ON THE ALGORITHMIC CONSTRUCTION OF CLASSIFYING SPACES AND THE ISOMORPHISM PROBLEM FOR BIAUTOMATIC GROUPS

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ABSTRACT. We show that the isomorphism problem is solvable in the class of central extensions of word-hyperbolic groups, and that the isomorphism problem for biautomatic groups reduces to that for biautomatic groups with finite centre. We describe an algorithm that, given an arbitrary finite presentation of an automatic group Γ, will construct explicit finite models for the skeleta of $K(\Gamma, 1)$ and hence compute the integral homology and cohomology of Γ.

For Fabrizio Catanese on his 60th birthday

There are several natural classes of finitely presented groups that cluster around the notion of non-positive curvature, ranging from hyperbolic groups to combable groups (see [6] for a survey and references). The isomorphism problem is solvable in the class of hyperbolic groups but is unsolvable in the class of combable groups [4]. It remains unknown whether the isomorphism is solvable in the intermediate classes, such as (bi)automatic groups and CAT(0) groups. Hyperbolic groups also form one of the very few interesting classes in which there is an algorithm that, given a finite presentation of a group Γ in the class, will construct finite models for the skeleta of a $K(\Gamma, 1)$. For finitely presented groups in general, one cannot even calculate $H^2(\Gamma, \mathbb{Z})$; see [13]. Our focus in this article will be on the isomorphism problem for biautomatic groups and the construction problem for classifying spaces of combable and automatic groups.

We remind the reader that the isomorphism problem for a class $\mathcal{G}$ of finitely presented groups is said to be solvable if there exists an algorithm that takes as input pairs of finite group presentations (P_1, P_2) and proceeding under the assumption that the groups $|P_i|$ belong to $\mathcal{C}$, decides whether or not $|P_1| \cong |P_2|$. The first purpose of this article is to point out that the isomorphism problem for biautomatic groups (or any subclass of such groups) can be reduced to the problem of determining

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isomorphism of the groups modulo their centres. We write $Z(G)$ to denote the centre of a group G.

Theorem A. *Let $\mathcal{G}$ be a class of biautomatic groups. Let $\mathcal{Q}$ be the class of groups $\{G/Z(G) \mid G \in \mathcal{G}\}$. If the isomorphism problem is solvable in $\mathcal{Q}$ then it is solvable in $\mathcal{G}$.*

Zlil Sela [16] proved that the isomorphism problem is solvable among finite presentations of freely-indecomposable, torsion-free hyperbolic groups, and his work has recently been extended by François Dahmani and Vincent Guirardel to cover all hyperbolic groups [9]; see also [8]. Walter Neumann and Lawrence Reeves [15] proved that central extensions of hyperbolic groups are biautomatic. Thus we have:

Corollary 0.1. *The isomorphism problem is solvable in the class of central extensions of hyperbolic groups.*

In Section 3 we describe an algorithm for constructing the skeleta of classifying spaces for combable groups (given an explicit fellow-traveller constant). From this we deduce:

Theorem B. *There is an algorithm that, given a positive integer d and any finite presentation of an automatic group Γ, will construct a compact $(d+1)$-skeleton for $K(\Gamma, 1)$, i.e. an explicit finite, connected cell complex K with $\pi_1 K \cong \Gamma$ and $\pi_i K = 0$ for $2 \leq i \leq d$.*

Theorem C. *There is an algorithm that, given any finite presentation of an automatic group Γ, will calculate $H_*(\Gamma, A)$ and $H^*(\Gamma, A)$, where A is any finitely generated abelian group and the action of Γ on A is trivial.*

The algorithm for calculating $H^2\Gamma$ furnishes the following major ingredient for the proof of Theorem A.

Corollary 0.2. *There is an algorithm that computes a complete irredundant list of central extensions of any given automatic group by a given finitely generated abelian group.*

Lee Mosher proved that central quotients of biautomatic groups are biautomatic [14]. By combining this with Theorems A and C, and some well-known facts about subgroups of biautomatic groups, we shall prove:

Theorem D. *The isomorphism problem among biautomatic groups is solvable if and only if the isomorphism problem is solvable among biautomatic groups with finite centre.*

It follows from Theorems C and D that if the isomorphism problem for biautomatic groups is unsolvable, then there must exist a recursive sequence of finite presentations such that each of the groups presented is biautomatic, all of the groups in the sequence have finite centre and isomorphic integral homology and cohomology groups, but one cannot tell which of the groups presented are isomorphic.

1. Determining the centre of a biautomatic group

The main purpose of this section is to prove the following:

Proposition 1.1. *There exists an algorithm that takes as input an arbitrary finite presentation of a biautomatic group and which gives as output the isomorphism type of the centre of the group, and a finite set of words that generate the centre.*

We shall assume that the reader is familiar with the basic vocabulary of automatic group theory, as set out in the seminal text [10]. It is convenient to fix the following notation.

Let Γ be a group with finite generating set $\mathcal{A}$. The free monoid on $\mathcal{A}$ is denoted $\mathcal{A}^*$, and the natural surjection $\mathcal{A}^* \to \Gamma$ is denoted μ. We assume that $\mathcal{A}$ is equipped with an involution, written $a \mapsto a^{-1}$ such that $\mu(a^{-1}) = \mu(a)^{-1}$.

