2A03 Sample Exam
sample sol^{ns}

1. (i) $(g_u, g_v) = \nabla g = \nabla(f \circ \underline{r}) = (\nabla f \circ \underline{r}) \cdot D\underline{r}$

$$= (f_x \circ \underline{r}, f_y \circ \underline{r}, f_z \circ \underline{r}) \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= (f_x \circ \underline{r} - f_y \circ \underline{r},\ f_y \circ \underline{r} - f_z \circ \underline{r})$$

so $g_u(u,v) = f_x(u, v-u, -v) - f_y(u, v-u, -v)$
$g_v(u,v) = f_y(u, v-u, -v) - f_z(u, v-u, -v)$

(ii) $\nabla f = (1, y, 2z)$

(iii) min value of f on plane = min value of g

$\nabla g = (1 - (v-u),\ (v-u) - 2(-v))$
$= (1 - v + u,\ 3v - u)$

$\nabla g = 0 \iff v + u = 1$ & $u = 3v$
$\iff v = 1/4,\ u = 3/4$

so min $= g(3/4, 1/4) = f(3/4, 1/4 - 3/4, -1/4)$
$= 3/4 + \frac{1}{4} + \frac{1}{16}$
$= \underline{\frac{17}{16}}$

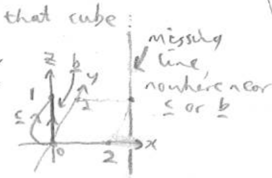
2. (i) $\underline{c}(0) = (0,0,0)$
$\underline{c}(1) = (0,0,1)$
curl $\underline{F} = \underline{0}$ on $\mathbb{R}^3$, which is simply connected
so $\underline{F}$ has path-independent integrals.

Let $\underline{b}(t) = (0,0,t)$ $t \in [0,1]$

Then $\int_{\underline{c}} \underline{F} \cdot d\underline{s} = \int_{\underline{b}} \underline{F} \cdot d\underline{s} = \int_0^1 \underline{F}(0,0,t) \cdot (0,0,1)\, dt$
$= \int_0^1 t^3 dt = \underline{1/4}$

(ii) Yes; $\underline{c}$ and $\underline{b}$ are both within e.g. $(-1, 1.5)^3$
and $\underline{F}$ is defined and has 0 curl on that cube;
so the above argument goes through.

(the situation looks something like this;



3. $\|DT\| = \rho \sin\phi$

$V^* := [0,1] \times [0,\pi] \times [0, 2\pi]$
$T: V^* \longrightarrow V$

$$\iiint_V f\, dV = \iiint_{V^*} (f \circ T) \|DT\|\, dV^*$$

$$= \iiint_{V^*} \rho^2 \cos^2\phi\, \rho \sin\phi\, dV^*$$

$$= \int_0^1 \int_0^{\pi} \int_0^{2\pi} \rho^3 \cos^2\phi \sin\phi\, d\theta\, d\phi\, d\rho$$

$$= 2\pi \int_0^1 \rho^3 \int_1^{-1} -u^2\, du\, d\rho \qquad \left(\begin{array}{l} u = \cos\phi \\ \frac{du}{d\phi} = -\sin\phi \end{array} \right)$$

$$= 2\pi \int_0^1 \rho^3 \left[\frac{-u^3}{3}\right]_1^{-1} d\rho$$

$$= \frac{4\pi}{3} \int_0^1 \rho^3\, d\rho$$

$$= \underline{\frac{4\pi}{12}}$$

4. (i) Parametrise the shadowed surface S
by $D :=$ unit disc $= \{(x,y) \mid x^2 + y^2 \leq 1\}$
$\underline{r}: D \to \mathbb{R}^3$
$\underline{r}(x,y) = (x, y, f(x,y))$

So $\underline{N} = \underline{r}_x \times \underline{r}_y = (1, 0, f_x) \times (0, 1, f_y) = (-f_x, -f_y, 1)$

so area $= \iint_S 1\, dS = \iint_D \|\underline{N}\|\, dA = \underline{\iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA}$

(ii) Min area when $f_x = 0 = f_y$, giving $\iint_D 1\, dA = \underline{\pi}$

so f has to be constant (horizontal flat surface)
e.g. $\underline{f = 0}$

5. Av divergence $= \dfrac{\iiint_V \operatorname{div} \underline{F}\, dV}{\operatorname{vol}(V)}$

$\operatorname{vol}(V) = 2\pi$

By Gauss, $\iiint_V \operatorname{div} \underline{F}\, dV = \iint_{\partial V} \underline{F} \cdot d\underline{S}$

∂V oriented outwards

3 pieces:

$\iint_{S_1} \underline{F} \cdot d\underline{S} = \iint_{S_1} ((1^2 - 1)e^{x^2} + 1^2)\, dS$ $\quad$ ($z = 1$ on S_1) ($\underline{n} = (0,0,1)$)

$= \iint_{S_1} 1\, dS = \operatorname{area}(S_1) = \pi$

$\iint_{S_3} \underline{F} \cdot d\underline{S} = \iint_{S_1} -((-1)^2)\, dS$ $\quad$ ($z = -1$, $\underline{n} = (0,0,-1)$)

$= -\pi$

so $\iint_S \underline{F} \cdot d\underline{S} = \iint_{S_2} \underline{F} \cdot d\underline{S}$

$$= \int_{-1}^{1} \int_0^{2\pi} (\cos\theta, \sin\theta, (z^2 - 1)e^{\cos^2\theta} + z^2) \cdot (\cos\theta, \sin\theta, 0)\, d\theta\, dz$$

(parametrising by $\underline{r}(z,\theta) = (\cos\theta, \sin\theta, z)$)

$$= \int_{-1}^{1} \int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\, d\theta\, dz$$

$= 4\pi$

so av. div. $= \dfrac{4\pi}{2\pi} = \underline{2}$