Math 2A03 Tutorial #5 (Oct. 14-15).

Exercise: Find a parametric equation of the following curves:

(i) The intersection of the planes $x+y-z=2$ and $2x-5y+z=3$ in $\mathbb{R}^3$.

(ii) The intersection of the cylinder $(x+2)^2+(z-2)^2=4$ and the plane $y=3$ in $\mathbb{R}^3$.

Solution: (i) We want to solve the system

$$x+y-z=2$$
$$2x-5y+z=3.$$

Since there are two equations and three unknowns we can parametrize one of the variables x, y or z. Choose, for instance $x=t$ where $t\in\mathbb{R}$ is a parameter. We get the system

$$t+y-z=2$$
$$2t-5y+z=3$$

which is equivalent to the system

$$y-z=2-t$$
$$-5y+z=3-2t.$$

Adding the two equations together, we get

$$-4y=5-3t$$

$$\Rightarrow \quad y=\frac{3t-5}{4}.$$

Since $z=y+t-2$, we can substitute the above for y:

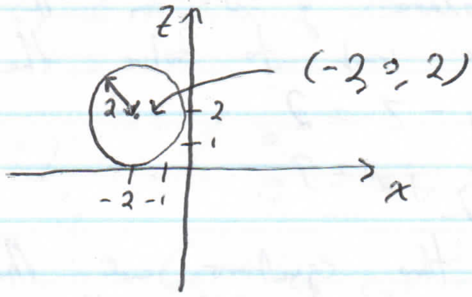
$$z=\frac{3t}{4}-\frac{5}{4}+t-2=\frac{3t-5+4t-8}{4}=\frac{7t}{4}-\frac{13}{4}.$$

Thus we get one possible parametrization:

$$c(t)=(x(t),y(t),z(t))$$
$$=\left(t,\frac{3t}{4}-\frac{5}{4},\frac{7t}{4}-\frac{13}{4}\right),\quad t\in\mathbb{R}.$$

(ii) Consider first the intersection of the cylinder $(x+2)^2 + (z-2)^2 = 4$ with the xz-plane i.e. where $y=0$. This intersection is given by a circle of radius 2:

$(y=0)$



Since a circle of radius 2 centered at the origin can be parametrized by

$$(2\cos t,\ 0,\ 2\sin t) \quad t \in [0, 2\pi]$$

The circle above can be parametrized as

$$c(t) = (-2, 0, 2) + (2\cos t, 0, 2\sin t)$$
$$= (-2 + 2\cos t,\ 0,\ 2 + 2\sin t),\quad t \in [0, 2\pi].$$

Since we want the intersection with the plane $y = 3$ we must translate the above circle 3 units (positively) along the y-axis). So we get:

$$c(t) = (-2 + 2\cos t,\ 3,\ 2 + 2\sin t),\quad t \in [0, 2\pi].$$

Exercise: (3.1, #25). Show that

$$c(t) = \left(\frac{3t}{(1+t^3)}, \frac{3t^2}{(1+t^3)} \right), \quad t \in \mathbb{R}$$

is a parametrization of the curve $x^3 + y^3 - 3xy = 0$.

Solution: We need to check that

$$x(t)^3 + y(t)^3 - 3x(t)y(t) = 0$$

where

$$x(t) = \frac{3t}{(1+t^3)}, \quad y(t) = \frac{3t^2}{(1+t^3)}.$$

Compute:

$$x^3 + y^3 - 3xy = \left(\frac{3t}{(1+t^3)}\right)^3 + \left(\frac{3t^2}{(1+t^3)}\right)^3 - 3\left(\frac{3t}{1+t^3}\right)\left(\frac{3t^2}{1+t^3}\right)$$

$$= \frac{27t^3}{(1+t^3)^3} + \frac{27t^6}{(1+t^3)^3} - \frac{27t^3}{(1+t^3)^2}$$

$$= \frac{27t^3(1+t^3) - 27t^3(1+t^3)}{(1+t^3)^3}$$

$$= 0.$$

Thus $c(t)$ is a parametrization of $x^3 + y^3 - 3xy = 0$.

Definition. • Let $c(t)$ be a differentiable path in $\mathbb{R}^2$ or $\mathbb{R}^3$. The velocity of $c(t)$ is given by $v(t) = c'(t)$, and the speed of $c(t)$ is given by the scalar function $\|v(t)\|$.
• If $c(t)$ is twice differentiable, then the acceleration of $c(t)$ is $a(t) = v'(t) = c''(t)$.

Exercise: (7.3 #1). Compute the velocity and the speed of the cycloid parametrized by

$$c(\theta) = (\theta - \sin\theta,\ 1-\cos\theta),\quad \theta \in \mathbb{R}.$$

Identify points where the speed is maximal.

