Second order Taylor

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I sketched in lectures a proof from the textbook of second-order Taylor's theorem under the assumption that the function is C^3. However, it's true under weaker assumptions; in particular, C^2 is enough. I give a proof of that here using the notation of the course, adapted from [Sch67, Thoreme 20, p.259] (which is rather more general).

Theorem 0.1. *Let f be a multivariate real-valued function. Suppose it is C^2 in a ball around a point a. Then*

$$f(a+h) = f(a) + \nabla f(a).h + \frac{1}{2}h^t Hf(a)h + R(h)$$

where $\lim_{h\to 0} \frac{R(h)}{||h||^2} = 0$.

Proof. Let $R(h) = f(a+h) - (f(a) + \nabla f(a).h + \frac{1}{2}h^t Hf(a)h)$.

Differentiating this with respect to h (for h small enough that f is differentiable at $a+h$), one obtains:

$$\nabla R(h) = \nabla f(a+h) - (\nabla f(a) + Hf(a).h).$$

Recall that $Hf = D\nabla f$, and ∇f is differentiable at a since f is C^2 at a. So by the definition of the derivative,

$$\lim_{h\to 0} \frac{\nabla R(h)}{||h||} = 0.$$

Now let $s(t)$ be smallest non-negative number which is greater than or equal to $||\nabla R(h)||$ for any h such that $||h|| \leq t$; by the definition of limit, $s(t)$ exists for all sufficiently small non-negative t, and $\lim_{t\to 0} \frac{s(t)}{t} = 0$.

Then by the mean value theorem applied to lines from 0, for all sufficiently small h,

$$|R(h)| = |R(h) - R(0)| \leq s(||h||)||h||.$$

So

$$\frac{|R(h)|}{||h||^2} \leq \frac{s(||h||)}{||h||} \to_{h\to 0} 0.$$

□

References

[Sch67] Laurent Schwartz. *Analyse mathématique. I.* Hermann, Paris, 1967.