\, dy$

$= \frac{1}{2}e - \frac{1}{2}\int_0^1 e^u\, du \quad (u = y^2)$

$= \frac{1}{2}e - \frac{1}{2}(e-1) = \underline{1/2}$

Example:
Evaluate $\int_0^2 \int_y^2 e^{x^2}\, dx\, dy$.

Again $\int e^{x^2} dx$ is not elementary,
so can't do directly.

But $\int_0^2 \int_y^2 e^{x^2}\, dx\, dy = \iint_D e^{x^2}\, dA$  D =

$= \int_0^2 \int_0^x e^{x^2}\, dy\, dx$

$= \int_0^2 x e^{x^2}\, dx = \frac{1}{2}[e^{x^2}]_0^2 = \frac{1}{2}(e^4 - 1)$

Lec 23

Triple integrals $\iiint_W f\, dV$

Def$^n$: $\cdot$ $W \subseteq \mathbb{R}^3$ is xy-simple
if $W = \{(x,y) \in D, K_1(x,y) \leq z \leq K_2(x,y)\}$
where D is elementary
$K_i : D \to \mathbb{R}$ are cont$^s$
and $K_1 \leq K_2$ on D

Then for f cont$^s$ : $D \to \mathbb{R}$,

$\iiint_W f\, dV = \iint_D \int_{K_1(x,y)}^{K_2(x,y)} f(x,y,z)\, dz\, dA$

so eg if D is x-simple $D = \{a \leq x \leq b, \phi(x) \leq y \leq \psi(x)\}$

$= \int_a^b \int_{\phi(x)}^{\psi(x)} \int_{K_1(x,y)}^{K_2(x,y)} f(x,y,z)\, dz\, dy\, dx$

- Analogously for yz-simple, xz-simple.
- W is elementary if it's xy-simple or yz-simple or xz-simple
- W is p.e. if it's the almost-disjoint union of finitely many elementary $W_i$, as in 2d case

Example: If W is the unit sphere
and $f : W \to \mathbb{R}$ is cont$^s$
let's express $\iiint_W f\, dV$ in terms of integrals in 1 variable.

$W = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$

$= \{(x,y,z) \mid x^2+y^2 \leq 1, -\sqrt{1-(x^2+y^2)} \leq z \leq \sqrt{1-(x^2+y^2)}\}$

$= \{(x,y,z) \mid -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, -\sqrt{1-(x^2+y^2)} \leq z \leq \sqrt{1-(x^2+y^2)}\}$

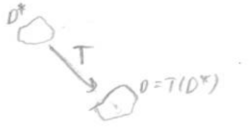
So $\iiint_W f\, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-(x^2+y^2)}}^{\sqrt{1-(x^2+y^2)}} f(x,y,z)\, dz\, dy\, dx$.

Area and Jacobians
~~Change of variables~~

Remark: $\text{Area}(D) = \iint_D 1\, dA$, $\text{Vol}(W) = \iiint_W 1\, dV$

Let $D^* \subseteq \mathbb{R}^2$ elementary
let $T : D^* \to \mathbb{R}^2$ be 1-1
consider $D := T(D^*) = \text{im}\, T = $ ~~image of~~ $F = \{ F(u,v) \mid (u,v) \in D^*\}$

Question: what is Area(D)?



Example: $T(u,v) = (u+3, v-7)$

Clearly $\text{Area}(T(D^*)) = \text{Area}(D^*)$
$= \iint_{D^*} 1\, dA^*$

Lec 23 cont$^d$

Example: $T(u,v) = (2u, 3v)$

$\text{Area}(D) = 6\,\text{Area}(D^*)$

Example $T(u,v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$ (affine)

Recall: If $A$ is an $n\times n$ matrix, $n = 2$ or $3$,
then the area/vol of the image of the unit box $[0,1]^n$
is $|\det A|$

(pf: e.g. $n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the matrix of $T$

$A\underline{i} = \begin{pmatrix} a \\ c \end{pmatrix}$ $A\underline{j} = \begin{pmatrix} b \\ d \end{pmatrix}$

want to see that the area of the parallelogram is $|\det A|$

Clear for $A = I$
check preserved by elementary col ops:
- scalar mult of a col ✓
- swapping cols ✓
- adding a multiple of one col to another ✓
(shear)

)

So $\text{Area}(D) = \left|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| \text{Area}(D^*) = \iint_{D^*} \left|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| dA^*$

General case:

Assume $T$ is $C^1$.

