STRUCTURE AND FINITENESS PROPERTIES OF SUBDIRECT PRODUCTS OF GROUPS

MARTIN R. BRIDSON AND CHARLES F. MILLER III

Abstract. We investigate the structure of subdirect products of groups, particularly their finiteness properties. We pay special attention to the subdirect products of free groups, surface groups and HNN extensions. We prove that a finitely presented subdirect product of free and surface groups virtually contains a term of the lower central series of the direct product or else fails to intersect one of the direct summands. This leads to a characterization of the finitely presented subgroups of the direct product of 3 free or surface groups, and to a solution to the conjugacy problem for arbitrary finitely presented subgroups of direct products of surface groups. We obtain a formula for the first homology of a subdirect product of two free groups and use it to show there is no algorithm to determine the first homology of a finitely generated subgroup.

A useful structure theory for subgroups of finite direct products of groups has yet to be developed. To begin to study such subgroups it is natural to assume one knows about the subgroups of the direct factors and to concentrate on subdirect products. Recall that G is termed a *subdirect product* of the groups $A_1, \dots, A_n$ if $G \subseteq A_1 \times \cdots \times A_n$ is a subgroup that projects surjectively to each factor.

Work by various authors has exposed the surprisingly rich structure to be found amongst the subdirect products of superficially-tame groups. For example, in contrast to the fact that subdirect products of abelian or nilpotent groups are again in the specified class, non-abelian free groups harbour a great diversity of subdirect products, including some with unsolvable decision problems [21]. This diversity has long been known, but it is only as a result of more recent work by Baumslag-Roseblade [5] and Bridson-Howie-Miller-Short [9] that it has been understood as a phenomenon that is intimately tied to the failure of various homological finiteness conditions.

From this background we pick out the three strains of thought to be developed in this article: the subtlety of subdirect products in general;

1991 *Mathematics Subject Classification.* 20F05, 20E07, 20F67, 20J05, 20F10.

Key words and phrases. subgroup structure, fibre product, subdirect product, free groups, surface groups, homology of groups, HNN extension.

the usefulness of finiteness properties in exploring this subtlety; and a special interest in the subdirect products of free groups and their associates such as surface groups. Further impetus for the study of subdirect products of surface groups comes from the work of Delzant and Gromov [15]: they proved that if a torsion-free group Γ is the fundamental group of a compact Kähler manifold and Γ has sufficient multi-ended splittings of the appropriate form, then there is a short exact sequence $1 \to \mathbb{Z}^n \to \Gamma_0 \to S \to 1$, where S is a subdirect product of surface groups and $\Gamma_0 \subset \Gamma$ is a subgroup of finite index.

Our first purpose in this article is to provide a systematic and clarifying treatment of the core issues that have emerged in the study of subgroups of direct products. We focus on subdirect products of groups as objects worthy of study in their own right. Following a discussion of some immediate consequences of the definition (Section 1), we address the question of when such groups are finitely generated and (more subtly) when they are finitely presented (Section 2). We illustrate the general theory with a string of explicit examples. In Section 3 we develop homological analogues of the results in Section 2. The main result in Section 3 provides one with a tool for analyzing the homology of subdirect products, a special case of which is the following:

Theorem A. *Suppose that $G \leq F_1 \times F_2$ is a subdirect product of two free groups F_1 and F_2. Let $L_i = G \cap F_i$. Then*

$$H_1(G, \mathbb{Z}) \cong H_1(F_2, \mathbb{Z}) \oplus H_2(F_1/L_1, \mathbb{Z}) \oplus C$$

where C is a subgroup of $H_1(F_1, \mathbb{Z})$ and hence is free abelian of rank at most the rank of F_1.

In the light of the Baumslag-Dyer-Miller construction [4], this yields:

Corollary B. *Let F_1 and F_2 be non-abelian free groups. Then there are continuously many subdirect products $G \leq F_1 \times F_2$ having non-isomorphic $H_1(G, \mathbb{Z})$.*

Gordon [17] showed that H_2 of a finitely presented group is not algorithmically computable. By combining his construction with the above theorem we deduce:

Corollary C. *If F_1 and F_2 are non-abelian free groups, there is no algorithm to compute for an arbitrary finitely generated subgroup $G \leq F_1 \times F_2$ the (torsion-free) rank of $H_1(G, \mathbb{Z})$, nor is there an algorithm to determine whether $H_1(G, \mathbb{Z})$ has non-zero torsion elements.*

In Section 4 we sharpen our focus on subdirect products of free groups and surface groups and prove:

Theorem D. *Let $\Sigma_1, \ldots, \Sigma_n$ be free groups or surface groups and let $G \leq \Sigma_1 \times \cdots \times \Sigma_n$ be a subdirect product which intersects each factor Σ_i non-trivially. If G is finitely presented, then each Σ_i contains a normal subgroup K_i of finite index such that*

$$\gamma_{n-1}(K_i) \subseteq G \cap \Sigma_i \subseteq K_i.$$

Thus the quotients $\Sigma_i/(G \cap \Sigma_i)$ are virtually nilpotent of class at most $n-2$, and hence both $\Sigma_i/(G\cap\Sigma_i)$ and $G/(G\cap\Sigma_i)$ are finitely presented.

In the case of three factors, a refinement of the analysis used to prove Theorem D yields the following characterization of finitely presented subgroups.

Theorem E. *Let $\Sigma_1, \Sigma_2, \Sigma_3$ be finitely generated free groups or surface groups and let $G \leq \Sigma_1 \times \Sigma_2 \times \Sigma_3$ be a subdirect product which intersects each factor Σ_i non-trivially. Then G is finitely presented if and only if each Σ_i contains a normal subgroup K_i of finite index such that the subgroup $G_0 = G \cap (K_1 \times K_2 \times K_3)$ satisfies the following condition: there is an abelian group Q and epimorphisms $\varphi_i : K_i \to Q$ such that $G_0 \cap \Sigma_i = \ker \varphi_i$ $(i = 1, 2, 3)$ and G_0 is the kernel of the map $\varphi_1 + \varphi_2 + \varphi_3$.*

Theorem D provides considerable control over the finitely presented subdirect products of free and surface groups. For example, in Section 5 we use it to prove:

Theorem F. *If D is a direct product of free and surface groups, then every finitely presented subgroup $G \subset D$ has a solvable conjugacy problem. The membership problem for G is also solvable.*

In Section 6 we shift our attention to subdirect products of HNN extensions and establish a criterion for proving that certain fibre products are not finitely presented. To exemplify the utility of this criterion, we combine it with an explicit calculation in group homology to prove the following (cf. [2]):

Theorem G. *Let $A = \mathrm{BS}(2,3) = \langle b, t \mid t^{-1}b^2t = b^3 \rangle$ and let $q : A \to \mathbb{Z} = \langle t \mid \ \rangle$ be the map defined by sending t to t and b to 1. Then the untwisted fibre product $G \subset A \times A$ associated to q is finitely generated but not finitely presented, and $H_2(G, \mathbb{Z}) = 0$.*

Added in proof: In recent joint work with J. Howie and H. Short [10], [11] we have proved that Theorems D, E and F extend to subdirect products of limit groups and that the bound on the nilpotency class in Theorem D is optimal.

1. Generalities about subdirect products

If $G \leq A_1 \times A_2$ is a subdirect product of groups and we put $L_i = G \cap A_i$, then G projects onto A_2 with kernel L_1. The composition of this map with $A_2 \to A_2/L_2$ maps G onto A_2/L_2 with kernel $L_1 \times L_2$. By symmetry, we have isomorphisms

$$A_1/L_1 \cong G/(L_1 \times L_2) \cong A_2/L_2.$$

Subdirect products of two groups are closely associated to the fibre product construction in the category of groups. Recall that, associated to each pair of short exact sequences of groups $1 \to L_i \to A_i \overset{p_i}{\to} Q \to 1,\ i = 1, 2$, one has the *fibre product* or *pullback*

$$P = \{(x, y) \in A_1 \times A_2 \mid p_1(x) = p_2(y)\}.$$

Observe that $L_1 \times L_2 \leq P$ and that P is generated by $L_1 \times L_2$ together with any set of lifts (t_1, t_2) of a set of generators $p_i(t_i)$ for Q.

It is clear that a fibre product is always a subdirect product. Conversely, given a subdirect product $G \leq A_1 \times A_2$, we can define $L_i := G \cap A_i$ and $Q := G/(L_1 \times L_2)$, take p_i to be the composition of the homomorphisms $A_i \to A_i/L_i \overset{\sim}{\to} Q$, and regard Q as "the diagonal subgroup" in $A_1/L_1 \times A_2/L_2$. In more detail, given $(x, y) \in A_1 \times A_2$, we have $(x, y) \in G$ if and only if $(x, y)(L_1 \times L_2) \in Q$; that is, if and only if $p_1(x) = (x, y)(L_1 \times L_2) = p_2(y)$. Thus G is the fibre product of p_1 and p_2.

We summarize this discussion as follows:

Proposition 1.1. *A subgroup $G \leq A_1 \times A_2$ is a subdirect product of A_1 and A_2 if and only if there is a group Q and surjections $p_i : A_i \to Q$ such that G is the fibre product of p_1 and p_2.* □

In the special case of a fibre product in which $A_1 = A_2$, $L_1 = L_2$, and $p_1 = p_2$, we shall call the fibre product *untwisted* (following [2]).

