MATH 3TP3 Assignment #7 Solutions

1. Read the derivation carefully, and get a feel for how it works.

2. (i)

$\forall x : \forall y : (x + Sy) = S(x + y)$	(Axiom)
$\forall y : (x + Sy) = S(x + y)$	(Spec)
$(x + S0) = S(x + 0)$	(Spec)
$\forall x : (x + 0) = x$	(Axiom)
$(x + 0) = x$	(Spec)
$S(x + 0) = Sx$	(Cong)
$(x + S0) = Sx$	(Trans)
$Sx = (x + S0)$	(Sym)
$\forall x : Sx = (x + S0)$	(Gen)

(ii)

$\forall x : (x \cdot 0) = 0$	(Axiom)
$(S0 \cdot 0) = 0$	(Spec)
[	
$\quad (S0 \cdot x) = x$	
$\quad \forall x : \forall y : (x \cdot Sy) = ((x \cdot y) + x)$	(Axiom)
$\quad \forall y : (S0 \cdot Sy) = ((S0 \cdot y) + S0)$	(Spec)
$\quad (S0 \cdot Sx) = ((S0 \cdot x) + S0)$	(Spec)
$\quad \forall x : Sx = (x + S0)$	(From (i))
$\quad S(S0 \cdot x) = ((S0 \cdot x) + S0)$	(Spec)
$\quad ((S0 \cdot x) + S0) = S(S0 \cdot x)$	(Sym)
$\quad (S0 \cdot Sx) = S(S0 \cdot x)$	(Trans)
$\quad S(S0 \cdot x) = Sx$	(Cong)
$\quad (S0 \cdot Sx) = Sx$	(Trans)
]	
$\langle (S0 \cdot x) = x \supset (S0 \cdot Sx) = Sx \rangle$	(Fantasy)
$\forall x : \langle (S0 \cdot x) = x \supset (S0 \cdot Sx) = Sx \rangle$	(Gen)
$\forall x : (S0 \cdot x) = x$	(Induction)

(iii) This one involves some awkward futzing with variables (see lines 15-19 and 39-44)...

1.	$\forall x : (x + 0) = x$	(Axiom)
2.	$((x + y) + 0) = (x + y)$	(Spec)
3.	$(y + 0) = y$	(Spec)
4.	$\forall x : x = x$	(Axiom)
5.	$x = x$	(Spec)

6.	$(x + (y + 0)) = (x + y)$	(Cong)
7.	$(x + y) = (x + (y + 0))$	(Sym)
8.	$((x + y) + 0) = (x + (y + 0))$	(Trans 2,7)
9.	$\forall y : ((x + y) + 0) = (x + (y + 0))$	(Gen)
10.	$\forall x : \forall y : ((x + y) + 0) = (x + (y + 0))$	(Gen)
11.	[	
12.	$\quad \forall x : \forall y : ((x + y) + z) = (x + (y + z))$	
13.	$\quad \forall y : ((x + y) + z) = (x + (y + z))$	(Spec)
14.	$\quad ((x + y) + z) = (x + (y + z))$	(Spec)
15.	$\quad \forall x : \forall y : (x + Sy) = S(x + y)$	(Axiom)
16.	$\quad \forall y : ((x + y') + Sy) = S((x + y') + y)$	(Spec)
17.	$\quad ((x + y') + Sz) = S((x + y') + z)$	(Spec)
18.	$\quad \forall y' : ((x + y') + Sz) = S((x + y') + z)$	(Gen)
19.	$\quad ((x + y) + Sz) = S((x + y) + z)$	(Spec)
20.	$\quad S((x + y) + z) = S(x + (y + z))$	(Cong 14)
21.	$\quad ((x + y) + Sz) = S(x + (y + z)$	(Trans)
22.	$\quad \forall y : (x + Sy) = S(x + y)$	(Spec 15)
23.	$\quad (x + S(y + z)) = S(x + (y + z))$	(Spec)
24.	$\quad S(x + (y + z)) = (x + S(y + z))$	(Sym)
25.	$\quad ((x + y) + Sz) = (x + S(y + z))$	(Trans 21,24)
26.	$\quad \forall y : (y' + Sy) = S(y' + y)$	(Spec 15)
27.	$\quad (y' + Sz) = S(y' + z)$	(Spec)
28.	$\quad \forall y' : (y' + Sz) = S(y' + z)$	(Gen)
29.	$\quad (y + Sz) = S(y + z)$	(Spec)
30.	$\quad S(y + z) = (y + Sz)$	(Sym)
31.	$\quad x = x$	(Carry 5)
32.	$\quad (x + S(y + z)) = (x + (y + Sz))$	(Cong)
33.	$\quad ((x + y) + Sz) = (x + (y + Sz))$	(Trans 25,32)
34.	$\quad \forall y : ((x + y) + Sz) = (x + (y + Sz))$	(Gen)
35.	$\quad \forall x : \forall y : ((x + y) + Sz) = (x + (y + Sz))$	(Gen)
36.	]	
37.	$\langle \forall x : \forall y : ((x + y) + z) = (x + (y + z)) \supset$ $\forall x : \forall y : ((x + y) + Sz) = (x + (y + Sz))\rangle$	(Fantasy)
38.	$\forall z : \forall x : \forall y : ((x + y) + z) = (x + (y + z))$	(Induction)
39.	$\forall x : \forall y : ((x + y) + z) = (x + (y + z))$	(Spec)

