Math 2A03 Tutorial #9 (Nov. 10 - 11)

Midterm 2, #4: Let $\vec{c}$ be the path given by
$$\vec{c}(t) = (t, t^2, t^3), \quad 0 \le t \le 1.$$
Suppose $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field defined on all of $\mathbb{R}^3$, and suppose $\operatorname{curl}\vec{F} = 0$ everywhere on $\mathbb{R}^3$. We are given the value of $\vec{F}(x,y,z)$ when $x=y=z$: For all $t \in \mathbb{R}$, $\vec{F}(t,t,t) = (t^2, t^2, t^2)$.
Find $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$.

Solution: We know that $\operatorname{curl}\vec{F} = 0$ on $\mathbb{R}^3$, which is simply connected. This is equivalent to saying that $\vec{F}$ is a gradient vector field, i.e. $\vec{F} = \nabla f$ for some real-valued function f. By properties of gradient vector fields, we also know that $\vec{F}$ is path-independent: For any two oriented simple curves $\vec{c}_1$ and $\vec{c}_2$ having the same initial and terminal points,
$$\int_{\vec{c}_1} \vec{F} \cdot d\vec{s} = \int_{\vec{c}_2} \vec{F} \cdot d\vec{s}.$$
The initial and terminal points of $\vec{c}(t)$ are $\vec{c}(0) = (0,0,0)$ and $\vec{c}(1) = (1,1,1)$. Since we only know the values of $\vec{F}(x,y,z)$ for $x=y=z$, we need to find a path with the same initial and terminal points as $\vec{c}$ which we can use to compute the path integral; the path $\vec{\gamma}(t) = (t,t,t)$, $0 \le t \le 1$ works since $\vec{\gamma}(0) = (0,0,0) = \vec{c}(0)$ and $\vec{\gamma}(1) = (1,1,1) = \vec{c}(1)$, and so
$$\begin{aligned}
\int_{\vec{c}} \vec{F} \cdot d\vec{s} &= \int_{\vec{\gamma}} \vec{F} \cdot d\vec{s} \\
&= \int_0^1 \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)\, dt \\
&= \int_0^1 F(t,t,t) \cdot (1,1,1)\, dt \\
&= \int_0^1 (t^2, t^2, t^2) \cdot (1,1,1)\, dt \\
&= \int_0^1 3t^2\, dt = t^3\Big|_0^1 = 1.
\end{aligned}$$

Exercise: Evaluate $\iint_D f \, dA$ for the following functions f and regions $D \subseteq \mathbb{R}^2$:

(a) $f(x,y) = xy^{-1} - x^2y^2$, $D = [0,2] \times [3,4]$.

(b) $f(x,y) = x^{-2/3}$, D is the region in the first quadrant bounded by the parabolas $y = x^2$, $y = -x^2 + 4$.

Solution: (a) The integral can be expressed as the iterated integral

$$\iint_D f \, dA = \int_3^4 \left(\int_0^2 xy^{-1} - x^2y^2 \, dx \right) dy$$

$$= \int_3^4 \left(\frac{x^2}{2} y^{-1} - \frac{x^3}{3} y^2 \right) \Big|_0^2 \, dy$$

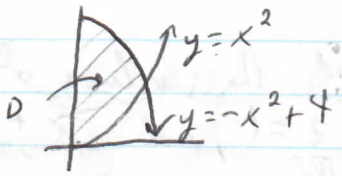
$$= \int_3^4 2y^{-1} - \frac{8}{3} y^2 \, dy$$

$$= \left(2 \ln y - \frac{8}{9} y^3 \right) \Big|_3^4$$

$$= \left(2 \ln 4 - \frac{512}{9} \right) - \left(2 \ln 3 - 24 \right)$$

$$= 2 \ln\left(\frac{4}{3}\right) - \frac{512}{9} + 24$$

$$\approx -32.3135.$$

(b) 