Proposition 1.2. *Let $G \leq A_1 \times \cdots \times A_n = D$ be a subdirect product of the groups $A_1, \ldots, A_n$ and let $L_i = G \cap A_i$. Then the following are equivalent:*

1. *G is normal in D;*
2. *each A_i/L_i is abelian;*
3. *G is the kernel of a homomorphism $\phi : D \to B$ where B is an abelian group.*

Proof. First suppose that G is normal in D. Let $x, y \in A_i$. Since G is a subdirect product there is an element of the form

$$\alpha = (a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \in G.$$

Since G is normal, it follows that

$$\beta = (a_1, \dots, a_{i-1}, y^{-1}xy, a_{i+1}, \dots, a_n) \in G.$$

Hence $\alpha^{-1}\beta = (1, \dots, 1, x^{-1}y^{-1}xy, 1 \dots, 1) \in G$. Thus $x^{-1}y^{-1}xy \in L_i = G \cap A_i$. Since x and y were arbitrary elements of A_i, it follows that A_i/L_i is abelian. Now suppose each A_i/L_i is abelian. Then G contains the derived group $[D, D]$ of D and hence G is the kernel of a homomorphism onto an abelian group. That (3) implies (1) is trivial. □

Corollary 1.3. *Let $G \leq A_1 \times A_2$ be a subdirect product of the groups A_1 and A_2 and let $L_i = G \cap A_i$. Then G is normal if and only if $G/(L_1 \times L_2)$ ($\cong A_1/L_1 \cong A_2/L_2$) is abelian.* □

Observe that in this Corollary the group G is the fibre product of two epimorphisms $p_1 : A_1 \to Q$ and $p_2 : A_2 \to Q$ where $Q = G/(L_1 \times L_2)$. Define $\phi : A_1 \times A_2 \to Q$ by $\phi(a_1, a_2) = p_1(a_1)p_2(a_2)^{-1}$. Since Q is abelian, ϕ is a homomorphism. Of course $\phi(a_1, a_2) = 1$ if and only if $p_1(a_1) = p_2(a_2)$, so G is the kernel of ϕ. Clearly ϕ is surjective. Thus $Q = G/(L_1 \times L_2) \cong (A_1 \times A_2)/G$.

A normal subgroup L of a group A is said to be *finitely normally generated* if it is the normal closure in A of finitely many of its elements. Observe that if A is a finitely presented group, then L is finitely normally generated if and only if A/L is finitely presented.

2. Generators and relations for a subdirect product

Suppose that $G \leq A \times B$ is a subdirect product of A and B. We are interested in obtaining a presentation for G. We define $L_A = G \cap A$ and $L_B = G \cap B$. Note that L_A and L_B are normal subgroups of G.

Let $x_1 = (a_1, b_1), x_2 = (a_2, b_2), \dots$ be a set of generators for G. Then the $a_1, a_2, \dots$ generate A and the $b_1, b_2, \dots$ generated B. Choose presentations $A = \langle a_1, a_2, \dots \mid r_1(a) = 1, r_2(a) = 1, \dots \rangle$ and $B = \langle b_1, b_2, \dots \mid s_1(b) = 1, s_2(b) = 1, \dots \rangle$ on these generators.

If $s_i(x)$ is the word on the x_i corresponding to $s_i(b)$, then $s_i(x) = (s_i(a), s_i(b)) = (s_i(a), 1)$ in $A \times B$ and hence $s_i(x) \in L_A$. More generally, if $w(x)$ is any word in the x_i then $w(x) \in L_A$ if and only if $w(b) =_B 1$. Hence L_A is normally generated by the words $s_1(x), s_2(x), \dots$. Similarly L_B is normally generated by the words $r_1(x), r_2(x), \dots$. Also $w(x) =_G 1$ if and only if $w(x) \in K_G = \mathrm{nm}(s_\ell(x)) \cap \mathrm{nm}(r_j(x))$ where for instance $\mathrm{nm}(s_\ell(x))$ denotes the normal closure of the $s_\ell(x)$ in the free group on the x_i. Thus G can be presented with generators x_i and defining relations any set of normal generators for K_G. Notice that K_G contains the commutator group $[\mathrm{nm}(s_\ell(x)), \mathrm{nm}(r_j(x))]$, reflecting the fact that L_A and L_B commute in G.

Of course if G is finitely generated, then A and B must be finitely generated. But the converse need not be true.

2.1. **Presenting finitely generated subdirect products.** As a consequence of this discussion, we observe the following:

Proposition 2.1. *Suppose that G is a subdirect product of two finitely generated groups A and B.*

(1) *If G is finitely generated and B is finitely presented, then $G \cap A$ is finitely normally generated in A (and hence in G and $A \times B$).*
(2) *If G is finitely presented, then B is finitely presented if and only if $G \cap A$ is finitely normally generated in A.* □

Now suppose that both G and B are finitely presented on the generators given above and that $G \cap A \neq 1$. Unfortunately, it need not be true that A is finitely presented (see Example 5 below).

2.2. **Examples illustrating a diversity of finiteness properties.** We now give a series of examples that illustrate a variety of possibilities for finite generation and finite presentation of subdirect products which intersect both factors.

Example 1: It is a consequence of a theorem of Baumslag and Roseblade [5] (see Theorem 4.1 below for a more complete statement), that a subdirect product of two non-abelian free groups which intersects both factors is finitely presented if and only if it has finite index. So "most" subdirect products of free groups are not finitely presented. Trying to better understand this and related phenomena has been one of our motivations for this study.

As an explicit example, let $Q = \langle t \mid \rangle$ be infinite cyclic. Let $C = \langle c_1, c_2 \mid \rangle$ be a free group mapping onto Q by $c_1 \mapsto t$ and $c_2 \mapsto 1$. Similarly let $D = \langle d_1, d_2 \mid \rangle$ be free mapping onto Q by $d_1 \mapsto t$ and $d_2 \mapsto 1$. Let $G \leq C \times D$ be the untwisted pullback of these two maps. Then G is generated by $g_1 = (c_1, d_1)$, $g_2 = (c_2, d_2)$, $g_3 = (c_2, 1)$ and $g_4 = (1, d_2)$. But G has infinite index in $C \times D$ and so is not finitely presentable. Notice that in this example $G \cap C$ and $G \cap D$ are both finitely normally generated.

Example 2: The second of our examples exhibits the nicest behaviour. Let $Q = \langle c_1, c_2 \mid [c_1, c_2] = 1 \rangle$, a free abelian group of rank two. Let $A = \langle a_1, a_2 \mid \rangle$ be a free group and let

$$B = \langle b_1, b_2 \mid [[b_1, b_2], b_1] = 1, [[b_1, b_2], b_2] = 1 \rangle$$

be free nilpotent group of class two and rank two. Map A and B onto Q via the maps $a_i \mapsto c_i$ and $b_i \mapsto c_i$. Let G be the pullback of these two maps. Then G is generated by the elements $x_1 = (a_1, b_1)$, $x_2 = (a_2, b_2)$, $x_3 = ([a_1, a_2], 1)$, $x_4 = (1, [b_1, b_2])$.

To use the notation of the discussion at the start of Section 2, we re-present A and B on the corresponding generators as

$$\begin{aligned} A &= \langle a_1, a_2, a_3, a_4 \mid a_4 = 1, [a_1, a_2]a_3^{-1} = 1\rangle \\ B &= \langle b_1, b_2, b_3, b_4 \mid [[b_1, b_2], b_1] = 1, \\ &\qquad [[b_1, b_2], b_2] = 1, b_3 = 1, [b_1, b_2]b_4^{-1} = 1\rangle. \end{aligned}$$

Then L_A is normally generated by the words we denoted $s_\ell(x)$ in our initial discussion, namely $[[x_1, x_2], x_1]$, $[[x_1, x_2], x_1]$, x_3, $[x_1, x_2]x_4^{-1}$. The subgroup L_B is normally generated by the two elements x_4, $[x_1, x_2]x_3^{-1}$ (these are the $r_j(x)$).

In this example G is finitely presented. To see this observe that A is free of rank two and L_B is infinite cyclic with generator $[b_1, b_2]$, and G is the split extension of L_B by A. (In fact L_B is central in $A \times B$, so $G \cong \mathbb{Z} \times F_2$.)

We record some general observations related to the salient points of the preceding example. We remind the reader that a group Q is said to be of type $\mathcal{F}_3$ if there is an Eilenberg-Maclane space $K(Q, 1)$ with a finite 3-skeleton.

Proposition 2.2. *Let A be a finitely generated free group, let $p_1 : A \to Q$ be an epimorphism, let B be a finitely presented group that fits into a short-exact sequence $1 \to N \to B \xrightarrow{p_2} Q \to 1$, and let $G \subset A \times B$ be the fibre product of p_1 and p_2. Then,*

1. *$G \cong N \rtimes A$ where the action is obtained by splitting the projection of G to the first factor in $A \times B$;*
2. *if N is finitely presented then G is finitely presented;*
3. *if N is finitely generated and Q is of type $\mathcal{F}_3$ then G is finitely presented.*

Assertion (1) is clear and assertion (2) follows. We omit the proof of (3), which is covered by the arguments used in [3] to prove the 1-2-3 Theorem, which provides a more subtle criterion for finite presentability of fibre products.

Example 3: The essential feature of this example is that although A and B are finitely presented, G fails to be even finitely generated as a result of Q not being finitely presented.

Let $Q = \langle c_1, c_2 \mid q_1(c), q_2(c), \ldots\rangle$ be a two generator group which is not finitely presentable. Let $A = \langle a_1, a_2 \mid \ \rangle$ and $B = \langle b_1, b_2 \mid \ \rangle$ be two free groups mapping onto Q via the maps $a_i \mapsto c_i$ and $b_i \mapsto c_i$. Let G be the untwisted pullback of these two maps. Then G is generated by the diagonal generators $x_1 = (a_1, b_1), x_2 = (a_2, b_2)$ together with the elements $(q_1(a), 1), (q_2(a), 1), \ldots$. (Notice that $(1, q_i(b)) = q_i(x)(q_i(a), 1)^{-1}$ so we do not need to add these as generators.) Now since Q is not finitely presentable, no finite subset of these generators suffices to generate G. For if (a_1, b_1), $(a_2, b_2))$, $(q_1(a), 1))$, ..., $(q_n(a), 1)$ generated G then $Q \cong A/L_A$ would be finitely presented with presentation $\langle a_1, a_2 \mid q_1(a) = 1, \ldots, q_n(a) = 1\rangle$. Thus G is not finitely generated.

