Math 2A03 Tutorial #12 (Dec. 1–2)

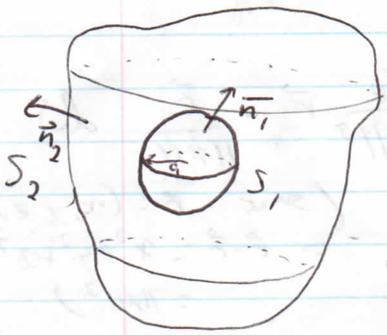
Example using the divergence theorem:

Let $\vec{E}$ be the electrostatic field in $\mathbb{R}^3 - \{(0,0,0)\}$ defined by

$$\vec{E}(x,y,z) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{(\sqrt{x^2+y^2+z^2})^3}(x,y,z)$$

due to a charge Q located at the origin.

Denote by S_1 a sphere of radius a centred at the origin and let S_2 be any closed surface containing S_1. Let $\vec{n}_1$ and $\vec{n}_2$ be the corresponding unit outward-pointing normal vectors.



Let W be the region in between S_1 and S_2. The boundary ∂W consists of S_1 and S_2 and is oriented by n_2 on S_2 (outward) and by n_1 on S_1 (inward).

By the divergence theorem we have

$$\iiint_W \operatorname{div}\vec{E}\, dV = \iint_{\partial W} \vec{E}\cdot d\vec{S} = \iint_{\partial W} \vec{E}\cdot\vec{n}\, dS$$

$$= \iint_{S_1} \vec{E}\cdot(-\vec{n}_1)\, dS + \iint_{S_2} \vec{E}\cdot\vec{n}_2\, dS.$$

But $\operatorname{div}\vec{E} = 0$:

$$\frac{\partial E_1}{\partial x} = \frac{\partial}{\partial x}\left(x\cdot(x^2+y^2+z^2)^{-3/2}\right)$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}.$$

Similarly $\dfrac{\partial E_2}{\partial y} = \dfrac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$ and

$\dfrac{\partial E_3}{\partial z} = \dfrac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$, so $\operatorname{div} E = \dfrac{\partial E_1}{\partial x} + \dfrac{\partial E_2}{\partial y} + \dfrac{\partial E_3}{\partial z} = 0.$

So by the divergence theorem we have

$$\iint_{S_2} \vec{E} \cdot \vec{n}_2 \, dS = \iint_{S_1} \vec{E} \cdot \vec{n}_1 \, dS .$$

We compute $\iint_{S_1} \vec{E} \cdot \vec{n}_1 \, dS$ by arguing as follows:

Let $\vec{r} = (x, y, z)$ denote the position vector of a point (x, y, z) on S_1. Since S_1 is a sphere, the outward normal at a point (x, y, z) on S_1 has the same direction as $\vec{r}$ and so the unit outward normal at (x, y, z) is

$$\vec{n}_1 = \vec{r}/\|\vec{r}\| .$$

$$\therefore \iint_{S_1} \vec{E} \cdot \vec{n}_1 \, dS = \iint_{S_1} \frac{1}{4\pi \varepsilon_0} \frac{Q}{\|\vec{r}\|^3} \vec{r} \cdot \frac{\vec{r}}{\|\vec{r}\|} \, dS$$

$$= \frac{Q}{4\pi \varepsilon_0} \iint_{S_1} \frac{\|\vec{r}\|^2}{\|\vec{r}\|^4} \, dS$$

(since $\vec{r} = (x, y, z)$ so $\vec{r} \cdot \vec{r} = x^2 + y^2 + z^2 = \|\vec{r}\|^2$)

$$= \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{\|\vec{r}\|^2} \iint_{S_1} dS$$

$$= \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{a^2} \underbrace{\iint_{S_1} dS}_{\text{surface area of } S_1}$$

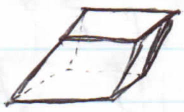
$$= \frac{Q}{4\pi \varepsilon_0} \cdot \frac{1}{a^2} \cdot 4\pi a^2$$

$$= \frac{Q}{\varepsilon_0} .$$

$$\therefore \iint_{S_2} \vec{E} \cdot \vec{n}_2 \, dS = \frac{Q}{\varepsilon_0}$$ for any closed surface S_2 containing S_1. Hence the flux of $\vec{E}$ does not depend on the choice of closed surface as long as the surface contains the charge Q (located at the origin).

Exercise: (8.2, #3): Let $\vec{F}$ be a vector field such that $\operatorname{div}\vec{F} = 3$ everywhere in $\mathbb{R}^3$. Find the flux of $\vec{F}$ out of the parallelepiped with sides of length 3, 2 and 5.