Solution: Compute:

$$v(\theta) = c'(\theta) = (1-\cos\theta,\ \sin\theta),$$

and so

$$\begin{aligned}\|v(\theta)\| &= \|c'(\theta)\| \\ &= \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} \\ &= \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} \\ &= \sqrt{2 - 2\cos\theta} \\ &= \sqrt{2(1-\cos\theta)} \\ &= \sqrt{2}\cdot\sqrt{1-\cos\theta}.\end{aligned}$$

Since $-1 \le \cos\theta \le 1$, $-1 \le -\cos\theta \le 1$ and so $0 \le 1-\cos\theta \le 2$ i.e. $0 \le \sqrt{1-\cos\theta} \le \sqrt{2}$. So the maximal value of $\sqrt{1-\cos\theta}$ is $\sqrt{2}$ which occurs when $-\cos\theta = 1$, i.e. when $\cos\theta = -1$; this occurs at all points

$$\theta = \pi + 2\pi k,\quad \text{where } k \text{ is an integer.}$$

Thus the maximal speed is 2; this occurs when $\theta = \pi + 2\pi k$.

Definition. Let $b(t)$ and $c(t)$ be two vector-valued functions of a single variable t. If $b'(t) = c(t)$, we call $b(t)$ the antiderivative of $c(t)$; we denote it by

$$b(t) = \int c(t)\,dt.$$

If $c(t) = (c_1(t), c_2(t), c_3(t))$ for real-valued functions c_1, c_2, c_3, then

$$b(t) = \int c(t)\,dt$$

$$= \left(\int c_1(t)\,dt, \int c_2(t)\,dt, \int c_3(t)\,dt \right).$$

Exercise: (3.2 # 5, 9). Find the velocity $v(t)$ and the position $c(t)$ of a particle given the following data:

(5) $a(t) = (-1, 1, 0)$, $v(0) = (1, 3, 0)$, $c(0) = (0, 3, 0)$.

(9) $a(t) = (t, t^2, t)$, $v(0) = (0, 3, -3)$, $c(0) = (4, 3, -6)$.

Solution: (5): Since $v(t)$ is the antiderivative of $a(t)$,

$v(t) = \int a(t)\,dt = \int (-1, 1, 0)\,dt = (-t + c_1, t + c_2, c_3)$

for some constants $c_1, c_2, c_3 \in \mathbb{R}$. Since $v(0) = (1, 3, 0)$, we have

$$(1, 3, 0) = (-0 + c_1, 0 + c_2, c_3) = (c_1, c_2, c_3),$$

i.e. $c_1 = 1$, $c_2 = 3$, $c_3 = 0$. Thus

$$v(t) = (-t + 1, t + 3, 0).$$

Now since $c(t)$ is the antiderivative of $v(t)$, we have

$$c(t) = \int v(t)\,dt = \int (-t + 1, t + 2, 0)\,dt$$

$$= \left(-\frac{t^2}{2} + t + d_1, \frac{t^2}{2} + 2t + d_2, d_3\right)$$

for some $d_1, d_2, d_3 \in \mathbb{R}$. Since ~~c(0) = (4, 3, -6)~~ $c(0) = (0, 3, 0)$ we get

$$(0, 3, 0) = \left(-\frac{t^2}{2} + t + d_1, \frac{t^2}{2} + 2t + d_2, d_3\right)\Big|_{t=0}$$

$$= (d_1, d_2, d_3), \quad \text{i.e. } d_1 = 0, d_2 = 3, d_3 = 0.$$

Thus $c(t) = \left(-\frac{t^2}{2} + t, \frac{t^2}{2} + 2t + 3, 0\right)$.

(9): $v(t) = \int a(t)\,dt = \int (t, t^2, t)\,dt$

$$= \left(\frac{t^2}{2} + c_1,\ \frac{t^3}{3} + c_2,\ \frac{t^2}{2} + c_3 \right),$$

$c_1, c_2, c_3 \in \mathbb{R}$. Since $v(0) = (0, 3 -3)$, we have

$$(0, 3 -3) = \left(\frac{0^2}{2} + c_1,\ \frac{0^3}{3} + c_2,\ \frac{0^2}{2} + c_3 \right)$$

$$= (c_1, c_2, c_3),$$

i.e. $c_1 = 0$, $c_2 = 3$, $c_3 = -3$. Thus

$$v(t) = \left(\frac{t^2}{2},\ \frac{t^3}{3} + 2,\ \frac{t^2}{2} - 3 \right).$$

Furthermore

$$c(t) = \int v(t)\,dt = \int \left(\frac{t^2}{2},\ \frac{t^3}{3} + 3\ \frac{t^2}{2} - 3 \right) dt$$

$$= \left(\frac{t^3}{6} + d_1,\ \frac{t^4}{12} + 2t + d_2,\ \frac{t^3}{6} - 3t + d_3 \right)$$

$d_1, d_2, d_3 \in \mathbb{R}$. Since $c(0) = (4, 3, -6)$, we have

$$(4, 3 -6) = (d_1, d_2, d_3),$$

i.e. $d_1 = 4$, $d_2 = 3$, $d_3 = -6$. Thus

$$c(t) = \left(\frac{t^3}{6} + 4,\ \frac{t^4}{12} + 2t + 3\ \frac{t^3}{6} - 3t - 6 \right).$$