Example 4: Let Q be not finitely presentable as in Example 3 and let

$$A = \langle a_1, a_2, a_3 \mid q_1(a) = 1, q_2(a) = 1, \ldots\rangle$$

where the words q_i are as in the previous example. (We are using the standard functional notation, so a is the ordered alphabet $a_1, a_2, \ldots$; note that none of the relations $q_i(a) = 1$ involve the generator a_3.) Similarly, let $B = \langle b_1, b_2, b_3 \mid q_1(b) = 1, q_2(b) = 1, \ldots\rangle$, so $B \cong A \cong Q * \langle a_3 \mid \ \rangle$. Map A onto Q by $a_1 \mapsto c_1$, $a_2 \mapsto c_2$ and $a_3 \mapsto 1$, and map B onto Q similarly. Since Q is a free factor, neither A nor B is finitely presentable.

Again let G be the untwisted pullback of these two maps. In this example G is finitely generated by the elements $x_1 = (a_1, b_1)$, $x_2 = (a_2, b_2), x_3 = (a_3, b_3)$ together with $(a_3, 1)$ and $(1, b_3)$ (one of these last two is redundant), but once again G is not finitely presented. This time, the lack of finite presentability can be seen as a special case of Proposition 2.1(2), for we are assuming B is not finitely presentable and $L_A = G \cap A$ is normally generated by the single element $(a_3, 1)$.

Example 5: Once again we take Q as in Example 3 and

$$A = \langle a_1, a_2, a_3 \mid q_1(a) = 1, q_2(a) = 1, \ldots\rangle$$

as in Example 3, but this time we let $B = \langle b_1, b_2 \mid \ \rangle$ be free of rank two. We map A onto Q by $a_1 \mapsto c_1$, $a_2 \mapsto c_2$ and $a_3 \mapsto 1$, and map B onto Q by $b_i \mapsto c_i$. Again let G be the pullback of these two maps. In this example G is generated by the finite collection of elements $x_1 = (a_1, b_1)$, $x_2 = (a_2, b_2)$, $x_3 = (a_3, 1)$. Again $L_A = G \cap A$ is the normal closure in A of $x_3 = (a_3, 1)$. Also $q_i(x) = (q_i(a), q_i(b)) = (1, q_i(b)) \in G$ and so L_B is the normal closure in B of the $(1, q_i(b))$.

In this example, it is more difficult to determine whether G is finitely presentable, but in fact it is not since L_A is not finitely generated (see Theorem 4.4 below).

2.3. **Criteria for finite generation.** In the discussion leading to Proposition 2.1 we *assumed* we had a set of generators for the subdirect product $G \leq A \times B$. We would like to know when G actually has a *finite* set of generators. Of course A and B must be finitely generated, so we assume this is the case. Since G is a subdirect product we can find finitely many elements $x_1 = (a_1, b_1), \ldots, x_n = (a_n, b_n)$ in G such that the a_i generate A and the b_i generate B. Denote by H the subgroup generated by $x_1, \ldots, x_n$. Then H is a subgroup of G but it may not contain all of either L_A or L_B. Observe that H itself is a subdirect product of A and B. Also note that if $(u(a), v(b)) \in G$ then $u(x)^{-1}(u(a), v(b)) = (1, u(b)^{-1}v(b)) \in L_B$ and $v(x)^{-1}(u(a), v(b)) = (v(a)^{-1}u(a), 1) \in L_A$. Thus $G = HL_A = HL_B$. Hence if either L_A or L_B is finitely normally generated, then G will be finitely generated. For instance, if L_A is finitely normally generated by $(z_1(a), 1), \ldots, (z_n(a), 1)$, then G is generated by the x_i and the $(z_j(a), 1)$.

In Example 4 the subgroup L_A is finitely normally generated but L_B is not.

We record the preceding general observation in the following proposition.

Proposition 2.3. *Suppose that $G \leq A \times B$ is a subdirect product of two finitely generated groups A and B. If either $G \cap A$ or $G \cap B$ is finitely normally generated, then G is finitely generated.* □

Combining this with Proposition 2.1, we conclude the following:

Corollary 2.4. *Suppose that $G \leq A_1 \times A_2$ is the subdirect product of two finitely presented groups A_1 and A_2. Let $L_i = G \cap A_i$. Then G is finitely generated if and only if one (and hence both) of A_1/L_1 and A_2/L_2 are finitely presented.* □

3. **Homological properties of subdirect products**

In this section we consider homological versions of the results from the previous section. Recall that if A is finitely presented, then the integral homology groups $H_1(A, \mathbb{Z})$ and $H_2(A, \mathbb{Z})$ are both finitely generated. On the other hand, there exist non-finitely generated groups that have $H_1(A, \mathbb{Z}) = 0$ as well as finitely generated groups G, with $H_2(G, \mathbb{Z})$ finitely generated, which are not finitely presentable (see, e.g., [1], [2] or Theorem G above). We also remind the reader that there exist finitely presented groups whose higher homology groups $H_n(G, \mathbb{Z})$ are not finitely generated; explicit examples due to Stallings and Bieri are described in the next section.

Consider a subdirect product $G \leq A \times B$ and let $L = G \cap A$. We think of G as an extension of L by B. Conjugation in G and A induces actions of these groups on their normal subgroup L and hence on $H_1(L, \mathbb{Z})$. Since G is a subdirect product, its image in Aut(L) is the same as that of A and hence

$$H_0(G/L, H_1(L, \mathbb{Z})) \cong H_0(A/L, H_1(L, \mathbb{Z})).$$

We make use of this observation to prove the following:

Theorem 3.1. *Let A and B be groups with both $H_1(-, \mathbb{Z})$ and $H_2(-, \mathbb{Z})$ finitely generated. Suppose that $G \leq A \times B$ is a subdirect product of A and B. Then $H_1(G, \mathbb{Z})$ is finitely generated if and only if one (and hence both) of $H_2(A/(G \cap A), \mathbb{Z})$ and $H_2(B/(G \cap B), \mathbb{Z})$ is finitely generated.*

Proof. Let $L = G \cap A$. The usual five term exact sequence for A/L gives the exactness of

$$H_2(A, \mathbb{Z}) \to H_2(A/L, \mathbb{Z}) \to H_0(A/L, H_1(L, \mathbb{Z})) \to H_1(A, \mathbb{Z}) \cdots .$$

By hypothesis $H_1(A, \mathbb{Z})$ and $H_2(A, \mathbb{Z})$ are both finitely generated, so $H_2(A/L, \mathbb{Z}))$ is finitely generated if and only if $H_0(A/L, H_1(L, \mathbb{Z}))$ is finitely generated.

Similarly, the five term exact sequence for G/L gives the exactness of

$$\cdots H_2(G/L, \mathbb{Z}) \to H_0(G/L, H_1(L, \mathbb{Z})) \to H_1(G, \mathbb{Z}) \to H_1(G/L, \mathbb{Z}) \to 0.$$

By hypothesis $B = G/L$ and so $H_1(G/L, \mathbb{Z})$ and $H_2(G/L, \mathbb{Z})$ are both finitely generated. Thus $H_1(G, \mathbb{Z})$ is finitely generated if and only if $H_0(G/L, H_1(L, \mathbb{Z}))$ is finitely generated.

Since $H_0(G/L, H_1(L, \mathbb{Z}))$ and $H_0(A/L, H_1(L, \mathbb{Z}))$ are isomorphic, it follows that $H_2(A/L, \mathbb{Z}))$ is finitely generated if and only if $H_1(G, \mathbb{Z})$ is finitely generated, as claimed. By the symmetric argument, the other assertion follows. □

If both factors A and B are free, the exact sequences in the above proof yield more precise information, since in this case the $H_1(-, \mathbb{Z})$ are free abelian and $H_2(-, \mathbb{Z}) = 0$. We record this as follows:

Corollary 3.2 (=Theorem A)**.** *Suppose that $G \leq F_1 \times F_2$ is a subdirect product of two free groups F_1 and F_2. Let $L_i = G \cap F_i$. Then*

$$H_1(G, \mathbb{Z}) \cong H_1(F_2, \mathbb{Z}) \oplus H_2(F_1/L_1, \mathbb{Z}) \oplus C$$

where $C = \ker(H_1(F_1, \mathbb{Z}) \twoheadrightarrow H_1(F_1/L_1, \mathbb{Z}))$ and hence is free abelian of rank at most the rank of F_1. □

Since it is known how to construct two-generator groups with prescribed countable $H_2(-,\mathbb{Z})$ (see [4]), one can apply the pull back construction to conclude the following result from [5]:

Corollary 3.3 (=Theorem B)**.** *Let F_1 and F_2 be non-abelian free groups. Then there are continuously many subdirect products $G \leq F_1 \times F_2$ having non-isomorphic $H_1(G,\mathbb{Z})$.* □

A theorem of Gordon asserts there is no algorithm to decide of a finitely presented group Γ whether or not $H_2(\Gamma,\mathbb{Z}) = 0$. This is proved (see [17] or [22]) by constructing a recursive collection of finite presentations of groups Γ_i for which no such algorithm exists; these presentations have a common finite set of symbols as generators, thus the Γ_i come equipped with a surjection $F \to \Gamma_i$ from a fixed finitely generated free group. It is easy to check from the construction that each of the groups involved is perfect. Moreover one can easily arrange that each Γ_i is either the trivial group or else $H_2(\Gamma_i,\mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ (or any other fixed finitely generated abelian group). We now apply the above corollary to the pullback G_i of two copies of the presentation map $F \to \Gamma_i$. Since each Γ_i is perfect, we have

$$H_1(G_i,\mathbb{Z}) \cong H_1(F,\mathbb{Z}) \oplus H_2(\Gamma_i,\mathbb{Z}) \oplus H_1(F,\mathbb{Z}).$$

This proves the following result.

