MATH 3TP3 Assignment #6
Due: Friday, November 2, in class

1. For each of the following sentences: if it is a PRED-theorem, give a PRED-derivation; if it is not, specify a structure in the language of arithmetic in which it is false.

   (a) $\langle \forall x : \exists y : (x + x) = y \supset \exists y : \forall x : (x + x) = y \rangle$

   (b) $\langle \exists y : \forall x : (x + x) = y \supset \forall x : \exists y : (x + x) = y \rangle$ Hint: First prove $\langle \forall x : (x + x) = y \supset \forall x : \exists y : (x + x) = y \rangle$, then try to prove the contrapositive of the sentence in the question

   (c) $\langle \exists y : \forall x : (x + x) = y \supset \exists y : \forall x : x = y \rangle$

   (d) $\forall x : Sx = (x + S0)$

   (e) $\langle \forall x : Sx = (x + S0) \supset \forall x : SSx = (x + SS0) \rangle$

2. For each of the following sentences: if it is a TNT'-theorem, give a TNT'-derivation; if it is not, specify a structure in the language of arithmetic which satisfies the axioms of TNT' but in which the sentence is false.

   Hint: recall the example $\mathrm{Mat}_2(\mathbb{N})$ given in lectures.

   (a) $\forall x : (x * S0) = x$

   (b) $(SS0 * SS0) = (SS0 + SS0)$

   (c) $\sim \forall x : \forall y : (x * y) = (y * x)$

   (d) $\forall x : \forall y : (x * y) = (y * x)$

Bonus Question

Using Gödel's Completeness Theorem, the fact that PRED-derivations are finite, and considering the obvious operations of addition, multiplication and successor on the set $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n for various n, show that there exists a structure in the language of arithmetic which satisfies all of the following sentences:

- $\forall x : x + 0 = x$
- $\forall x : \forall y : x + Sy = S(x + y)$

- $\forall x : \forall y : \langle Sx = Sy \supset x = y \rangle$
- $\exists x : \langle \sim x = 0 \wedge x + x = 0 \rangle$
- The infinite collection of sentences: $\sim S0 = 0$, $\sim SS0 = 0$, $\sim SSS0 = 0$, $\ldots$