Heterosexuality, and Hall's Marriage Theorem

Basic "marriage" problem

A set A of men;
a set B of women;
certain pairings (a, b) are "compatible".

A **matching** is a choice of some compatible pairings ("marriages"), such that no man is paired to multiple women and no woman to multiple men.

Can we find a matching in which every woman is paired with some man?

Graphical formulation

(You can skip this if you don't know what a graph is.)

Bipartite graph: two collections of A and B of vertices; all edges have one vertex in A and the other in B.

A matching is a set of edges such that no vertex is incident to more than one of the edges.

Can we find a matching such that every $b \in B$ is incident to one of the edges?

SDR formulation

Given a family of subsets $(A_i)_{i \in B}$ of a set A,
can we find elements $a_i \in A_i$ which are distinct, i.e. $a_i \neq a_j$ when $i \neq j$?

Such a selection of $a_i \in A_i$ for $i \in B$ is called a system of distinct representatives (SDR).

Other instances

Example:
Given an $m \times n$ board with certain squares missing,
can we place a rook on each row such that no row or column contains two rooks?

Here we marry rows and columns, consecrating the marriage with the symbolic placing of a rook.

Example:
Certain jobs are to performed by certain people, one job by each;
not all people are suitable for all jobs.
Can we assign each job to some person?

Example:
Various classes are to be scheduled at various times in various rooms;
some classes require certain rooms,
and some times are unsuitable for some classes.

Suppose all classes are to start on the hour.

Can all classes be scheduled?

Here, we marry classes and hour-room pairs.

Hall's Marriage Theorem

Definition:
A family $(A_i)_{i\in B}$ of sets satisfies the marriage condition if for any k, the union of any k of the sets in the family has size at least k;
i.e. for every $B' \subseteq B$,

$$\left|\bigcup_{i\in B'} A_i\right| \geq |B'|.$$

Theorem [Philip Hall, 1935]:
A family $(A_i)_{i\in B}$ of finite sets has a system of distinct representatives iff it satisfies the marriage condition.

Proof:
The marriage condition is necessary, since if $a_i \in A_i$ is an SDR and $B' \subseteq B$

$\bigcup_{j\in B'} A_j \supseteq \{a_j \mid j \in B'\}$

so, by distinctness,

$\left|\bigcup_{j\in B'} A_j\right| \geq |\{a_j \mid j \in B'\}| = |B'|.$

Now suppose the marriage condition holds, and suppose inductively that the theorem holds when B is smaller.

Suppose first that for all $B' \subsetneq B$,

$$\left|\bigcup_{i\in B'} A_i\right| \geq |B'| + 1.$$

Let $i_0 \in B$,
and let $a_0 \in A_{i_0}$. We proceed by deleting i_0 and a.
Let $A'_i := A_i \setminus \{a_0\}$;
then $(A'_i)_{i\in B\setminus\{i_0\}}$ also satisfies the marriage condition.
Indeed, for any $B' \subseteq B \setminus \{i_0\}$,

$|\bigcup_{j\in B'} A'_i| \geq |\bigcup_{j\in B'} A_i| - 1$
$\geq (|B'| + 1) - 1$
$= |B'|.$

So by the inductive hypothesis, this family has an SDR,
adjoining a_0 to which yields an SDR for $(A_i)_{i\in B}$.

Otherwise, say $B^0 \subsetneq B$ with

$$\left|\bigcup_{i\in B^0} A_i\right| = |B^0|.$$

Let $A^0 := \bigcup_{i\in B^0} A_i$.
We proceed by deleting B^0 and A^0.
Let $A'_i := A_i \setminus A^0$, and consider $(A'_i)_{i\in(B\setminus B^0)}$.
Then for $B' \subseteq B \setminus B^0$,

$$\begin{aligned} |\textstyle\bigcup_{j\in B'} A'_i| &\geq |\textstyle\bigcup_{j\in B'\cup B^0} A_i| - |B^0| \\ &\geq |B'\cup B^0| - |B^0| \\ &= |B'| \end{aligned}$$

So by the inductive hypothesis, this family has an SDR.

Now $(A_i)_{i\in B^0}$ also satisfies MC,
and $B^0 \subsetneq B$,
so by the inductive hypothesis,
this family also has an SDR.

Since $A'_i \cap A_j = \emptyset$ for $j \in B^0$,
the union of these two SDRs is an SDR for $(A_i)_{i\in B}$.
$\square$

Dominoes on a chessboard

Consider an $m \times n$ chequered board,
meaning that each square is either white or black, and no two neighbouring squares are of the same colour,
and suppose we delete some squares.

e.g.

```
  , . ,   ,
, .   . , .
.     , .
,     . , .
```

(here I use dots for white squares, commas for black squares, and spaces for deleted squares.).

Can we put 2×1 dominoes on the non-deleted squares of the board, such that the dominoes don't overlap and every non-deleted square is covered?

e.g. here's a solution in the case of the above example, using 8 dominoes denoted by the corresponding numbers:

```
  3 3 4   5
2 2   4 6 5
1     7 6
1     7 8 8
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.

We can view this as a marriage problem: we want to marry white squares to black squares, and only adjacent squares are compatible.

So we can solve the problem iff there are as many white squares as black squares and the family of sets of (say) black squares adjacent to the white squares satisfies MC.

Exercise:
Draw the corresponding bipartite graph for the example board above.

Find some boards which don't satisfy the marriage condition,
then try (and fail) to cover them with dominoes.

Latin squares

Definition:
A Latin rectangle is an $m \times c$ array with each entry an element of $\{1, ..., n\}$, such that no number appears twice in any row or column.

Theorem:
Any $m \times n$ Latin rectangle with $m < n$ can be completed by adding rows to form an $n \times n$ Latin square.

Proof:
By induction, it suffices to show that we can add a row to form an $m \times n+1$ Latin rectangle.

Let A_i, for $i = 1, ..., n$, be the set of numbers in $\{1, ..., n\}$ **not** appearing in the ith column.

Then $(A_i)_{i\in\{1,...,n\}}$ satisfies the marriage condition.
Indeed suppose $I \subseteq \{1, ..., n\}$.
Each number in $\{1, ..., n\}$ occurs in each of the m rows,
and so occurs in at most $n - m$ of the A_i.
Each A_i has size $n - m$.
So

$$|I|(n - m) = \sum_{i\in I} |A_i| \leq (n - m)|\bigcup_{i\in I} A_i|,$$

so

$$|\bigcup_{i\in I} A_i| \geq |I|.$$

So by Hall's Marriage Theorem, an SDR $(a_i \in A_i)_{i\in\{1,...,n\}}$ exists;
by distinctness and the definition of A_i, adding this as a row yields an $m \times n + 1$ Latin rectangle. $\square$