Corollary 3.4 (=Corollary C)**.** *If F_1 and F_2 are non-abelian free groups, there is no algorithm to compute for an arbitrary finitely generated subgroup $G \leq F_1 \times F_2$ the (torsion-free) rank of $H_1(G,\mathbb{Z})$, nor is there an algorithm to determine whether $H_1(G,\mathbb{Z})$ has non-zero torsion elements.* □

As another application we note the following example. Let F be a finitely generated free group and suppose that F/L has finitely generated $H_2(F/L,\mathbb{Z})$ but is not finitely presented (cf. Theorem G). Let $G \leq F \times F$ be the pullback or fibre product corresponding to this presentation. Then $H_1(G,\mathbb{Z})$ is finitely generated by Theorem 3.1, but G is not finitely generated by Corollary 2.4. We record this as the following:

Corollary 3.5. *There is a subdirect product $G \leq F \times F$ of two finitely generated free groups such that $H_1(G,\mathbb{Z})$ is finitely generated but G is not finitely generated.* □

4. **Subdirect products of free and surface groups**

In this section and the next we focus on subdirect products of free and surface groups.

4.1. **Background.** The results at the end of the previous section indicate how wild the finitely generated subgroups of the direct product of two free groups can be. But the following result of Baumslag and Roseblade shows that the only finitely presented subgroups are "the obvious ones".

Theorem 4.1 (Baumslag and Roseblade [5]). *Let $F_1 \times F_2$ be the direct product of two free groups F_1 and F_2. Suppose that $G \leq F_1 \times F_2$ is a subgroup and define $L_i = G \cap F_i$.*

(1) *If either $L_i = 1$ then G is free.*
(2) *If both L_i are non-trivial and one of them is finitely generated, then $L_1 \times L_2$ has finite index in G.*
(3) *Otherwise, G is not finitely presented.*

This result contains Grunewald's earlier result [18] that in a direct product of two isomorphic free groups, the untwisted fibre product $P \leq F \times F$ of a finite presentation of an infinite group is finitely generated but not finitely presented. We remind the reader that such fibre products can have unsolvable membership problem and unsolvable conjugacy problem (see for instance [22]).

The following construction shows that finitely presented subgroups of the direct product of more than two free groups can be considerably more complicated than in the case of two factors.

Examples of Stallings and Bieri: Let $F_1 = \langle a_1, b_1 \mid \rangle$, ..., $F_n = \langle a_n, b_n \mid \rangle$ be free groups of rank 2 and let $Q = \langle c \mid \rangle$ be an infinite cyclic group. Let ϕ_n be the map from the direct product $F_1 \times \cdots \times F_n$ to Q defined by $a_i \mapsto c$ and $b_i \mapsto c$. Define $\mathrm{SB}_n = \ker \phi_n$. It is easy to check that SB_n is a subdirect product of the F_i and that (for $n > 1$) SB_n is finitely generated by the elements $a_i b_i^{-1}$, $a_i a_j^{-1}$ and $b_i b_j^{-1}$. Moreover, it can be shown [6] that SB_n is of type FP_{n-1} (even better, type $\mathcal{F}_{n-1}$). But it is not of type FP_n, indeed $H_n(\mathrm{SB}_n, \mathbb{Z})$ is not finitely generated. (See [26] and [6] for details.)

In order to relate these observations to our consideration of subdirect products of two groups, we observe that projection onto the first $n-1$ factors maps SB_n surjectively onto $F_1 \times \cdots \times F_{n-1}$ with kernel L_n which is the normal closure in F_n of $a_n b_n^{-1}$. Further $\mathrm{SB}_{n-1} = \mathrm{SB}_n \cap (F_1 \times \cdots \times F_{n-1})$. Note that for $n > 2$ the group SB_n is finitely presented.

In the light of the diverse behaviour we have seen among the finitely generated subgroups of the direct product of two free groups, one might expect that the above examples are just the first in a menagerie of increasingly exotic finitely presented subgroups in the case of three or more factors. However, somewhat to our surprise, this does not appear to be the case.

The first sign of tameness among the finitely presented subgroups of direct products of arbitrarily many free groups comes from the following theorem of Bridson, Howie, Miller and Short [9], which shows that whatever wildness exists may be detected at the level of homology.

Theorem 4.2 ([9])**.** *Let $F_1, \dots, F_n$ be free groups. A subgroup $G \leq F_1 \times \cdots \times F_n$ is of type* FP_n *if and only if it has a subgroup of finite index that is itself a direct product of (at most n) finitely generated free groups.*

Thus, on the one hand, we know that the only homologically-tame subgroups of a direct product of free groups are the obvious ones. On the other hand, we have a specific method for constructing examples of homologically-wild subgroups coming from the construction of Stallings and Bieri. Moreover one has essentially complete knowledge of the latter situation, because the BNS invariants of direct products of free groups have been calculated [24], providing a complete classification of the finiteness properties of the kernels of maps from $F_1 \times \cdots \times F_n$ to abelian groups.

Our repeated failure to construct finitely-presented subgroups that that are neither FP_∞ nor of Stallings-Bieri type led us to pose the following:

Question 4.3. *Let $D = F_1 \times \cdots \times F_n$ be a direct product of free groups (of various ranks) and let $p_i : D \to F_i$ be the natural projection. Let $G \subset D$ be a subgroup that intersects each F_i non-trivially.*

If G is finitely presented but not of type FP_n*, then does G have a subgroup of finite index G_0 which is normal in $p_1(G_0) \times \cdots \times p_n(G_0)$ with abelian quotient?*

Added in proof: It turns out this question has a negative answer. Indeed our recent joint work with J. Howie and H. Short [11] shows the bound on the nilpotency class obtained in Theorem D is optimal.

We will make use of the following theorem from [23], which was used there to give a straightforward proof of the Baumslag-Roseblade Theorem:

Theorem 4.4 (Miller[23])**.** *Let $A \times F$ be the direct product of a group A with a free group F. Suppose that $G \leq A \times F$ is a subgroup which intersects F non-trivially. If G is finitely presented, then $L = G \cap A$ is finitely generated.* □

4.2. **Preparatory results for the surface case.** Since we want our results to apply to both free and surface groups, we must also prove the analog of the above result for compact surfaces. One ingredient of the proof is the well known fact [25] that a finitely generated, non-trivial normal subgroup of a compact surface group of negative Euler characteristic must have finite index. A second important ingredient is the following substitute for the use made in [23] of a theorem of M. Hall.

Lemma 4.5. *Let Π be the fundamental group of a closed surface S and let $H \leq \Pi$ be a non-cyclic 2-generator subgroup. Then there exist elements $a_1, b_1 \in H$ that, in a finite index subgroup $\Pi_0 \leq \Pi$, serve as the beginning generators in a standard presentation*

$$\Pi_0 = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.$$

Equivalently, there a finite-sheeted covering $\hat{S}$ of S such that $\langle a_1, b_1 \rangle \cap \pi_1 \hat{S}$ is the fundamental group of a subsurface of positive genus.

Proof. By a theorem of Scott [25], given a finitely generated subgroup H of a surface group, one may pass to a finite-sheeted cover $\overline{S}$ so that H is the fundamental group of a subsurface T onto which $\overline{S}$ retracts. If H is a 2-generator free group, then this surface is either a once-punctured torus or a thrice-punctured sphere. In the former case, we are done. In the latter case, there is a 4-sheeted cover $\hat{T}$ of T that is a 4-punctured torus. Pulling back the covering $\hat{T} \to T$ along the retraction $\overline{S} \to T$ provides the desired covering $\hat{S} \to S$. □

An alternative proof of this lemma can be derived from a theorem of Bridson and Howie (Corollary 3.2 of [8]). The type of "positive-genus" argument we used has been exploited in [12] to obtain results about elementarily free groups.

We are now ready to prove the analog of Theorem 4.4 for surface groups.

Theorem 4.6. *Let Σ be the fundamental group of a compact surface other than the Klein bottle and torus, and let A be an arbitrary group. Let $G \leq A \times \Sigma$ be a subgroup that intersects Σ non-trivially. If G is finitely presented, then $G \cap A$ is finitely generated.*

Proof. Let $L = G \cap A$ and let $p : G \to \Sigma$ be the standard projection. If the surface is a sphere or projective plane, the statement is trivial. More generally, if $G \cap \Sigma$ has finite index in $p(G)$, then G contains $(G \cap A) \times (G \cap \Sigma)$ as a subgroup of finite index. And since L is a retract of this subgroup, it is finitely presented.

The finitely generated group $p(G) \subset \Sigma$ is either free, in which case we are done by theorem 4.4, or else it is again the fundamental group of a closed surface. Thus there is no loss of generality in assuming that $p(G) = \Sigma$. This forces $G \cap \Sigma$ to be normal in Σ. Thus we are reduced to the case where $G \cap \Sigma \subset \Sigma$ is a non-trivial normal subgroup of infinite index. It follows from Lemma 4.5 that by replacing Σ with a subgroup of finite index and taking the preimage in G, we may assume that Σ has a presentation of the form

$$\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

where a_1 and b_1 both lie in $G \cap \Sigma$. The defining relation is equivalent to the equation

$$a_1^{-1} b_1 a_1 = b_1 [a_2, b_2] \cdots [a_g, b_g]$$

and so we regard Σ as the HNN extension with stable letter a_1 which conjugates the cyclic subgroup $\langle b_1 \rangle$ to the cyclic subgroup generated by the right hand side. Note that $b_1, a_2, b_2, \ldots, b_g$ freely generate a free subgroup of Σ.

We proceed as in [23]. For each $i = 2, \ldots, g$ pick a lift $\hat{a}_i \in p^{-1}(a_i)$ and $\hat{b}_i \in p^{-1}(b_i)$ in G . Observe that

$$[a_2, b_2] \cdots [a_g, b_g] =_G [\hat{a}_2, \hat{b}_2] \cdots [\hat{a}_g, \hat{b}_g] \cdot c_1$$

for some element $c_1 \in L = G \cap A$.

Since $b_1, \hat{a}_2, \hat{b}_2, \ldots, \hat{b}_g$ are the pre-image of a free basis, they freely generate a free subgroup of G. Hence the subgroup H of G generated by L together with these elements has the structure of an HNN extension of L. The associated subgroup for each stable letter is L itself and we note that b_1 acts trivially on L since $b_1 \in G \cap \Sigma$.

