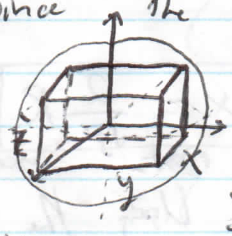
Exercise: Find the volume of the largest rectangular box that can be inscribed into the sphere of radius $R > 0$.

Solution: Assume the box in $\mathbb{R}^3$ is placed so that its center is at the origin and its sides are parallel to the coordinate planes. Label its length, width and height by x, y and z respectively. Since the box is inscribed into the sphere of radius R, centered at the origin, each of the eight vertices of the box must be touching the sphere.



Since the box is centered at the origin, these eight points are given by $(\pm x/2, \pm y/2, \pm z/2)$. Since they lie on the sphere, they must satisfy

$$\left(\pm \frac{x}{2}\right)^2 + \left(\pm \frac{y}{2}\right)^2 + \left(\pm \frac{z}{2}\right)^2 = R^2,$$

which gives ~~us~~ us the constraint

$$x^2 + y^2 + z^2 = 4R^2.$$

Thus we must maximize the function

$$V(x,y,z) = xyz$$

subject to the constraint

$$g(x,y,z) = x^2 + y^2 + z^2 = 4R^2.$$

We use the method of Lagrange multipliers: Using $\nabla V(x,y,z) = \lambda \cdot \nabla g(x,y,z)$, we get

$$(yz, xz, xy) = \lambda(2x, 2y, 2z)$$

and so we get the system

$$\begin{aligned} yz &= \lambda 2x \\ xz &= \lambda 2y \\ xy &= \lambda 2z \end{aligned}$$

Since $x, y, z > 0$, we can divide the first equation by the second equation to get

$$\frac{yz}{xz} = \frac{\lambda 2x}{\lambda 2y} \Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow \boxed{x^2 = y^2}.$$

Similarly we can divide the first equation by the third equation to get

$$\frac{yz}{xy} = \frac{\lambda 2x}{\lambda 2z} \Rightarrow \frac{z}{x} = \frac{x}{z} \Rightarrow \boxed{x^2 = z^2}.$$

Since our constraint says $x^2+y^2+z^2 = 4R^2$, we get

$$x^2 + x^2 + x^2 = 4R^2$$

and so $x^2 = \frac{4}{3}R^2$, i.e. $\boxed{x = \frac{2}{\sqrt{3}}R}$

(we can disregard the negative root since we are working with lengths). Using the previous equations we get $x = y = z = \frac{2}{\sqrt{3}}R$.

$\therefore$ the largest rectangular box which can be inscribed in the sphere of radius R is the cube with sides of length $\frac{2}{\sqrt{3}}R$, and its volume is $\left(\frac{2}{\sqrt{3}}R\right)^3 = \frac{8}{3^{3/2}} R^3$.

Exercise: Find the arc-length function of the path $\vec{c}(t) = \left(t\sin 2t,\ t\cos 2t,\ \frac{4}{3}t^{3/2}\right)$, $0 \le t \le 2\pi$.

Solution: First we need to compute the speed $\|\vec{c}\,'(t)\|$:

$$\vec{c}\,'(t) = \left(\sin 2t + 2t\cos 2t,\ \cos 2t - 2t\sin 2t,\ 2\sqrt{t}\right),$$

so

$$\|\vec{c}\,'(t)\| = \sqrt{(\sin 2t + 2t\cos 2t)^2 + (\cos 2t - 2t\sin 2t)^2 + (2\sqrt{t})^2}$$

$$= \big(\sin^2 2t + 4t\sin 2t\cos 2t + 4t^2\cos^2 2t + \cos^2 2t - 4t\cos 2t\sin 2t + 4t^2\sin^2 2t + 4t\big)^{1/2}$$

$$= \left(1 + 4t^2 + 4t\right)^{1/2}.$$

Note that $4t^2 + 4t + 1 = (2t+1)^2$, and so $\|\vec{c}\,'(t)\| = \sqrt{(2t+1)^2} = 2t+1$.

$\therefore$ The arc-length function $s(t)$ is given by

$$s(t) = \int_0^t \|\vec{c}\,'(\tau)\|\,d\tau$$

$$= \int_0^t 2\tau + 1 \;d\tau$$

$$= \left(\tau^2 + \tau\right)\Big|_0^t$$

$$= t^2 + t,$$

where $0 \le t \le 2\pi$.

Exercise: Compute the work done by the force $\vec{F}(x,y) = (x^3, x+y)$ acting on the particle that moves from $(0,0)$ to $(1, \pi^2/4)$ along the curve $\vec{c}(t) = (\sin t, t^2)$.

Solution: The work is given by the path integral $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$, where $\vec{F}(x,y)$ is the above vector field and where $\vec{c}(t) = (\sin t, t^2)$ $0 \le t \le \frac{\pi}{2}$.

Recall that $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$.