Given a language $\mathcal{L} \subseteq \mathcal{A}^*$, we define the language $\mathcal{L}^{-1} \subseteq \mathcal{A}^*$ to be the set of formal inverses of $\mathcal{L}$, that is, $a_1 \cdots a_n \in \mathcal{L}^{-1}$ if and only if $a_n^{-1} \cdots a_1^{-1} \in \mathcal{L}$. A language $\mathcal{L} \subset \mathcal{A}^*$ is called a ***biautomatic structure*** for Γ if the restriction of $\mu : \mathcal{A}^* \twoheadrightarrow \Gamma$ to each of $\mathcal{L}$ and $\mathcal{L}^{-1}$ is an automatic structure — see [10, Definition 2.5.4].

We remind the reader that associated to a biautomatic structure $\mathcal{L} \twoheadrightarrow \Gamma$ one has a ***fellow-traveller*** constant $k > 0$ with the property that for letters $a, a' \in \mathcal{A}^{\pm 1}$, words $w, w' \in \mathcal{L}$ with $\mu(aw) = \mu(w'a')$ and positive integers t, one has $d(\mu(aw_t), \mu(w'_t)) \leq k$, where d is the word metric associated to $\mathcal{A}^* \twoheadrightarrow \Gamma$ and u_t denotes the prefix of length t in the word u.

In [10, Chapter 5] an algorithm is described which, given a finite presentation for an automatic group, will produce an automatic structure for the group. Although not explicitly stated in [10], the following generalization is a straightforward consequence of their proof.

Proposition 1.2. *There exists an algorithm that, given a finite presentation of a biautomatic group, will construct a biautomatic structure for the group and calculate a fellow-traveller constant for that structure.*

Proof. We describe the changes needed to the algorithm in section 5.2 of [10].

Given a finite state automaton W with accepted language $\mathcal{L}$ (over the given generating set), the algorithm given in [10] will, if $\mathcal{L}$ is part of an automatic structure, eventually terminate and give a full description of the automatic structure (i.e., all multiplier automata and the equality checker). Let W^{-1} be a finite state automaton with accepted language $\mathcal{L}^{-1}$. Applying this algorithm to both W and W^{-1} gives an algorithm which will terminate if $\mathcal{L}$ is a biautomatic structure.

Using a standard diagonal argument, this procedure is applied 'simultaneously' to all pairs of finite state automata (W, W^{-1}) with the given input alphabet. The fact that the group is biautomatic ensures that at some point the algorithm terminates. One can then obtain a fellow-traveller constant directly from the geometry of the automata (cf. Lemma 2.3.2 [10]). □

Lemma 1.3. *There is an algorithm that, given a finite presentation of an automatic group, will list all of the words that represent the identity in order of increasing length.*

Proof. As above, we first calculate an automatic structure for the group. Let $\mathcal{A}$ denote the generating set of the presentation. Given an enumeration of all words over $\mathcal{A}$ in order of increasing length, we can use the equality checker to test whether each is equal in the group to the identity (which is represented by the empty word). □

Lemma 1.4. *There exists an algorithm that, given a biautomatic structure ($\mathcal{L} \twoheadrightarrow \Gamma$) for a group Γ, will calculate the sublanguage of $\mathcal{L}$ that evaluates onto the centre of Γ (i.e., the language $\mathcal{L} \cap \mu^{-1}(Z(\Gamma))$).*

Proof. Denote by $C(g)$ the centralizer of a group element g. Setting $\mathcal{L}_a = \mathcal{L} \cap \mu^{-1}(C(\mu(a)))$, we have $\mathcal{L} \cap \mu^{-1}(Z(\Gamma)) = \cap_{a \in \mathcal{A}} \mathcal{L}_a$. If the fellow-traveller constant of the biautomatic structure is k, then by definition $d(1, \mu(p^{-1}ap)) \leq k$ for all prefixes p of words $w \in \mathcal{L}$ such that $\mu(wa) = \mu(aw)$. Thus, writing $\mathcal{P}_a^k$ for the language of words $w \in \mathcal{A}^*$ such that $d(1, \mu(p^{-1}ap)) \leq k$ for all prefixes p of w, we see that $\mathcal{L}_a \subseteq \mathcal{P}_a^k$. Therefore, since the intersection of finitely many regular languages is regular, it suffices to construct a finite state automaton over $\mathcal{A}$ with accepted language $\mathcal{P}_a^k \cap \mu^{-1}(C(\mu(a)))$.

