Permutations

A permutation of a finite set S is an ordered list of its elements.

An r-permutation of S is an ordered list of r of its elements.

Warning:
there is another, related, meaning of 'permutation': an element of the group of bijections of S. We won't use that meaning in this course.

$P(n, r) :=$ number of r-permutations of a set of size n.

e.g. $P(26, 5)$ = number of strings of 5 distinct letters from the Roman alphabet.

By the multiplication principle,
$P(n, r) = n * (n-1) * ... * (n-(r-1))$ (n choices for first, $n-1$ for second...)

$P(n, r) = n!/(n-r)!$

$P(n, n) = n!$

Remark:
Can interpret "$P(n, r) = n!/(n-r)!$" as follows:
We can obtain an r-permutation of S by taking the first r elements of a permutation of S.
Partition the permutations of S according to the r-permutation which results from this: we see that the elements of each set of the partition correspond to the permutations of the left-over $n-r$ elements, so we recover the formula by the division principle.

A circular r-permutation of a set is a way of putting r of its elements around a circle, with two such considered equal if one can be rotated to the other.

We can obtain a circular r-permutation from an r-permutation by "joining the ends into a circle". Each circular r-permutation is obtained from r different r-permutations, so by the division principle:
number of circular r-permutations of n elements
$= P(n, r)/r$
$= n!/r(n-r)!$

Example:
How many different kinds of necklace can be made from 7 spherical beads of different colours? Consider two necklaces to be of the same kind when they can be non-destructively manipulated to look the same.

Solution:
There are $7!/7 = 6!$ circular permutations of the 13 colours. Each kind of necklace is obtained from exactly **two** circular permutations, because flipping the necklace in space doesn't change the kind. So the answer is
$6!/2 = 360$.

Example:
How many ways can 13 people be sat around a round table, if Professor Q is not to be sat next to his arch-nemesis Inspector P?

Solution:
Without the restriction, there would be 12! seating arrangements.
Consider seating everyone but P; each such arrangement yields two forbidden arrangements of all 13, one by placing P to Q's right and one by placing P to Q's left. We count each forbidden arrangement once in this way.

So the answer is $12! - 2 * 11! = 10 * 11! = 399168000$

Subsets ("Combinations")

An <u>r-subset</u>, or <u>r-combination</u>, of a set S is a subset of size r.

$C(n, r) = \binom{n}{r}$ = number of r-subsets of a set of size n.

e.g. $\binom{26}{5}$ = number of unordered selections of 5 letters from the roman alphabet

Theorem:
$\binom{n}{r} = n!/r!(n-r)!$

Proof:
The r-permutations of a set are precisely the permutations of the r-subsets.
Each r-subset has $r!$ permutations, so

$$P(n, r) = r! * \binom{n}{r}.$$

So

$$\binom{n}{r} = P(n, r)/r! = n!/r!(n-r)!.$$

$\binom{n}{r}$ is also called a "binomial coefficient".

Example:
If we expand out $(x+y)^n$ and collect terms to obtain

$$a_0x^n + a_1x^{n-1}y + ... + a_{n-1}xy^{n-1} + a_ny^n,$$

what are the coefficients a_k?

Solution:
a_k is the number of ways of choosing y k times when we have to choose either x or y from each factor of the product

$$(x+y)(x+y)...(x+y) \text{ (n times)},$$

which is the number of subsets of this set of n factors.

So $a_k = \binom{n}{k}$.

Theorem [Pascal's Formula]:
If $0 < k < n$,
$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Proof:
$|S| = n$.
Fix $x \in S$; let $S' := S \setminus \{x\}$.
Partition the k-subsets of S according to whether they contain x.
Those which don't correspond to k-subsets of S',
those which do correspond to $(k-1)$-subsets of S'.

Theorem:
$\Sigma_{k=0}^{n} \binom{n}{k} = 2^n$

Proof:
$|S| = n$.
$\Sigma_{k=0}^{n} \binom{n}{k}$ = number of subsets of S.
But to choose a subset of S is to choose for each element of S whether it should or should not go in to the subset. That's two choices for each of the n elements, so by the multiplication principle there are $2 * 2 * ... * 2 = 2^n$ subsets of S.