Now G is an extension of H with stable letter a_1 and is finitely presented, so it can be generated by $a_1, b_1, \hat{a}_2, \ldots, \hat{b}_g$ together with finitely many elements $c_1, \ldots, c_n \in L$ (including the previously chosen c_1). Thus we can present G as

$$G = \langle a_1, b_1, \hat{a}_2, \ldots, \hat{b}_g, c_1, \ldots, c_n \mid a_1^{-1} b_1 a_1 = b_1 [\hat{a}_2, \hat{b}_2] \cdots [\hat{a}_g, \hat{b}_g] \cdot c_1,$$

$$a_1^{-1} d a_1 = d \ (d \in L), \text{ relations of } H \rangle.$$

The associated subgroup for the stable letter a_1 is $L \times \langle b_1 \rangle$. Observe that H is generated by the generators other than a_1 since the action of a_1 on L is trivial. Hence, because G is finitely presented, $L \times \langle b_1 \rangle$ is finitely generated, and so L is finitely generated. This completes the proof. $\square$

4.3. **The Virtually-Nilpotent Quotients Theorem.** The following theorem controls the way in which a finitely presented subdirect product of free and surface groups can intersect the direct factors of the ambient group.

The reader will recall that the m-th term of the lower central series of a group H is defined inductively by $\gamma_1(H) = H$ and $\gamma_{m+1}(H) = [\gamma_m(H), H]$. And H is defined to be nilpotent of class c if $\gamma_{c+1}(H) = 1$.

Theorem 4.7 (=Theorem D)**.** *Let $\Sigma_1, \dots, \Sigma_n$ be free or surface groups and let $G \leq D = \Sigma_1 \times \cdots \times \Sigma_n$ be a subdirect product which intersects each factor Σ_i non-trivially. If G is finitely presented, then each Σ_i contains a normal subgroup K_i of finite index such that*

$$\gamma_{n-1}(K_i) \subseteq G \cap \Sigma_i \subseteq K_i.$$

Thus the quotients $\Sigma_i/(G \cap \Sigma_i)$ and

$$D/((G \cap \Sigma_1) \times \cdots \times (G \cap \Sigma_n))$$

are virtually nilpotent of class at most $n-2$. Hence both $\Sigma_i/(G \cap \Sigma_i)$ and $G/(G \cap \Sigma_i)$ are finitely presented, and consequently the projection of G into the product of any $j < n$ factors is finitely presented.

Note that in the case $n = 2$ the conclusion is that $K_i = G \cap \Sigma_i$ which implies the Baumslag-Roseblade Theorem. In case $n = 3$, the conclusion is that $[K_i, K_i] \subseteq G \cap \Sigma_i \subseteq K_i$ so that the $\Sigma_i/(G \cap \Sigma_i)$ are virtually abelian, as happens for the Stallings-Bieri examples SB_n.

Proof. Let $p_i : G \to \Sigma_i$ be the induced projection maps and put $N_i = \ker p_i$ and $L_i = G \cap \Sigma_i$. Since G is finitely presented, by Theorem 4.4 each N_i is finitely generated. Because G is subdirect, each Σ_i is finitely generated, and the L_i are normal in Σ_i as well as in G.

Again since G is subdirect, for $j \neq i$, the projection $p_j(N_i)$ is normal in Σ_j. Now $L_j \subseteq p_j(N_i)$ so $p_j(N_i)$ is a non-trivial finitely generated normal subgroup of the free group Σ_j, and hence has finite index in Σ_j.

For notational simplicity we focus on the case $j = 1$ and note that similar arguments work for the remaining $j = 2, \dots, n$. Define

$$K_1 = p_1(N_2) \cap \cdots \cap p_1(N_n).$$

We note that $L_1 = N_2 \cap \cdots \cap N_n \subseteq K_1$. For any choice of $n-1$ elements $x_2, \dots, x_n \in K_1$, there are elements $y_i \in N_i$ with $p_1(y_i) = x_i$ for $i = 2, \dots, n$. Observe for example that y_2 has the form $(x_2, 1, z_{2,3}, \dots, z_{2,n})$ and y_3 has the form $(x_3, z_{3,2}, 1, \dots, z_{3,n})$. Hence their commutator is

$$[y_2, y_3] = ([x_2, x_3], 1, 1, [z_{2,4}, z_{3,4}], \dots, [z_{2,n}, z_{3,n}]) \in N_2 \cap N_3.$$

On forming an $(n-1)$-fold commutator such as $[y_2, y_3, \ldots, y_n]$ one obtains

$$[y_2, y_3, \ldots, y_n] = ([x_2, x_3, \ldots, x_n], 1, 1, \ldots, 1) \in G \cap \Sigma_1 = L_1,$$

and similarly for other commutator arrangements. Hence $\gamma_{n-1}(K_1) \subseteq L_1$. But we know $L_1 \subseteq K_1$, so

$$\gamma_{n-1}(K_1) \subseteq L_1 \subseteq K_1$$

as desired.

The remaining assertions follow from the fact that finitely generated nilpotent groups are finitely presented, since this implies that any normal subgroup of a finitely generated free group that contains a term of the lower central series is finitely normally generated. This completes the proof. □

In case $n = 3$ we can actually characterize the finitely presented subdirect products of $\Sigma_1 \times \Sigma_2 \times \Sigma_3$.

Lemma 4.8. *Let $\Sigma_1, \Sigma_2, \Sigma_3$ be free groups or surface groups and let $G \leq \Sigma_1 \times \Sigma_2 \times \Sigma_3$ be a subdirect product which intersects each factor Σ_i non-trivially. If G is finitely presented, then each Σ_i contains a normal subgroup K_i of finite index such that the projections of $G_0 = G \cap (K_1 \times K_2 \times K_3)$ to pairs of factors,*

$$p_{ij} : G_0 \to K_i \times K_j$$

$i < j$, are surjective. In particular the projections of G_0 to the K_i are also surjective.

Proof. Since the hypotheses are the same, we may continue with the notation of the proof of the previous theorem. Let $k_1 \in K_1$. By the definition of K_1 there are triples $(k_1, 1, y_2) \in N_2 \subset G$ and $(k_1, x_3, 1) \in N_3 \subset G$. Thus $(1, x_3^{-1}, y_2) \in N_1$. Therefore $x_3 \in p_2(N_1) \cap p_2(N_3) = K_2$ and $y_2 \in p_3(N_1) \cap p_3(N_2) = K_3$.

It follows that $(k_1, 1, y_2) \in N_2 \cap (K_1 \times K_2 \times K_3)$ and hence $(k_1, 1) \in p_{12}(G_0)$. Also $(k_1, x_3, 1) \in N_3 \cap (K_1 \times K_2 \times K_3)$ and so $(k_1, 1) \in p_{13}(G_0)$. Similar calculations apply for other factors K_i and so G_0 projects onto pairs as claimed. □

Here is a lemma about 3 factors with projections onto any 2 factors which works for any groups.

Lemma 4.9. *Let $G \leq A_1 \times A_2 \times A_3$ be a subdirect product which projects surjectively onto any product of two factors and let $L_i = G \cap A_i$. Then there is an abelian group Q and epimorphisms $\varphi_i : A_i \to Q$ such that $L_i = \ker \varphi_i$ $(i = 1, 2, 3)$ and G is the kernel of the map $\varphi_1 + \varphi_2 + \varphi_3$.*

Proof. Since the projection to $A_2 \times A_3$ is surjective, we can think of G as a subdirect product of the two groups A_1 and $(A_2 \times A_3)$. Hence $A_1/L_1 \cong (A_2 \times A_3)/(G \cap (A_2 \times A_3))$. If we put $Q = A_1/L_1$ then G can also be viewed as the pullback (fibre product) of the quotient map $\varphi_1 : A_1 \to Q$ and a surjection $\psi : A_2 \times A_3 \to Q$.

Now ψ restricts to homomorphisms $\theta_2 : A_2 \to Q$ and $\theta_3 : A_3 \to Q$ with $\psi((a_2, a_3)) = \theta(a_2) \cdot \theta(a_3)$. Since the projection from G to $A_1 \times A_3$ is surjective, for any $x \in A_1$ there is a $z \in A_3$ such that $(x, 1, z) \in G$. Since G is the pullback of φ_1 and ψ, we have $\varphi_1(x) = \psi((1, z)) = \theta_3(z)$. Because $x \in A_1$ was arbitrary, it follows that θ_3 has the same image as φ_1 and hence maps A_3 onto Q with kernel L_3.

Similarly since the projection from G to $A_1 \times A_2$ is surjective, θ_2 maps A_2 surjectively onto Q with kernel L_2. But the images of A_2 and A_3 commute in Q and so Q must be abelian. Changing to additive notation and defining $\varphi_2 = -\theta_2$ and $\varphi_3 = -\theta_3$, it follows that G is the kernel of the map $\varphi_1 + \varphi_2 + \varphi_3$. This completes the proof. □

Meinert [24] has calculated the Bieri-Neumann-Strebel invariants for direct products D of finitely many finitely generated free groups. In particular he has calculated which homomorphisms from D to an abelian group have a finitely presented kernel. By combining his result with the preceding two lemmas we obtain a characterization of finitely presented subdirect products of 3 free or surface groups.

Theorem 4.10 (=Theorem E). *Let $\Sigma_1, \Sigma_2, \Sigma_3$ be finitely generated free groups or surface groups and let $G \leq \Sigma_1 \times \Sigma_2 \times \Sigma_3$ be a subdirect product which intersects each factor Σ_i non-trivially. Then G is finitely presented if and only if each Σ_i contains a normal subgroup K_i of finite index such that the subgroup $G_0 = G \cap (K_1 \times K_2 \times K_3)$ satisfies the following condition: there is an abelian group Q and epimorphisms $\varphi_i : K_i \to Q$ such that $G_0 \cap \Sigma_i = \ker \varphi_i$ $(i = 1, 2, 3)$ and G_0 is the kernel of the map $\varphi_1 + \varphi_2 + \varphi_3$.*

Proof. By passing to a double cover if necessary, we may assume that all of the surface groups are orientable. The necessity of the given condition is established by the preceding two lemmas and its sufficiency in the case of free groups is a special case of Meinert's theorem [24]. Thus we need only argue that sufficiency in the case of surface groups follows from Meinert's result.