Fact: $\text{Area}(D) = \iint_{D^*} |\det(DT(u,v))|\, dA^*$

Jacobian determinant

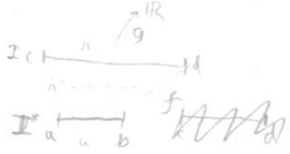
why? Near any point $(u_0, v_0)$
$T$ is approximately affine
$T(u_0+u, v_0+v) \approx DT(u_0,v_0)\begin{pmatrix} u \\ v \end{pmatrix} + T(u_0,v_0)$
so "stretching factor" near $(u_0,v_0) \approx |\det(DT(u_0,v_0))|$

Example – Area of unit disc

Jac. gp
(1): $\begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix}$ (2) $\left(\frac{x^2(y+1)}{2}, \frac{y^2(x+1)}{2}\right)$

Analogue in 1d:

Let $f: I^* = [a,b] \to [c,d] = I$ be $C^1$ and 1-1



Then

$\text{length}(I) = d - c = \int_c^d 1\,dx = \int_a^b |f'(u)|\,du$ $\left(\begin{array}{l} x = f(u) \\ \frac{dx}{du} = f'(u) \end{array}\right.$

(need $|f'(u)|$: if $f$ is decreasing,
$\int_c^d 1\,dx = \int_b^a f'(u)\,du = \int_a^b |f'(u)|\,du$ )

We know more: for $g: [c,d] \to \mathbb{R}$ integrable
$$\int_c^d g(x)\,dx = \int_a^b g(f(u))\,|f'(u)|\,du$$

Lec 4

Fact [Change of Variables]:

Let $D$ and $D^*$ be piecewise el$^y$ regions in $\mathbb{R}^2$
Let $T: D^* \to D$ be $C^1$, onto, and 1-1 on the interior of $D^*$
Let $g: D \to \mathbb{R}$ be integrable
then $\iint_D g(x,y)\,dA = \iint_{D^*} g(T(u,v))\,\|DT(u,v)\|\,dA^*$

book's notation: $\iint_{D^*} g(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

The corresponding statement holds in 3d $\iiint_W h(x,y,z)\,dV = \iiint_{W^*} h(T(u,v,w))\,\|DT(u,v,w)\|\,dA^*$
(and in 1d as above)
(generally: $\int_Y g(\underline{x})\,d\underline{x} = \int_X g(\underline{u})\,\|DT(\underline{u})\|\,d\underline{u}$ $T: X \to Y$ $C^1$ bij$^n$)

Example: Find $\iint_D e^{-(x^2+y^2)/2}\,dA$ where $D$ is the disc

$T$ polar → cartesian
$T(r,\theta) = (r\cos\theta, r\sin\theta)$
$|DT| = \left|\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| = |r\cos^2\theta + r\sin^2\theta| = |r| = r$

so $\iint_D e^{-(x^2+y^2)/2}\,dA = \iint_{D^*} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)/2}\, r\,dA^*$ $D^* = [0,1] \times [0, 2\pi]$

$= \int_0^b \int_0^{2\pi} e^{-r^2/2}\, r\,d\theta\,dr = 2\pi \int_0^{b^2/2} e^{-u}\,du = 2\pi\left(1 - e^{-b^2/2}\right)$

$\xrightarrow[b\to\infty]{} 2\pi$

( so $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dA = 2\pi$ ! )

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11.5 → 13.5

2A03 Lec 25, which actually is Lec 29
(numbering messed up somewhere...)

Def^n ; A parametrisation of a surface $S \subseteq \mathbb{R}^3$
is a map $\underline{r} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
which is 1-1 on the interior of D,
and whose image is S, $\underline{r} : D \twoheadrightarrow S$

Examples:

- $f : \mathbb{R}^2 \to \mathbb{R}$
  $\underline{r}(x,y) := (x, y, f(x,y))$
  graph of f

- unit sphere:
  $\underline{r}(\phi, \theta) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$
  $r : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3$

$\left(\begin{array}{l}\phi = \text{"inclination" / "zenithal angle"} \\ \theta = \text{"azimuth"}\end{array}\right)$

Suppose $\underline{r}$ is $C^1$.

