MATH 3TP3 Assignment #2 Solutions

1. *Consider the following formal system:*

   - *Alphabet:* {`-`, `!`}*;*
   - *Axioms:* {`-!--`}*;*
   - *Production rules:*

     (I) `-!`$x \mapsto$ `-!-`x

     (II) x`!`$y \mapsto x$`-!-`y

     (III) x`!`$y \mapsto y$`!`x

   (a) Find derivations in the system of `--!----` *and* `---!-`*.*

   (b) Consider the language consisting of strings of the form x`!`y *where* x *and* y *are (non-empty) strings of hyphens.*

   Give an interpretation for this language with respect to which the system is consistent and complete. Briefly explain your reasoning.

   (a)

   `-!--`, `-!---`, `--!----`;

   `-!--`, `-!---`, `---!-`

   (b) Interpret `-`^{n}`!-`^{m} as "n is not equal to m".

   The system is consistent with respect to this interpretation, since the axiom is true ($1 \neq 2$) and the production rules preserve truth:

   (I) $1 \neq n \implies 1 \neq n+1$ for a *positive* integer n;

   (II) $n \neq m \implies n+1 \neq m+1$;

   (III) $n \neq m \implies m \neq n$.

   To see that the system is complete with respect to the interpretation, suppose $n \neq m$ are positive integers; we show that `-`^{n}`!-`^{m} is a theorem. First, suppose $n < m$. Then by $(m-n-1)$ applications of rule (I) starting from the axiom, `-!-`$^{m-(n-1)}$ is a theorem. By $(n-1)$ applications of rule (II) to this, so is `-`^{n}`!-`^{m} as required.

   Finally: if $n > m$, apply rule (III) to the theorem `-`^{m}`!-`^{n}.

2. *We have seen that the* **pq***-system captures addition of positive integers $n + m = k$, and the* **tq***-system captures multiplication $nm = k$. Devise a formal system (with finitely many axioms and production rules) which captures exponentiation $n^m = k$.*

   Argue informally that your system is consistent and complete with respect to the intended interpretation.

   Hint: we want a formal system on an alphabet including e, q *such that* $\texttt{-}^n\texttt{e-}^m\texttt{q-}^k$ *is a theorem iff $n^m = k$. Your first thought might be to copy the pattern of the previous systems and have a production rule "given x*e*y*q*z, produce x*e*y*-q*$zz\ldots z$ with as many copies of z as x has hyphens" - but sadly this is* **not** *a typographical rule of the kind allowed in our formal systems. So you need to find a way around this. You may find it helpful to make use of systems we have already developed, as we did when we captured compositeness.*

   Note: the intention of the question was to capture exponentiation of **positive** integers, hence ignoring the annoying question of what 0^0 is. The answer below is for that.

   - Alphabet: {t, e, q, -}
   - Axioms: {-t-q-, -e-q-}
   - Production rules:
     (I) xt-q$z \mapsto$ -xt-qz-
     (II) xtyq$z \mapsto x$ty-qzx
     (III) xe-q$y \mapsto$ -xe-qy-
     (IV) (xeyqz, xtzqw) $\mapsto x$ey − qw

   This embeds the tq-system, so we know that $-^n t -^m q-^k$ is a theorem iff $nm = k$.

   Consistency: the remaining axiom is true ($1^1 = 1$), and the remaining rules (III) and (IV) preserve truth: $n^1 = m \implies (n+1)^1 = n+1$, and if $n^m = k$ and $nk = l$, then $n^{m+1} = l$.

   Completeness: we prove by induction on m that if n, m are positive integers, then $\texttt{-}^n\texttt{e-}^m\texttt{q-}^{n^m}$ is a theorem.

   For $m = 1$: repeatedly applying (III) to the axiom -e-q-, we find that $\texttt{-}^n\texttt{e-q-}^n$ is a theorem.

Suppose $\texttt{-}^n\texttt{e-}^m\texttt{q-}^{n^m}$ is a theorem. By completeness of the tq-system, $\texttt{-}^n\texttt{t-}^{n^m}\texttt{q-}^{n^{m+1}}$ is a theorem. Applying (IV) to these, we deduce that $\texttt{-}^n\texttt{e-}^{m+1}\texttt{q-}^{n^{m+1}}$ is a theorem.

3. *Determine which of the following strings are well formed. For those that are, produce their parse (formation) trees.*

   The first and fourth are not well-formed; the second and third are.

4. *For each of the following wff's, determine whether it is true or false according to an interpretation under which P and Q are true and R is false*

   All but the third are true.