To this end, for $i = 1, 2, 3$ we choose an epimorphism $\pi_i : F_i \to K_i$ where F_i is a finitely generated free group and the kernel is either trivial or the normal closure of a single product c_i of commutators (i.e. the standard surface relation). For notational convenience we define $c_i = 1$ if Σ_i is free.

Consider the composition $\Phi_i := \phi_i \pi_i : F_i \to Q$. Meinert's theorem tells us that the kernel $\Gamma_0 \subset F_1 \times F_2 \times F_3$ of $\Phi_1 + \Phi_2 + \Phi_3$ is finitely presented.

The kernel I of the map $F_1 \times F_2 \times F_3 \to K_1 \times K_2 \times K_3$ induced by the π_i is the normal closure of $C = \{c_1, c_2, c_3\}$, and $G_0 = \Gamma_0 / I$. Since Γ_0 is subdirect, the normal closure of C in Γ_0 is the same as its normal closure in $F_1 \times F_2 \times F_3$. It follows that a presentation of G_0 can be obtained by adding just three relations to a presentation for Γ_0. In particular, G_0 is finitely presented. □

5. Decision problems for finitely presented subgroups

The results in this section should be contrasted with the fact that if Σ_1 and Σ_2 are non-abelian free or surface groups, then there are finitely generated subgroups $H \subset \Sigma_1 \times \Sigma_2$ for which the conjugacy problem and membership problem are unsolvable [21].

5.1. **The conjugacy problem.** We shall prove that finitely presented subgroups of direct products of surface groups have a solvable conjugacy problem. Our proof relies heavily on the structure of such subgroups as described in Theorem 4.7. With this structure in hand, we can adapt the argument used in the proof of Theorem 3.1 of [7], where it was proved that if G is a bicombable group, $N \subset G$ is a normal subgroup and the generalized word problem for G/N is solvable, then N has a solvable conjugacy problem.

The class of bicombable groups contains the hyperbolic groups and is closed under finite direct products and the passage to subgroups of finite index. It follows that subgroups of finite index in direct products of free and surface groups are bicombable. The properties of bicombable groups that we need are all classical and easy to prove in the case of such subgroups, but we retain the greater generality with an eye to future applications.

With some effort, one can give an elementary proof of the following lemma using induction on the nilpotency class. A more elegant argument due to Lo (Algorithm 6.1 of [20]) provides an algorithm that is practical for computer implementation.

Lemma 5.1. *If Q is a finitely generated nilpotent group, then there is an algorithm that, given finite sets $S, T \subset Q$ and $q \in Q$, will decide if $q\langle S\rangle$ intersects $\langle T\rangle$ non-trivially.* □

The adaptation of Theorem 3.1 of [7] that we need in the present context is the following.

Proposition 5.2. *Let Γ be a bicombable group, let $H \subset \Gamma$ be a subgroup, and suppose that there exists a subgroup $L \subset H$ normal in Γ such that Γ/L is nilpotent. Then H has a solvable conjugacy problem.*

Proof. The properties of a bicombable group Γ that we need here are (1) the conjugacy problem is solvable in Γ and (2) there is an algorithm that, given $g \in \Gamma$ as a word in the generators, will calculate a finite generating set for the centralizer of g. (The second fact is proved in [7] and has its origins in the work of Gersten and Short [16]. The running time of the algorithm depends on the length of the word representing g and on the fellow-traveller constant of the bicombing.) The reader should have little difficulty in supplying their own proof of these facts in the case where Γ is a product of free and surface groups.

Given $x, y \in H$ (as words in the generators of Γ) we use the positive solution to the conjugacy problem in Γ to determine if there exists $\gamma \in \Gamma$ such that $\gamma x \gamma^{-1} = y$. If no such γ exists, we stop and declare that x, y are not conjugate in H. If γ does exist then we find it and consider

$$\gamma C = \{g \in \Gamma \mid gxg^{-1} = y\},$$

where C is the centralizer of x in Γ. Note that x is conjugate to y in H if and only if $\gamma C \cap H$ is non-empty.

We employ the algorithm from (2) to compute a finite generating set $\hat{S}$ for C. We then employ Lo's algorithm (Lemma 5.1) in the nilpotent group Γ/L to determine if the image of γC intersects H/L. Since $L \subset H$, this intersection is non-trivial (and hence x is conjugate to y) if and only if $\gamma C \cap H$ is non-empty. ☐

We need a further lemma in order to make full use of the preceding proposition. We remind the reader that the solvability of the conjugacy problem does not in general pass to subgroups or overgroups of finite index [14].

A group G is said to have *unique roots* if for all $x, y \in G$ and $n \neq 0$ one has $(x^n = y^n) \implies (x = y)$. Torsion-free hyperbolic groups and their direct products have this property.

Lemma 5.3. *Suppose G is a group in which roots are unique and $H \subset G$ is a subgroup of finite index. If the conjugacy problem for H is solvable, then the conjugacy problem for G is solvable.*

Proof. Let m_0 be the index of H in G and let $m = m_0!$. Given $x, y, g \in G$, since roots are unique, $x^m = gy^mg^{-1}$ if and only if $x = gyg^{-1}$. Thus

x, y are conjugate in G if and only if x^m, y^m are conjugate in G. Note $x^m, y^m \in H$.

If $c_1, \dots, c_{m_0}$ are coset representatives for H in G and $x_i := c_i x c_i^{-1}$, then x^m is conjugate to y^m in G if and only if y^m is conjugate to at least one of x_i^m in H.

Combining these two observations, we see that deciding if x is conjugate to y in G reduces to deciding if one of finitely many conjugacy relations holds in H. This completes the proof. □

Theorem 5.4. *If D is a direct product of free and surface groups, then every finitely presented subgroup of D has a solvable conjugacy problem.*

Proof. Projecting D away from direct factors that intersect G trivially, and replacing each of the remaining factors by the projection of G to that factor, we see that there is no loss of generality in assuming that G is a subdirect product of $D = \Sigma_1 \times \cdots \times \Sigma_n$ and that each $L_i = \Sigma_i \cap G$ is non-trivial.

Theorem 4.7 tells us that $L = L_1 \times \cdots \times L_n$ is normal in D and D/L is virtually nilpotent. Let N be a nilpotent subgroup of finite index in D/L, let D_0 be its inverse image in D and let $G_0 = D_0 \cap G$.

We are now in the situation of Proposition 5.2 with $\Gamma = D_0$ and $H = G_0$. Thus G_0 has a solvable conjugacy problem.

Finally, since roots are unique in surface groups, they are unique in D. Therefore Lemma 5.3 applies and we conclude that the conjugacy problem for G is solvable. □

5.2. **The membership problem.** In the course of proving our next theorem we will need the following technical observation.

Lemma 5.5. *If Σ is a finitely generated free or surface group, then there is an algorithm that, given a finite set $X \subset \Sigma$, will output a finite presentation for the subgroup generated by X.*

Proof. Let G be the subgroup generated by X. The lemma is a simple consequence of the fact that Σ has a subgroup of finite index that retracts onto G. This fact is due to M. Hall [19] in the case of free groups and P. Scott [25] in the case of surface groups.

In more detail, running through the finite-index subgroups $\Sigma_0 \subset \Sigma$, one calculates a presentation for Σ_0 and one attempts to express the elements $x \in X$ as words u_x in the generators of Σ_0 by listing all words in these generators and checking for equality in Σ by testing if each is (freely) equal to a product of conjugates of the defining relations. When words u_x have been found for all $x \in X$, one begins a naive search for homomorphisms $\phi : \Sigma_0 \to G$: words v_b in the letters $X^{\pm 1}$

are chosen as putative images for the generators b of Σ_0 and a naïve search is implemented to verify if the putative map sends the relations of Σ_0 to words representing the identity and if $\phi(u_x) = x$ for all $x \in X$.

One applies this procedure to all Σ_0 simultaneously, proceeding along finite diagonals. The theorems of Hall and Scott assure us that this procedure will eventually terminate, at which point we have a presentation $\Sigma_0 = \langle B \mid R\rangle$ and words $\{u_x,\, v_b \mid x \in X,\, b \in B\}$. The desired presentation of G is then $\langle B \cup X \mid R,\, xu_x^{-1},\, bv_b^{-1}\ (x \in X,\, b \in B)\rangle$. $\square$

Theorem 5.6. *If D is the direct product of finitely many finitely generated free and surface groups and $G \subset D$ is a finitely presented subgroup, then the membership problem for G is decidable, i.e. there is an algorithm which, given $h \in D$ (as a word in the generators) will determine whether or not $h \in G$.*

Proof. We proceed by induction on the number of factors in $D = \Sigma_1 \times \cdots \times \Sigma_n$. For $n = 1$ the assertion of the theorem is well-known, in particular it follows from Scott's theorem that surface groups are subgroup separable [25].

In more detail, given a finite generating set X for $G \subset \Sigma_1$ and given $h \in \Sigma_1$, one knows by Scott's theorem that if $h \notin G$ then there is a finite quotient $\pi : \Sigma_1 \to Q$ of Σ_1 such that $\pi(h) \notin \pi(G)$; that is, π *separates h from G*. To determine whether $h \in G$ it is enough to run two simultaneous processes: on the one hand one enumerates the finite quotients of Σ_1, looking for one that separates h from G; on the other hand one tries to show that $h \in G$ by simply testing if $h^{-1}w = 1$ where w runs through an enumeration of the words in the generators X of G.

Now, proceeding by induction on n, we assume that there is a solution to the membership problem for each finitely presented subgroup of a direct product of $n-1$ or fewer free and surface groups. Let $D = \Sigma_1 \times \cdots \times \Sigma_n$ and suppose that G is a finitely presented subdirect product of D. Define $L_i = G \cap \Sigma_i$.

