MATH 3TP3 Assignment #4 Solutions

1. Yet more truth tables.

2. (a) I'll give a detailed proof, although I expected and accepted much less detailed answers.

   First we prove:

   Claim. *Given a truth assignment f, the truth value of α with respect to f is that of σ with respect to the truth assignment f' which is the same as f except that it assigns True to P iff τ is true with respect to f.*

   Proof. Inductively assume the claim when σ is replaced by a shorter wff.

   If $\sigma = P$, then $\alpha = \tau$ and the claim is clear.

   If σ is a different propositional variable, then $\alpha = \sigma$ and the claim is again clear.

   If $\sigma = \langle\sigma' \wedge \sigma''\rangle$, then let α' and respectively α'' be the result of substituting all instances of P in σ' and respectively σ'' with τ; the inductive hypothesis applies to these. Now $\alpha = \langle\alpha' \wedge \alpha''\rangle$, and

   $$\begin{aligned} &\alpha \text{ is true wrt } f' && \iff \\ &\alpha' \text{ is true wrt } f' \text{ and } \alpha'' \text{ is true wrt } f' && \iff \\ &\sigma' \text{ is true wrt } f \text{ and } \sigma'' \text{ is true wrt } f && \iff \\ &\sigma \text{ is true wrt } f. \end{aligned}$$

   The other cases are similar. □

   Now we show that α is a tautology. Given a truth assignment f, by the claim α is true wrt f iff σ is true wrt f'; but σ is a tautology, so it is true wrt f'; hence α is true wrt f.

   (b) Apply part (a) to 1(i) twice, first replacing P with $\langle P \supset \langle R \wedge P\rangle$ and then Q with $\langle Q \supset P\rangle$.

3. (i) [

   $\sim\langle\sim P \vee \sim Q\rangle$

   $\langle\sim\sim P \wedge \sim\sim Q\rangle$

$\sim\sim P$
P
$\sim\sim Q$
Q
$\langle P \wedge Q \rangle$
$\sim\sim \langle P \wedge Q \rangle$
]
$\langle \sim \langle \sim P \vee \sim Q \rangle \supset \sim\sim \langle P \wedge Q \rangle \rangle$
$\langle \sim \langle P \wedge Q \rangle \supset \langle \sim P \vee \sim Q \rangle \rangle$

(ii) Not a tautology, so by the Soundness theorem, no PROP-derivation exists.

(iii) [
$\langle P \supset \langle Q \wedge \sim Q \rangle \rangle$
[
$\sim\sim P$
P
$\langle P \supset \langle Q \wedge \sim Q \rangle \rangle$
$\langle Q \wedge \sim Q \rangle$
[
$\sim\sim \langle P \supset P \rangle$
$\sim Q$
]
$\langle \sim\sim \langle P \supset P \rangle \supset \sim Q \rangle$
$\langle Q \supset \sim \langle P \supset P \rangle \rangle$
Q
$\sim \langle P \supset P \rangle$
]
$\sim\sim P \supset \sim \langle P \supset P \rangle$
$\langle P \supset P \rangle \supset \sim P$
[
P
]
$\langle P \supset P \rangle$
$\sim P$
]
$\langle \langle P \supset \langle Q \wedge \sim Q \rangle \rangle \supset \sim P \rangle$

4. (i) [

P

$\sim\sim P$

]

$\langle P \supset \sim\sim P \rangle$

(ii) trivial

(iii) trivial

(iv) [

$\langle \sim P \wedge \sim Q \rangle$

$\sim P$

[

$\langle P \wedge Q \rangle$

P

]

$\langle \langle P \wedge Q \rangle \supset P \rangle$

$\langle \sim P \supset \sim \langle P \wedge Q \rangle \rangle$

$\sim \langle P \wedge Q \rangle$

]

$\langle \langle \sim P \wedge \sim Q \rangle \supset \sim \langle P \wedge Q \rangle \rangle$

(v) [

$\langle P \wedge Q \rangle$

Q

[

P

Q

]

$\langle P \supset Q \rangle$

]

$\langle \langle P \wedge Q \rangle \supset \langle P \supset Q \rangle \rangle$

(vi) $\vdash \langle \langle \sim\sim P \wedge \sim Q \rangle \supset \sim \langle \sim P \vee Q \rangle \rangle$ by a substitution instance of (iv). So it suffices to find a derivation using this as an axiom:

[

$\langle P \wedge \sim Q \rangle$

P

$\sim\sim P$

$\sim Q$

$\langle \sim\sim P \supset \sim Q \rangle$

$\langle \langle \sim\sim P \wedge \sim Q \rangle \supset \sim \langle \sim P \vee Q \rangle \rangle (axiom)$

$\sim \langle \sim P \vee Q \rangle$

[

$\langle P \supset Q \rangle$

[

$\sim\sim P$

P

$\langle P \supset Q \rangle$

Q

]

$\langle \sim\sim P \supset Q \rangle$

$\langle \sim P \vee Q \rangle$

]

$\langle \langle P \supset Q \rangle \supset \langle \sim P \vee Q \rangle \rangle$

$\langle \sim \langle \sim P \vee Q \rangle \supset \sim \langle P \supset Q \rangle \rangle$

$\sim \langle P \supset Q \rangle$

]

$\langle \langle P \wedge \sim Q \rangle \supset \sim \langle P \supset Q \rangle \rangle$

(vii) [

$\langle \sim P \wedge \sim Q \rangle$

$\sim P$

[

$\sim Q$

$\sim P$

]

$\langle \sim Q \supset \sim P \rangle$

$\langle P \supset Q \rangle$

]

$\langle \langle \sim P \wedge \sim Q \rangle \supset \langle P \supset Q \rangle \rangle$

(viii) [

$\langle P \wedge Q \rangle$
Q
[
$\sim P$
Q
]
$\langle \sim P \supset Q \rangle$
$\langle P \vee Q \rangle$

]
$\langle \langle P \wedge Q \rangle \supset \langle P \vee Q \rangle \rangle$

(ix) Essentially the same as the last one.

(x) Easy.