We need to find the point where the parabolas intersect. Combining $y = x^2$ and $y = -x^2 + 4$, we get $x^2 = 2$ and so $x = \pm\sqrt{2}$. Since we are in the first quadrant, $x > 0$ and so $x = \sqrt{2}$ is the point of intersection. So we can compute the integral as the iterated integral

$$\iint_D f\,dA = \int_0^{\sqrt{2}} \left(\int_{x^2}^{-x^2+4} x^{-2/3}\,dy \right) dx$$

$$= \int_0^{\sqrt{2}} x^{-2/3} \cdot y \Big|_{x^2}^{-x^2+4} dx$$

$$= \int_0^{\sqrt{2}} x^{-2/3} (-x^2 + 4 - x^2)\,dx$$

$$= \int_0^{\sqrt{2}} x^{-2/3} (4 - 2x^2)\,dx$$

$$= \int_0^{\sqrt{2}} (4x^{-2/3} - 2x^{4/3})\,dx$$

$$= \left(12x^{1/3} - \frac{6}{7} x^{7/3} \right) \Big|_0^{\sqrt{2}}$$

$$= 12(2)^{1/6} - \frac{6}{7}(2)^{7/6}$$

$$\approx 11.5453.$$

Exercise: (6.3 #29) Find the volume of the solid between the planes $x+y+z=1$ and $x+y+2z=1$ in the first octant in $\mathbb{R}^3$.

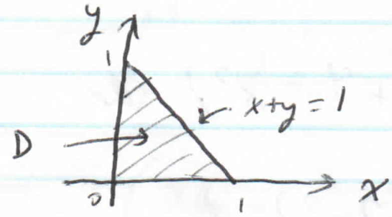
Solution: Write the equations of the planes as

$$z = 1-x-y$$
$$z = \frac{1}{2}(1-x-y).$$

We want the line of intersection of these two planes: this happens when

$$1-x-y = \frac{1}{2}(1-x-y).$$

This holds precisely when $x+y=1$. So the solid is bounded above by $z=1-x-y$ bounded below by $z=\frac{1}{2}(1-x-y)$ and is located over the region D in the xy-plane which is bounded by the lines $x=0$, $y=0$, and $x+y=1$:

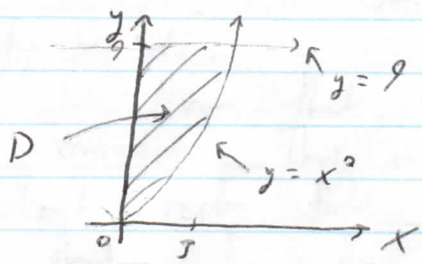


∴ the volume of the solid is given by

$$\iint_D \left(1-x-y-\frac{1}{2}(1-x-y)\right) dA$$
$$= \iint_D \left(\frac{1}{2}-\frac{1}{2}x-\frac{1}{2}y\right) dA$$
$$= \frac{1}{2}\int_0^1 \left(\int_0^{1-y} 1-x-y \, dx\right) dy$$
$$= \frac{1}{2}\int_0^1 \left(x-\frac{x^2}{2}-xy\Big|_0^{1-y}\right) dy$$
$$= \frac{1}{2}\int_0^1 \left((1-y)-\frac{(1-y)^2}{2}-(1-y)y\right) dy$$
$$= \frac{1}{2}\int_0^1 \left(1-y-\frac{1}{2}+y-\frac{y^2}{2}-y+y^2\right) dy$$
$$= \frac{1}{2}\int_0^1 \left(\frac{1}{2}-y+\frac{y^2}{2}\right) dy$$
$$= \frac{1}{2}\left(\frac{1}{2}y-\frac{y^2}{2}+\frac{y^3}{6}\right)\Bigg|_0^1 = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{1}{12}.$$

Exercise: (6.3 #19) Evaluate $\int_0^3 \left(\int_{x^2}^9 x \cos(2y^2)\, dy \right) dx$ by reversing the order of integration.

Solution: First we must determine the region D which we are integrating over: D is defined by $0 \le x \le 3$ and $x^2 \le y \le 9$:



Since we want to change the order of integration, we see that $0 \le y \le 9$, $0 \le x \le \sqrt{y}$ defines the same region D. Reversing the order of integration, we get

$$\int_0^3 \left(\int_{x^2}^9 x \cos(2y^2)\, dy \right) dx$$

$$= \int_0^9 \left(\int_0^{\sqrt{y}} x \cos(2y^2)\, dx \right) dy$$

$$= \int_0^9 \left(\frac{x^2}{2} \cdot \cos(2y^2) \right) \Big|_0^{\sqrt{y}}\, dy$$

$$= \frac{1}{2} \int_0^9 y \cos 2y^2\, dy$$

$\left(u = 2y^2 \right.$ $\left. du = 4y\, dy \right)$ $\rightarrow$ $= \frac{1}{2} \int_0^{162} \frac{1}{4} \cos u\, du$

$$= \frac{1}{8} \left(\sin u \Big|_0^{162} \right)$$

$$= \frac{1}{8} \sin 162 .$$