There is no loss of generality in assuming that elements $h \in D$ are given as words in the generators of the factors, and thus we write $h = (h_1, \ldots, h_n)$. We assume that the generators of G are given likewise.

We first deal with the case where some L_i is trivial, say L_1. The projection of G to $\Sigma_2 \times \cdots \times \Sigma_n$ is then isomorphic to G, so in particular it is finitely presented and our induction provides an algorithm that determines if $(h_2, \ldots, h_n)$ lies in this projection. If it does not, then $h \notin G$. If it does, then by a naïve search we eventually find a word w in the generators of G so that $h^{-1}w$ projects to $1 \in \Sigma_2 \times \cdots \times \Sigma_n$. Since $L_1 = G \cap \Sigma_1 = \{1\}$, we deduce that in this case $h \in G$ if and only if

$h^{-1}w = 1$, and the validity of this equality can be checked because the word problem is solvable in D.

It remains to consider the case where G intersects each factor non-trivially. Again we are given $h = (h_1, \ldots, h_n)$. The projection G_i of G to Σ_i is finitely generated and the Σ_i are subgroup separable, so we can determine algorithmically if $h_i \in G_i$. If $h_i \notin G_i$ for some i then $h \notin G$ and we stop. Otherwise, we replace D by the direct product of the G_i. Lemma 5.5 allows us to compute a finite presentation for G_i and hence D.

We are now reduced to the case where G is a subdirect product of D and all of the intersections L_i are non-trivial. Theorem 4.7 tells us that $Q = D/L$ is virtually nilpotent, where $L = L_1 \times \cdots \times L_n$. Let $\phi : D \to Q$ be the quotient map.

Virtually nilpotent groups are subgroup separable, so if $\phi(h) \notin \phi(G)$ then there is a finite quotient of Q (and hence D) that separates h from G. But $\phi(h) \notin \phi(G)$ if $h \notin G$ because $L = \ker \phi$ is contained in G. Thus, as in the second paragraph of the proof, an enumeration of the finite quotients of D provides an effective procedure for proving that $h \notin G$ if this is the case. (Note that we need a finite presentation of D in order to make this enumeration procedure effective; hence our earlier invocation of Lemma 5.5.)

We now have a procedure that will terminate in a proof if $h \notin G$. Once again, we run this procedure in parallel with a simple-minded enumeration of $h^{-1}w$ that will terminate with a proof that $h \in G$ if this is true. □

Remark 5.7. Subsequent to our work, Bridson and Wilton [13] used results from [10] to prove that in the profinite topology on any finitely generated residually free group, all finitely presented subgroups are closed. This provides a *uniform* solution to the membership problem for the subgroups considered in Theorem 5.6.

6. Fibre products and HNN extensions

Consider an HNN-extension of the form

$$A = \langle B, t \mid t^{-1}ht = \phi(h)\ (h \in H) \rangle$$

where ϕ is an isomorphism between subgroups H and $\phi(H)$ of the base group B. Recall that A is said to be an *ascending HNN-extension* if either $H = B$ or $\phi(H) = B$. In this case either t or t^{-1} conjugates B into a subgroup of itself.

The metabelian Baumslag-Solitar groups $\mathrm{BS}(1,p) = \langle b, t \mid t^{-1}bt = b^p \rangle$ are examples of ascending HNN-extensions. In [2] it is shown, for instance, that the untwisted fibre product of two copies of $\mathrm{BS}(1,p)$ mapping onto the infinite cycle $Q = \langle t \mid \ \rangle$ is finitely presented.

The group $\mathrm{BS}(2,3) = \langle b, t \mid t^{-1}b^2t = b^3\rangle$ is a non-ascending HNN-extension of $\langle b \mid \ \rangle$. We are going to show, in contrast to [2], that the untwisted fibre product G of two copies of $\mathrm{BS}(2,3)$ mapping onto Q is finitely generated but not finitely presented. We also show $H_2(G, \mathbb{Z}) = 0$, so that homology does not detect the lack of a sufficient finite set of relators.

The following gives a large collection of untwisted fibre products which are finitely generated but not finitely presented.

Theorem 6.1. *Let A_1 and A_2 be non-ascending HNN extensions with finitely presented base groups, finitely generated amalgamated subgroups and stable letters t_1 and t_2. Let $q_i : A_i \to Q = \langle t \mid \ \rangle$ be the map defined by sending t_i to t and the base groups to 1. Then the fibre product G of q_1 and q_2 is finitely generated but not finitely presented.*

Proof. To simplify notation, even though they are not assumed to be isomorphic, we suppress the subscripts on A_1 and A_2 for the first part of the proof, adding subscripts when we consider the fibre product. By hypothesis we have a non-ascending HNN extension $A = \langle B, t \mid t^{-1}ht = \phi(h)\ (h \in H)\rangle$ where B is finitely presented and H is finitely generated. Of course A is finitely presented as well.

It is easy to see that the kernel L of ϕ has the structure of a two-way infinite, proper amalgamated free product. Setting $B^i = t^{-i}Bt^i$, we observe that B^0 and B^1 generate their amalgamated free product

$$\langle B^0, B^1\rangle = \langle B, t^{-1}Bt\rangle = B \underset{\phi(H)=t^{-1}Ht}{\star} (t^{-1}Bt).$$

Conjugating by t translates this amalgamation decomposition, and L has an amalgamation decomposition indexed by the integers as

$$L = \cdots \star B^{-1} \underset{t\phi(H)t^{-1}=H}{\star} B^0 \underset{\phi(H)=t^{-1}Ht}{\star} B^1 \star \cdots .$$

Notice that each amalgamated subgroup is properly contained in each factor because of our assumption that A is not an ascending HNN-extension. It follows that L is not finitely generated.

At this point we reinstate the subscripts on the various objects associated with A_1 and A_2.

Now the pullback $G \subset A_1 \times A_2$ is generated by B_1 and B_2 together with $\vec{t} = (t_1, t_2)$. Also $G \cap A_i = L_i$, the kernel of the map onto the infinite cycle, has a decomposition as above. Furthermore, G is the split extension of $L_1 \times L_2$ by the infinite cyclic group generated by $\vec{t} = (t_1, t_2)$.

A presentation for G can be obtained by taking as generators $\vec{t}$ together with generators for B_1 and B_2, and taking as relations:

(1) the finite sets of relations of both B_1 and B_2;
(2) the relations $\vec{t}^{\,-1}h_1\vec{t} = \phi_1(h_1)$ and $\vec{t}^{\,-1}h_2\vec{t} = \phi_2(h_2)$ where h_1 ranges over a finite set of generators for H_1 and h_2 ranges over a finite set of generators for H_2;
(3) the relations $u_1v_2 = v_2u_1$ where u_1 ranges over a set of generators for L_1 and v_2 ranges over a set of generators for L_2.

The relations in (1) and (2) are finite in number, but the relations in (3) are necessarily infinite in number since neither L_1 nor L_2 is finitely generated.

Now, in order to obtain a contradiction, assume that G is finitely presented. Then in the above presentation some finite subset S of the relations in (3) together with (1) and (2) suffice to present G. We can assume there is a finite portion of the decomposition of each L_i of the form

$$B_i^{(m,n)} = B_i^m \underset{(\phi(H))^m = H^{m+1}}{\star} \cdots \underset{(\phi(H))^{n-1} = H^n}{\star} B_i^n$$

so that all the generators that appear in S lie in either $B_1^{(m,n)}$ or $B_2^{(m,n)}$. We choose the interval (m,n) large enough to contain both 0 and 1. Of course each $B_i^{(m,n)}$ is finitely presented. We are free to add finitely many relations to those already present, so we now enlarge S to contain the finitely many relations required to say that $B_1^{(m,n)}$ and $B_2^{(m,n)}$ commute.

Observe that the group presented in this way can also be described as the HNN-extension of $B_1^{(m,n)} \times B_2^{(m,n)}$ by the stable letter $\vec{t}$ which acts in the same way on each direct factor and is non-ascending on each. But then the normal closure of $B_1^{(m,n)} \times B_2^{(m,n)}$ has a two-way infinite amalgamated free product decomposition of the form

$$\cdots \star (B_1^{(m,n)} \times B_2^{(m,n)}) \underset{M^{(m,n)}}{\star} (B_1^{(m+1,n+1)} \times B_2^{(m+1,n+1)}) \star \cdots$$

By our non-ascending assumption, there are elements $x_1 \in B_1^{(m,n)}$ and $y_2 \in B_2^{(m+1,n+1)}$ which do not lie in the amalgamated subgroup $M^{(m,n)}$. Hence $x_1y_2x_1^{-1}y_2^{-1} \neq 1$ or equivalently $x_1y_2 \neq y_2x_1$ in this group. But these elements clearly must commute in G, which is a contradiction. Hence G could not have been finitely presented. $\square$

The following gives another example of the type given by Baumslag in [1].

Theorem 6.2 (=Theorem G)**.** *Let* $A = \mathrm{BS}(2,3) = \langle b, t \mid t^{-1}b^2t = b^3\rangle$ *and let* $q : A \to Q = \langle t \mid \ \rangle$ *be the map defined by sending* t *to* t *and* b

to 1. Then the untwisted fibre product G of two copies of the map q is finitely generated but not finitely presented, and has $H_2(G, \mathbb{Z}) = 0$.

Proof. An easy calculation shows that $H_1(A, \mathbb{Z}) = \mathbb{Z}$ and by the Mayer-Vietoris sequence for HNN-extensions (or the homology theory of one-relator groups) we have $H_2(A, \mathbb{Z}) = 0$. Define $L = [A, A]$ the derived group of A. Then A is also a split extension of L by Q. The E^2 term of the spectral sequence for this extension has zero maps as differentials so $E^2 = E^\infty$. Since $H_1(A, \mathbb{Z}) = \mathbb{Z}$, it follows from the spectral sequence that $H_0(Q, H_1(L, \mathbb{Z})) = 0$. Also since $H_2(A, \mathbb{Z}) = 0$ it follows that $H_1(Q, H_1(L, \mathbb{Z})) = 0$ and $H_0(Q, H_2(L, \mathbb{Z})) = 0$.