40. $\forall y : ((x + y) + z) = (x + (y + z))$ (Spec)
41. $((x + y) + z) = (x + (y + z))$ (Spec)
42. $\forall z : ((x + y) + z) = (x + (y + z))$ (Gen)
43. $\forall y : \forall z : ((x + y) + z) = (x + (y + z))$ (Gen)
44. $\forall x : \forall y : \forall z : ((x + y) + z) = (x + (y + z))$ (Gen)

(iv) Here's a relatively neat way to get this. Note that in the inductive step, the inductive hypothesis (the premise of the fantasy) isn't actually used!

$\forall x : x = x$
$0 = 0$
$\langle 0 = 0 \supset \langle 0 \cdot 0 = 0 \supset 0 = 0 \rangle\rangle$ (Tautology)
$\langle 0 \cdot 0 = 0 \supset 0 = 0 \rangle$
[
$\quad\langle (x \cdot x) = 0 \supset x = 0 \rangle$
$\quad$[
$\quad\quad(Sx \cdot Sx) = 0$
$\quad\quad\forall x : \forall y : (x \cdot Sy) = ((x \cdot y) + x)$
$\quad\quad\forall y : (Sx \cdot Sy) = ((Sx \cdot y) + Sx)$
$\quad\quad(Sx \cdot Sx) = ((Sx \cdot x) + Sx)$
$\quad\quad\forall x : \forall y : (x + Sy) = S(x + y)$
$\quad\quad\forall y : ((Sx \cdot x) + Sy) = S((Sx \cdot x) + y)$
$\quad\quad((Sx \cdot x) + Sx) = S((Sx \cdot x) + x)$
$\quad\quad(Sx \cdot Sx) = S((Sx \cdot x) + x)$
$\quad\quad S((Sx \cdot x) + x) = (Sx \cdot Sx)$
$\quad\quad S((Sx \cdot x) + x) = 0$
$\quad$]
$\quad\langle (Sx \cdot Sx) = 0 \supset S((Sx \cdot x) + x) = 0 \rangle$
$\quad\forall x : \sim Sx = 0$
$\quad\sim S((Sx \cdot x) + x) = 0$
$\quad\langle \sim S((Sx \cdot x) + x) = 0 \supset \sim (Sx \cdot Sx) = 0 \rangle$
$\quad\sim (Sx \cdot Sx) = 0$
$\quad\langle \sim (Sx \cdot Sx) = 0 \supset \langle (Sx \cdot Sx) = 0 \supset Sx = 0 \rangle\rangle$ (Tautology)
$\quad\langle (Sx \cdot Sx) = 0 \supset Sx = 0 \rangle$
]
$\langle\langle (x \cdot x) = 0 \supset x = 0 \rangle \supset \langle (Sx \cdot Sx) = 0 \supset Sx = 0 \rangle\rangle$
$\forall x : \langle\langle (x \cdot x) = 0 \supset x = 0 \rangle \supset \langle (Sx \cdot Sx) = 0 \supset Sx = 0 \rangle\rangle$

$\forall x : \langle (x \cdot x) = 0 \supset x = 0 \rangle$

3. Consider the structure $\mathbb{N}^* :=< \mathbb{N} \cup \{\infty\}; S, +, \cdot, 0 >$ where we extend the usual definitions of $S, +, \cdot$ on $\mathbb{N}$ to ∞ as follows:

$$\begin{aligned} S(\infty) &:= \infty; & & \\ x + \infty &:= \infty & & \text{for all } x; \\ \infty + x &:= \infty & & \text{for all } x; \\ x \cdot \infty &:= \infty & & \text{for all } x; \\ \infty \cdot 0 &:= 0; & & \\ \infty \cdot x &:= \infty & & \text{for all } x \neq 0. \end{aligned}$$

Then we can check that $\mathbb{N}^*$ satisfies the axioms of TNT'. Note that $\infty \cdot 0 = 0$, but $0 \cdot \infty = \infty$... this is necessary to make the axioms work out.

Now does $\mathbb{N}^*$ satisfy

$$\forall x : \langle x = 0 \vee \exists y : Sy = x \rangle ?$$

Yes, it does! ∞ *is* the successor of something, namely ∞!

We can't set $S\infty$ to anything else without contradicting one the axioms - 0 is not allowed to be a successor, and nothing can be the successor of two distinct elements.

So in fact, the hint I gave in the question was entirely misleading! Sorry about that.

However, our old friend $\mathrm{Mat}_2(\mathbb{N})$ **does** provide an example. Thanks to Michael Birch for pointing this out!

$$\mathrm{Mat}_2(\mathbb{N}) \not\models \forall x : \langle x = 0 \vee \exists y : Sy = x \rangle,$$

since e.g. consider

$$\begin{matrix} 10 \\ 00 \end{matrix}$$

The structure $\mathbb{N}^*$ is still an interesting model of TNT' to consider, though. It demonstrates, for example, that

$$TNT' \not\vdash \forall x :\sim Sx = x.$$