Solution: Let S be the above parallelepiped. By the divergence theorem we have



$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}\vec{F}\, dV.$$

Since we are given that $\operatorname{div}\vec{F} = 3$ everywhere in $\mathbb{R}^3$, we can compute:

$$\begin{aligned}
\iint_S \vec{F}\cdot d\vec{S} &= \iiint_W 3\, dV \\
&= 3\cdot \iiint_W dV \\
&= 3\cdot \text{volume}(W) \\
&= 3\cdot (3\cdot 2\cdot 5) \\
&= 90.
\end{aligned}$$

<u>Exercise</u>: (8.3, #5). Find the circulation $\int_C \vec{F}\cdot d\vec{s}$ of the vector field

$$\vec{F}(x,y,z) = (2x+y,\ 0,\ -3x+y+z)$$

where $\vec{c}$ is the boundary of the triangle cut out from the plane $x+4y+3z=1$ by the first octant, oriented clockwise as seen from the origin.

<u>Solution</u>: First we compute the curl of $\vec{F}$:

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+y & 0 & -3x+y+z \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 0 & -3x+y+z \end{vmatrix} - \hat{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ 2x+y & -3x+y+z \end{vmatrix} + \hat{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ 2x+y & 0 \end{vmatrix}$$

$$= \hat{i}(1) - \hat{j}(-3+0) + \hat{k}(-1)$$

$$= (1, 3, -1).$$

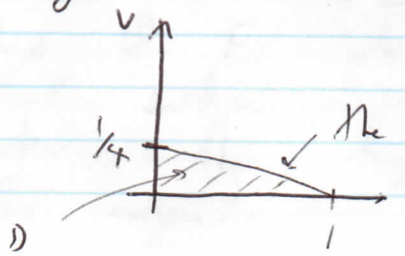
By Stokes' theorem we have

$$\int_C \vec{F}\cdot d\vec{s} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_S (1,3,-1)\cdot d\vec{S}$$

where S is the part of the plane $x+4y+3z=1$ in the first octant. Orient S by a normal vector that points away from the origin, so that the normal has positive $\hat{k}$ component.

We can parametrize S as the graph of $z = \frac{1}{3}(1-x-4y)$:

$\vec{r}(u,v) = (u, v, \frac{1}{3}(1-u-4v))$, where (u,v) belongs to the following region D in the uv-plane:



the line $v = -\frac{1}{4}u + \frac{1}{4}$, given by the intersection of the above plane with $z=0$.

The surface normal $\vec{N}$ is given by

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1/3 \\ 0 & 1 & -4/3 \end{vmatrix} = \left(\tfrac{1}{3}, \tfrac{4}{3}, 1\right);$$

since the $\hat{k}$ component is positive, this coincides with our choice of orientation of S.

$$\therefore \iint_S (1, 3, -1) \cdot d\vec{S}$$

$$= \iint_D (1, 3, -1) \cdot \left(\tfrac{1}{3}, \tfrac{4}{3}, 1\right) dA$$

$$= \iint_D \tfrac{1}{3} + 4 - 1 \quad dA$$

$$= \frac{10}{3} \iint_D dA = \frac{10}{3} \cdot \text{area}(D) = \frac{10}{3} \cdot \frac{1}{8} = \frac{5}{12}.$$

$\therefore$ by Stokes' theorem we have $\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \frac{5}{12}$.

Exercise: ~~Let $\vec{F}(x,y,z) = (0, \ldots$~~

Let $\vec{F}(x,y,z) = (-y, x, 0)$ and let $\vec{C}$ be the boundary of the square with vertices $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$ and $(0, 0, 1)$, oriented counterclockwise as seen from above. Compute $\int_{\vec{C}} \vec{F} \cdot d\vec{s}$.

Solution: The curl of $\vec{F}$ is given by

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(1+1) = (0,0,2).$$

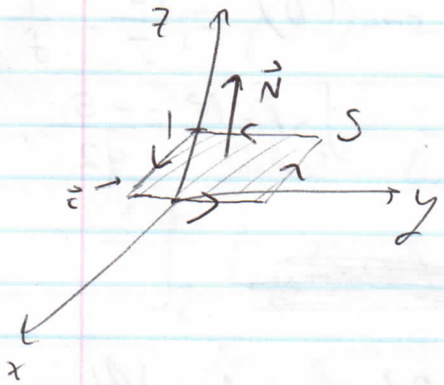
$\therefore$ by Stokes' theorem

$$\int_C \vec{F}\cdot d\vec{s} = \iint_S (0,0,2)\cdot d\vec{S},$$

where S is the square $[0,1]\times[0,1]$ lying in the plane $z=1$. We can parametrize S by

$$\vec{r}(u,v) = (u,v,1)$$

where $0\le u\le 1$ and $0\le v\le 1$.



Hence the surface normal is

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1);$$

this matches the orientation of the boundary (since the $\hat{k}$ component is positive).

$$\therefore \int_C \vec{F}\cdot d\vec{s} = \iint_S (0,0,2)\cdot d\vec{S}$$

$$= \iint_{[0,1]\times[0,1]} (0,0,2)\cdot(0,0,1)\, dA$$

$$= 2\cdot \iint_{[0,1]\times[0,1]} dA$$

$$= 2\cdot \text{area}([0,1]\times[0,1])$$

$$= 2.$$