Ramsey Theory

Example:
Given 6 people,
either there are 3 who all like each other,
or there are 3 no two of whom like each other.

Abstract version:
$K_n :=$ "complete graph on n vertices"
$= n$ points with an edge between each pair.

Colour the edges of K_6 each either red or blue,
then there's a red copy of K_3 or there's a blue copy of K_3;
i.e. there is a monochromatic triangle.

Denote this fact
$$K_6 \to K_3, K_3$$

Proof:
Pick a vertex v_0.
Consider the 5 edges from it.
3 of them are red or 3 of them are blue, since $5 > (3-1) + (3-1)$.
Say 3 are red, and consider the 3 other vertices of these red edges.
If the edges between them are all blue, they form a blue triangle and we're done.
Else, some edge is red; but then it along with the edges from v_0 form a red triangle, and we're done.

Ramsey's Theorem for 2-coloured graphs:
Given n and m positive integers,
there exists r such that for any red-blue colouring of the edges of K_r,
there are n vertices all edges between which are red or there are m vertices all edges between which are blue.

Notation:
We write
$$K_r \to K_n, K_m$$
to mean that r has this property,
and we let $r(m, n)$ ("the (m,n)th Ramsey number") be the least such r.

Remarks:
We saw that $K_6 \to K_3, K_3$;
it's easy to see that $K_5 \not\to K_3, K_3$,
so $r(3, 3) = 6$.

It has been shown that
$$r(3, 4) = 9$$

$r(3,5) = 14$
$r(4,4) = 18$

$r(5,5)$ is unknown! All we know is
$43 \le r(5,5) \le 49$.

Erdös:
"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

Proof of Theorem:
Suppose inductively that
$K_b \to K_{n-1}, K_m$
and
$K_c \to K_n, K_{m-1}$.

We show that
$K_{b+c} \to K_n, K_m$.

So colour K_{b+c}, and suppose there's no red K_n and no blue K_m.

Pick a vertex v_0; consider the $b+c-1$ edges from it.
Since $b+c-1 > (b-1)+(c-1)$,
b of the edges are red or c of the edges are blue.

Say b are red.
Consider the K_b formed by the vertices these edges connect to v_0.
By the inductive hypothesis, it contains a red K_{n-1} or a blue K_m.
If it contains a red K_{n-1}, adjoining v_0 yields a red K_n; contradiction.
If it contains a blue K_m, then so does our original K_{b+c}; contradiction.

A symmetrical argument applies in the case that c of the edges from v_0 are blue.

Remark:
This proof yields a recursive upper bound on the Ramsey numbers:
$r(m,n) \le r(n-1,m) + r(n,m-1)$
(but this is far from sharp).