$D\underline{r}(\underline{u}) : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad (\underline{u} = (u,v))$
linear

$D\underline{r}(\underline{u})\begin{pmatrix} a \\ b \end{pmatrix}$ is a tangent vector to S at $\underline{r}(\underline{u})$

$\underline{T}_u(\underline{u}) := D\underline{r}(\underline{u})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial \underline{r}}{\partial u}(\underline{u})$

$\underline{T}_v(\underline{u}) := D\underline{r}(\underline{u})\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial \underline{r}}{\partial v}(\underline{u})$

$\underline{N}(\underline{u}) := \underline{T}_u(\underline{u}) \times \underline{T}_v(\underline{u})$ is, if non-zero, normal to S at $\underline{r}(\underline{u})$

Note $\underline{N}(\underline{u}) = \underline{0}$ iff $\dim(\operatorname{Im} D\underline{r}(\underline{u})) < 2$

Def^n: A param^n $\underline{r} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is smooth
if it is $C^1$ and $\underline{N}(\underline{u}) \neq 0$ for all $\underline{u} \in D$.
A surface is smooth if it has a smooth param^n.

Spherical co-ordinates - leaving to Jamal

$\alpha : [0,\infty] \times [0,\pi] \times [0,2\pi] \longrightarrow \mathbb{R}^3$
$\alpha(\rho, \phi, \theta) := (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$
is onto $\mathbb{R}^3$, and 1-1 on the interior of dom α, and $C^1$

$$D\alpha = \begin{pmatrix} \cos\theta\sin\phi & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\theta & -\rho\sin\phi & 0 \end{pmatrix}$$

$\|D\alpha\| = \rho^2 \sin\phi$

Example: $B :=$ unit ball
$$\iiint_B x^2y^2z^2 \, dV = \iiint_{[0,1]\times[0,\pi]\times[0,2\pi]} \rho^6 \cos^2\theta \sin^2\theta \sin^4\phi \cos^2\phi \, \rho^2 \sin\phi \, dV^*$$

Example: $f : \mathbb{R}^2 \to \mathbb{R}$
$\underline{r}(u,v) = (u, v, f(u,v))$
parametrises the graph of f
and is smooth: $\underline{N}(u,v) = \underline{T}_u(u,v) \times \underline{T}_v(u,v)$
$= (1, 0, f_u) \times (0, 1, f_v)$
$= (-f_u, -f_v, 1) \neq \underline{0}$

$$D\underline{r} = \begin{array}{cc} T_\phi & T_\theta \\ \left(\begin{array}{c} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{array}\right. & \left.\begin{array}{c} -\sin\theta\sin\phi \\ \cos\theta\sin\phi \\ 0 \end{array}\right) \end{array}$$

$\underline{N}(\phi,\theta) = T_\phi \times T_\theta = (\cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \underbrace{\cos^2\theta\cos\phi\sin\phi + \sin^2\theta\cos\phi\sin\phi}_{\cos\phi\sin\phi})$

$= \underline{0}$ iff $\sin\phi = 0$ iff $\phi = 0$ or $\pi$
not smooth $\quad$ (iff $T_\theta = 0$)
(hairy ball)

Orientation

Def^n: An orientation of a surface S
is a continuous choice of a unit normal vector at each point of S, i.e. $\underline{n} : S \to \mathbb{R}^3$ cont^s s.t. $\|\underline{n}(s)\| = 1$ and normal.
S is orientable (or two-sided) if it has an orientati

Note it then has two.

Remark: Non-orientable (one-sided) surfaces exist,
e.g. Möbius strip

Def^n: A param^n $\underline{r} : D \twoheadrightarrow S$
preserves a chosen orientation $\underline{n}$ on S
if $\underline{N}(\underline{u}) = \underline{n}(\underline{r}(\underline{u})) \ \forall \underline{u}$
reverses