Now L has a decomposition as a two-way infinite amalgamated free product. If we put $b_i = t^{-i}bt^i$ for $i \in \mathbb{Z}$ then $b_{i+1}^2 = b_i^3$ and the decomposition of L is

$$\cdots \star \langle b_{-1}, b_0 \mid b_{-1}^3 = b_0^2 \rangle \underset{\langle b_0 \rangle}{\star} \langle b_0, b_1 \mid b_0^3 = b_1^2 \rangle \underset{\langle b_1 \rangle}{\star} \langle b_1, b_2 \mid b_1^3 = b_2^2 \rangle \cdots$$

Computing $H_1(L, \mathbb{Z})$ by abelianizing this group, one can show that $H_1(L, \mathbb{Z})$ is locally cyclic. Using the (abelian) calculation $(b_1 b_{-1} b_0^2)^6 = b_1^6 b_{-1}^6 b_0^{-12} = b_0^9 b_0^4 b_0^{-12} = b_0$ it follows that $H_1(L, \mathbb{Z}) \cong \mathbb{Z}[\frac{1}{6}]$ and the action induced by conjugation by t (respectively t^{-1}) on $H_1(L, \mathbb{Z})$ is multiplication by $\frac{3}{2}$ (respectively $\frac{2}{3}$).

Let $K_{m,n}$ be the subgroup of L generated by $\{b_m, \ldots, b_n\}$. Again an easy Mayer-Vietoris sequence calculation shows that $H_1(K_{0,1}, \mathbb{Z}) = \mathbb{Z}$ and $H_2(K_{0,1}, \mathbb{Z}) = 0$ (it is the group of the trefoil knot). Using the amalgamated free product decomposition $K_{m,n+1} = K_{m,n} \underset{b_n = b_n}{\star} K_{n,n+1}$, the Mayer-Vietoris sequence shows inductively that $H_1(K_{m,n}, \mathbb{Z}) = \mathbb{Z}$ and $H_2(K_{m,n}, \mathbb{Z}) = 0$. Since homology commutes with direct limits of groups, it follows that $H_2(L, \mathbb{Z}) = 0$.

Let A_1 and A_2 be two copies of A and view G as a subgroup of $A_1 \times A_2$. Then $L_i = G \cap A_i$ and G is the split extension of $L_1 \times L_2$ by the infinite cycle generated by $\vec{t} = (t_1, t_2)$ which we can identify with Q. The previous theorem shows that G is finitely generated but not finitely presented.

Now $H_1(L_1 \times L_2, \mathbb{Z}) = H_1(L_1, \mathbb{Z}) \oplus H_1(L_2, \mathbb{Z}) = \mathbb{Z}[\frac{1}{6}] \oplus \mathbb{Z}[\frac{1}{6}]$. Since $H_2(L, \mathbb{Z}) = 0$ the Künneth theorem shows $H_2(L_1 \times L_2, \mathbb{Z}) \cong H_1(L_1, \mathbb{Z}) \otimes H_1(L_2, \mathbb{Z})$ where conjugation by $\vec{t}$ induces the diagonal action on the right hand tensor product. Since $H_1(L, \mathbb{Z}) = \mathbb{Z}[\frac{1}{6}]$, the calculation $\frac{1}{6^n} \otimes 1 = \frac{1}{6^n} \otimes \frac{6^n}{6^n} = \frac{6^n}{6^n} \otimes \frac{1}{6^n} = 1 \otimes \frac{1}{6^n}$ implies $H_1(L_1, \mathbb{Z}) \otimes H_1(L_2, \mathbb{Z}) \cong H_1(L, \mathbb{Z})$. Hence we have $H_2(L_1 \times L_2, \mathbb{Z}) \cong H_1(L, \mathbb{Z})$.

We now examine the E^2 terms $H_p(Q, H_q(L_1 \times L_2, \mathbb{Z}))$ of the spectral sequence for G with $p + q = 2$. Since Q is the infinite cyclic group

$H_2(Q, H_0(L_1 \times L_2, \mathbb{Z})) = 0$. Using our observation in the first paragraph of the proof about the spectral sequence for A we have

$$H_1(Q, H_1(L_1 \times L_2, \mathbb{Z}) \cong H_1(Q, H_1(L_1, \mathbb{Z}) \oplus H_1(L_2, \mathbb{Z})$$
$$\cong H_1(Q, H_1(L_1, \mathbb{Z})) \oplus H_1(Q, H_1(L_1, \mathbb{Z})) = 0 \oplus 0 = 0$$

and

$$H_0(Q, H_2(L_1 \times L_2, \mathbb{Z})) \cong H_0(Q, H_1(L, \mathbb{Z})) = 0.$$

Since all three of these terms with $p + q = 2$ are 0, it follows that $H_2(G, \mathbb{Z}) = 0$. This completes the proof. $\square$

Acknowledgements: The following organizations provided financial support at various stages of this project: the Australian Research Council, the EPSRC of the United Kingdom, the Swiss National Science Foundation and l'Alliance Scientific (grant # PN 05.004). Bridson was also supported in part by a Royal Society Nuffield Research Merit Award. We are grateful to all of these organizations.

We also thank l'Université de Provence Aix-Marseilles, the EPFL (Lausanne) and l'Université de Genève for their hospitality during the writing of this article. We particularly thank Hamish Short for his extended hospitality.

References

[1] G. Baumslag, *A finitely generated, infinitely related group with trivial multiplicator*, Bull. Austral. Math. Soc. **5** (1971), 131–136.

[2] G. Baumslag, M. R. Bridson, D. F. Holt and C. F. Miller III, *Finite presentation of fibre products of metabelian groups*, J. Pure Appl. Algebra **181** (2003), 15–22.

[3] Baumslag G., Bridson M.R., Miller C.F., Short H., *Fibre products, non-positive curvature and decision problems*, Comm. Math. Helv. **75** (2000), 457–477.

[4] G. Baumslag, E. Dyer and C. F. Miller III, *On the integral homology of finitely presented groups*, Topology, **22** (1983), 27-46.

[5] G. Baumslag and J. E. Roseblade, *Subgroups of direct products of free groups*, J. London Math. Soc. (2) **30** (1984), 44–52.

[6] R. Bieri, "Homological dimension of discrete groups", Queen Mary College Mathematics Notes (1976).

[7] M. R. Bridson, *On the subgroups of semihyperbolic groups*, in "Essays on geometry and related topics", pp. 85–111, Monogr. Enseign. Math. **38**, Geneva, 2001.

[8] M. R. Bridson and J. Howie, *Subgroups of direct products of elementarily free groups*, Geom. Funct. Anal. (GAFA) **17** (2007), 385–403.

[9] M. R. Bridson, J. Howie, C. F. Miller III and H. Short, *The subgroups of direct products of surface groups*, Geometriae Dedicata **92** (2002), 95–103.

[10] M. R. Bridson, J. Howie, C. F. Miller III and H. Short, *Subgroups of direct products of limit groups*, Ann. of Math., in press.

[11] M. R. Bridson, J. Howie, C. F. Miller III and H. Short, *Finitely presented residually free groups*, preprint.

[12] M. R. Bridson, M. Tweedale and H. Wilton, *Elementarily free are measure equivalent to free groups*, Ergodic Theory Dynam. Systems **27** (2007), 703–712.

[13] M. R. Bridson and H. Wilton, *Subgroup separability in residually free groups*, Math. Z., DOI 10.1007/s00209-007-0256-7.

[14] D. J. Collins and C. F. Miller III, *The conjugacy problem and subgroups of finite index*, Proc. London Math. Soc. **34** (1977), 535–556.

[15] T. Delzant and M. Gromov, *Cuts in Kähler groups*, in: Infinite groups: geometric, combinatorial and dynamical aspects, (L. Bartholdi, T. Ceccherini-Silberstein, T. Smirnova-Nagnibeda and A. Zuk, eds) 31–55, Progr. Math., 248, Birkhäuser, Basel, 2005.

[16] S. M. Gersten and H. B. Short, *Rational subgroups of biautomatic groups*, Ann. of Math. **134** (1991), 125–158.

[17] C. McA. Gordon, *Some embedding theorems and undecidability questions for groups*, in Combinatorial and Geometric Group Theory (Edinburgh 1993), LMS Lecture Notes **204**. Cambridge 1995, pp. 105–110.

[18] F. Grunewald, *On some groups which cannot be finitely presented*, J. London Math.Soc. (2) **17** (1978), 427–436.

[19] M. Hall Jr., *Subgroups of finite index in free groups*, Canad. J. Math. **1** (1949), 187–190.

[20] E. H. Lo, *Finding intersections and normalizers in finitely generated nilpotent groups*, J. Symbolic Comput. **25** (1998), 45–59.

[21] C. F. Miller III, "On group-theoretic decision problems and their classification", Annals of Mathematics Studies, No. 68, Princeton University Press (1971).

[22] C. F. Miller III, *Decision problems for groups: survey and reflections*, in Algorithms and Classification in Combinatorial Group Theory (eds. G Baumslag and C. F. Miller III), MSRI Publications No. 23, Springer-Verlag (1992), 1-59.

[23] C. F. Miller III, *Subgroups of direct products with a free group*, Quarterly Journal of Maths **53** (2002), 503–506.

[24] H. Meinert, *The geometric invariants of direct products of virtually free groups*, Comment. Math. Helv. **69** (1994), 39–48.

[25] G. P. Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2) **17** (1978), 555–565.

[26] J. R. Stallings, *A finitely presented group whose 3–dimensional homology group is not finitely generated*, Amer. J. Math. , **85**, (1963) 541–543.

Martin R. Bridson, Mathematics Department, Huxley Building, Imperial College London, London SW7 2AZ, U.K.

Current address: Mathematical Institute, 24-29 St Giles', Oxford OX1 3LB, U.K.

E-mail address: `bridson@maths.ox.ac.uk`

Charles F. Miller III, Department of Mathematics and Statistics, University of Melbourne, Melbourne 3010, Australia

E-mail address: `c.miller@ms.unimelb.edu.au`