- $\vec{F}(\vec{c}(t)) = \vec{F}(\sin t, t^2) = (\sin^3 t, \sin t + t^2)$.
- $\vec{c}\,'(t) = (\cos t, 2t)$.

$$\therefore \quad W = \int_0^{\pi/2} (\sin^3 t, \sin t + t^2) \cdot (\cos t, 2t)\, dt$$

$$= \int_0^{\pi/2} (\sin^3 t \cos t + 2t \sin t + 2t^3)\, dt.$$

For the first integral, we the substitution $u = \sin t$, $du = \cos t\, dt$. Then

$$\int_0^{\pi/2} \sin^3 t \cos t\, dt = \int_0^1 u^3\, du = \frac{u^4}{4}\Big|_0^1 = \frac{1}{4}.$$

For the second integral, we integration by parts with $u = t$ $dv = \sin t\, dt$ (so $du = dt$ and $v = -\cos t$) to get

$$\int_0^{\pi/2} t \sin t\, dt = -t\cos t\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos t\, dt$$

$$= -\frac{\pi}{2}\cos\frac{\pi}{2} + 0\cdot\cos 0 + \sin t\Big|_0^{\pi/2}$$

$$= \sin\frac{\pi}{2} - \sin 0 = 1.$$

The third integral is

$$\int_0^{\pi/2} 2t^3\, dt = \frac{t^4}{2}\Big|_0^{\pi/2} = \frac{\pi^4}{32}.$$

$$\therefore \quad W = \frac{1}{4} + 2\cdot 1 + \frac{\pi^4}{32} = \frac{9}{4} + \frac{\pi^4}{32}.$$

Exercise: Find $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ where $\vec{F}(x,y) = (2xye^y, x^2e^y(1+y))$ and $\vec{c}$ is the straight-line segment from $(0,0)$ to $(3,-2)$

(a) using a parametrization for $\vec{c}$;

(b) using the fact that $\text{curl}\,\vec{F} = \mathbf{0}$.

Solution: (a) Since $\vec{c}$ is a straight-line segment, we can parametrize it as

$$\vec{c}(t) = (3t, -2t), \quad 0 \le t \le 1.$$

Then

$$\vec{F}(\vec{c}(t)) = \vec{F}(3t, -2t)$$
$$= (2 \cdot 3t \cdot (-2t) \cdot e^{-2t}, (3t)^2 e^{-2t}(1-2t))$$
$$= (-12t^2 e^{-2t}, (9t^2 - 18t^3) e^{-2t}),$$

and $\vec{c}\,'(t) = (3, -2)$, so

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^1 (-12t^2 e^{-2t}, (9t^2 - 18t^3) e^{-2t}) \cdot (3, -2)\, dt$$
$$= \int_0^1 (-36t^2 e^{-2t} - 18t^2 e^{-2t} + 36t^3 e^{-2t})\, dt$$
$$= \int_0^1 (-54t^2 e^{-2t} + 36t^3 e^{-2t})\, dt.$$

Using integration by parts for both integrals, we get

$$\int_0^1 t^2 e^{-2t}\, dt = \frac{1}{4}(1 - 5e^{-2})$$

and

$$\int_0^1 t^3 e^{-2t}\, dt = \frac{1}{8}(3 - 19e^{-2}).$$

$$\therefore \int_{\vec{c}} \vec{F} \cdot d\vec{s} = -54\left(\frac{1}{4}(1 - 5e^{-2})\right) + 36\left(\frac{1}{8}(3 - 19e^{-2})\right)$$
$$= -18e^{-2}.$$

(b) First check that curl $\vec{F} = 0$:

$$\text{curl } \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= -\frac{\partial}{\partial y}(2xye^y) + \frac{\partial}{\partial x}(x^2e^y + yx^2e^y)$$

$$= -2x(e^y + ye^y) + 2e^yx + y \cdot e^y \cdot 2x$$

$$= -2xe^y - 2xye^y + 2xe^y + 2xye^y$$

$$= 0.$$

Since the domain of $\vec{F}$ is all of $\mathbb{R}^2$ (which is simply connected), we know that $\vec{F}$ is a gradient vector field, i.e. $\vec{F} = \nabla f$ for some real-valued function f. Since $\vec{F} = \nabla f$ we have

$$(2xye^y, x^2e^y(1+y)) = (f_x, f_y)$$

and so

$$f_x = 2xye^y, \quad f_y = x^2e^y(1+y).$$

$\Rightarrow$ $\int f_x \, dx = x^2ye^y + g(y)$; taking the derivative of this expression with respect to y yields

$$x^2(e^y + ye^y) + g'(y) = f_y = x^2(e^y + ye^y).$$

So $g'(y) = 0$ and so $g(y) = c$, a constant. So any function of the form

$$f(x,y) = x^2ye^y + c$$

(where c is a constant) satisfies $\vec{F} = \nabla f$.

If we apply the generalization of the Fundamental Theorem of Calculus, we get

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a)) = f(3,-2) - f(0,0) = -18e$$

Exercise: Is it true that if two vector fields $\vec{F}$ and $\vec{G}$ have the same circulation along a given closed curve, then $\vec{F} = \vec{G}$?

Solution: No. For a simple counterexample, parametrize the unit circle by

$$\vec{c}(t) = (\cos t, \sin t), \quad 0 \leq t \leq 2\pi$$

and let

$$\vec{F}(x, y) = (1, 0)$$
$$\vec{G}(x, y) = (0, 1).$$

Then

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (1, 0) \cdot (-\sin t, \cos t)\, dt$$

$$= -\int_0^{2\pi} \sin t \, dt = 0$$

and

$$\int_{\vec{c}} \vec{G} \cdot d\vec{s} = \int_0^{2\pi} (0, 1) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} \cos t \, dt = 0.$$

So $\vec{F}$ and $\vec{G}$ have the same circulation along $\vec{c}$, but $\vec{F} \neq \vec{G}$.