Theorem (Equality of mixed partials). Let f be a real-valued function of m variables $x_1, \dots, x_m$, and suppose f is of class C^2 (i.e. all second-order partial derivatives of f are continuous). Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

for all $i, j = 1, \dots, m$.

Exercise (4.1, #16) Explain why there is no C^2 function $f(x,y)$ such that

$$f_x(x,y) = e^x + xy, \quad f_y(x,y) = e^x + xy.$$

Solution: Suppose such a C^2 function $f(x,y)$ did exist. Then by equality of mixed partials, we must have

$$f_{xy}(x,y) = f_{yx}(x,y).$$

But

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(e^x + xy)$$

$$= x,$$

while

$$f_{yx} = \frac{\partial}{\partial x}(e^x + xy)$$

$$= e^x + y.$$

So

$$f_{xy}(x,y) = e^x + y \neq x = f_{yx}(x,y),$$

which contradicts the fact that we must have $f_{xy}(x,y) = f_{yx}(x,y)$. Thus such a function can't exist.

Exercise: (4.1, #25). Show that the gravitational potential function

$$V(x,y,z) = \frac{-GMm}{\sqrt{x^2+y^2+z^2}}$$

satisfies Laplace's equation:

$$V_{xx} + V_{yy} + V_{zz} = 0$$

(whenever $(x,y,z) \neq \vec{0}$).

Solution: First we compute V_x using the chain rule:

$$\begin{aligned} V_x(x,y,z) &= \frac{\partial}{\partial x}\left(-GMm \cdot (x^2+y^2+z^2)^{-1/2}\right) \\ &= -GMm \cdot -\frac{1}{2} \cdot (x^2+y^2+z^2)^{-3/2} \cdot 2x \\ &= GMm \cdot x(x^2+y^2+z^2)^{-3/2}. \end{aligned}$$

We then compute $V_{xx}(x,y,z)$ using both the product rule and the chain rule:

$$\begin{aligned} V_{xx}(x,y,z) &= \frac{\partial}{\partial x}\left(GMm\, x(x^2+y^2+z^2)^{-3/2}\right) \\ &= GMm\left((x^2+y^2+z^2)^{-3/2} + x \cdot -\frac{3}{2}(x^2+y^2+z^2)^{-5/2} \cdot 2x\right) \\ &= GMm\left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right) \\ &= GMm\left(\frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}}\right) \\ &= GMm \cdot \left(\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\right). \end{aligned}$$

Similarly (by symmetry), we can compute

$$V_{yy} = GMm \cdot \left(\frac{x^2 - 2y^2 + z^2}{(x^2+y^2+z^2)^{5/2}} \right),$$

$$V_{zz} = GMm \cdot \left(\frac{x^2 + y^2 - 2z^2}{(x^2+y^2+z^2)^{5/2}} \right).$$

Thus:

$$V_{xx} + V_{yy} + V_{zz} = GMm \left(\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \right)$$

$$+ \; GMm \left(\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \right) + GMm \left(\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \right)$$

$$= GMm \left(\frac{-2x^2+y^2+z^2 + x^2-2y^2+z^2+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \right)$$

$$= GMm \left(\frac{-2x^2 + 2x^2 - 2y^2 + 2y^2 - 2z^2 + 2z^2}{(x^2+y^2+z^2)^{5/2}} \right)$$

$$= 0.$$

$\therefore$ $V(x,y,z)$ satisfies Laplace's equation.

Definition. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function, and let $(x_0, y_0) \in \mathbb{R}^2$. The first-order and second-order <u>Taylor polynomials</u> of f at (x_0, y_0) are given by

$$T_1(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

and

$$T_2(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$+ \frac{1}{2}\Big[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)$$
$$+ f_{yy}(x_0,y_0)(y-y_0)^2\Big].$$

We can write $T_2(x,y)$ in a more compact form as:

$$T_2(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$
$$+ \frac{1}{2}\begin{bmatrix} x-x_0 & y-y_0 \end{bmatrix} Hf(x_0,y_0) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix},$$

where

$$Hf(x_0,y_0) = \begin{bmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{bmatrix}$$

is the <u>Hessian matrix</u> of f at (x_0, y_0).

Exercise (4.2 #29). Find the first-order and second-order Taylor polynomials of

$$f(x,y) = \sqrt{x+4y-1}$$

at the point $(5,3)$. Compare the two approximations of $f(4.9, 3.1)$ with the value of the function at that point.

Solution: First we see that

$$f(5,3) = \sqrt{5+4\cdot 3-1}$$
$$= \sqrt{16} = 4.$$

We need to compute f_x, f_y, f_{xx}, f_{xy}, and f_{yy}:

$$f_x(x,y) = \frac{\partial}{\partial x}\left((x+4y-1)^{1/2}\right)$$
$$= \frac{1}{2}\cdot(x+4y-1)^{-1/2}\cdot 1$$
$$= \frac{1}{2\cdot\sqrt{x+4y-1}},$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left((x+4y-1)^{1/2}\right)$$
$$= \frac{1}{2}\cdot(x+4y-1)^{-1/2}\cdot 4$$
$$= \frac{2}{\sqrt{x+4y-1}}.$$

So
$$f_x(5,3) = \frac{1}{8}, \quad f_y(5,3) = \frac{1}{2}.$$

Next we compute the second partials:

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{1}{2}\cdot(x+4y-1)^{-1/2}\right)$$
$$= \frac{1}{2}\cdot -\frac{1}{2}\cdot(x+4y-1)^{-3/2}$$
$$= \frac{-1}{4(x+4y-1)^{3/2}} \quad ;$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{2}\cdot(x+4y-1)^{-1/2}\right)$$
$$= \frac{1}{2}\cdot\frac{-1}{2}\cdot(x+4y-1)^{-3/2}\cdot 4$$
$$= \frac{-1}{(x+4y-1)^{3/2}} \quad ;$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(2\cdot(x+4y-1)^{-1/2}\right)$$
$$= 2\cdot\frac{-1}{2}\cdot(x+4y-1)^{-3/2}\cdot 4$$
$$= \frac{-4}{(x+4y-1)^{3/2}}.$$

So

$$f_{xx}(5,3) = \frac{-1}{256} = -\frac{1}{2^8},$$

$$f_{xy}(5,3) = -\frac{1}{64} = -\frac{1}{2^6},$$

$$f_{yy}(5,3) = -\frac{1}{16} = \frac{-1}{2^4}.$$

Thus:

$$T_1(x,y) = 4 + \frac{1}{8}(x-5) + \frac{1}{2}(y-3)$$
$$= \frac{15}{8} + \frac{1}{8}x + \frac{1}{2}y \; ;$$

$$T_2(x,y) = 4 + \frac{1}{8}(x-5) + \frac{1}{2}(y-3)$$
$$- \frac{1}{512}(x-5)^2 - \frac{1}{64}(x-5)(y-3)$$
$$- \frac{1}{32}(y-3)^2$$
$$= \frac{671}{512} + \frac{49}{256}x + \frac{49}{64}y$$
$$- \frac{1}{512}x^2 - \frac{1}{64}xy - \frac{1}{32}y^2.$$

We compute

$$T_1(4.9, 3.1) = 4.0375000,$$
$$T_2(4.9, 3.1) = 4.0373242 \quad \text{and}$$
$$f(4.9, 3.1) \approx 4.0373258 \quad \text{(roughly)}.$$