The set of states of the desired automaton is the ball $B(k, 1)$ of radius k about $1 \in \Gamma$ in the word metric, together with one other (fail) state ϕ. The state corresponding to $\mu(a)$ is both the start and (unique) accept state. The transitions are given by

$$B(k,1) \ni g \overset{b\in\mathcal{A}}{\longmapsto} \begin{cases} \mu(b)^{-1}g\mu(b) & \text{if } \mu(b)^{-1}g\mu(b) \in B(k,1) \\ \phi & \text{otherwise} \end{cases}$$

$$\phi \overset{b\in\mathcal{A}}{\longmapsto} \phi$$

That is, if the machine is in state g when it reads the letter b from the input tape, then it moves to the fail state if conjugation by $\mu(b)$ sends g to an element outside the ball $B(k,1)$, and it moves to $\mu(b)^{-1}g\mu(b)$ if it is in the ball. ☐

Proposition 1.5. *There is an algorithm that, given a finite presentation $\langle \mathcal{A} \mid \mathcal{R}\rangle$ for a biautomatic group Γ, will calculate a finite set of words in $\mathcal{A}^*$ that generates the centre of Γ, and will give a finite presentation of $Z(\Gamma)$ in terms of these generators.*

Proof. Proposition 1.2 yields an explicit biautomatic structure for Γ, together with a fellow-traveller constant $k > 0$ for that structure. In the course of the preceding proof we implicitly showed that $Z(\Gamma)$ is generated by words from $\mathcal{A}^*$ that have length at most $2k+1$. Thus, in order to obtain an explicit set of generators for $Z(\Gamma)$ we need only check which words of length at most $2k+1$ commute with all of the generators $\mathcal{A}$ of Γ. And this one can do by listing all of the words of length at most $4k+4$ that represent the identity in Γ, using Lemma 1.3.

In fact, the preceding proof shows that $Z(\Gamma)$ is a quasiconvex subgroup of Γ (a result originally due to Gersten and Short [12]) with a quasiconvexity constant K that can be calculated from k (see [2] pages 94–95). It follows that one obtains a finite presentation for $Z(\Gamma)$ by simply calculating which concatenations of strings of generators of $Z(\Gamma)$, with total length $2(2k+1)+2(2K+[K(2k+1)]$ in $\mathcal{A}^*$, represent $1 \in \Gamma$. ☐

We can now complete the proof of Proposition 1.1. Given a presentation of a biautomatic group, we calculate a finite presentation of $Z(\Gamma)$ as above. The isomorphism problem for abelian groups is solvable and one can make an explicit list (P_n) of finite presentations, exactly one for each isomorphism type of finitely generated abelian group. One then looks for an isomorphism between $Z(\Gamma)$ and the groups on this list by simply enumerating homomorphisms from $Z(\Gamma)$ to each of the groups and *vice versa*, looking for an inverse pair, as in Lemma 1.7. In more detail: for each of the presentations on the list (P_n), the construction of Lemma 1.7 provides a partial algorithm that will successfully terminate if the group $G_n = |P_n|$ is isomorphic to $Z(\Gamma)$; the algorithm that we are describing here runs in a diagonal manner — first it runs one step of the procedure that looks for an isomorphism between $Z(\Gamma)$ and G_1, then one step of the procedure comparing $Z(\Gamma)$ to G_2, then a further two steps comparing $Z(\Gamma)$ to G_1, to G_2, and to G_3; then a further three steps of the procedures comparing $Z(\Gamma)$ to G_1, to G_2, to G_3, and to G_4, and so on.

This completes the proof of Proposition 1.1.

Remark 1.6. In deciding the isomorphism class of $Z(\Gamma)$ above, we appealed to the general solution for the isomorphism problem for finitely generated abelian groups. In the final section we present some results which provide a more efficient search that exploits the rational structure on $Z(\Gamma)$.

In the proof of Proposition 1.1 we used the following general and well-known result, which we make explicit for the sake of clarity.

Lemma 1.7. *There is a partial algorithm that, given two finite presentations, will search for an isomorphism between the two groups: if the presentations define isomorphic groups then this procedure will eventually halt; if the groups are not isomorphic then it will not terminate.*

Proof. Given finite presentations $\langle A_1 \mid R_1\rangle$ and $\langle A_2 \mid R_2\rangle$ defining groups G_1 and G_2, one can enumerate all homomorphisms from G_1 to G_2 by running through all choices of words $\{u_a \mid a \in A_1\}$ in the free group on A_2, treating these as putative images of the generators of G_1 in G_2: one freely reduces the words obtained by substituting u_a for each occurrence of a in the relations R_1 – call these words $\{\rho_r \mid r \in R_1\}$; one tries to verify that $a \mapsto u_a$ defines a homomorphism $G_1 \to G_2$ by listing all products of conjugates of the relations $R_2^{\pm 1}$, freely reducing them and comparing the freely-reduced form to the words ρ_r; the assignment $a \mapsto u_a$ defines a homomorphism $G_1 \to G_2$ if and only if this procedure eventually produces all of the words ρ_r.

One applies the same process with the indices reversed to enumerate all homomorphisms from G_2 to G_1. In parallel, one tests all pairs of homomorphisms $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_1$ that are found in order to see if they are mutually inverse. Again this test is carried out by a naive search: the homomorphisms are described by explicit formulae saying where they send the generators, so to check that $f_2 \circ f_1 = \mathrm{id}_{G_1}$, for example, one simply has to check if a list of words $(w_a : a \in A_1)$ defines the same indexed set of elements of G_1 as $(a : a \in A_1)$;

one is interested only in a positive answer, so one does not need a solution to the word problem for this, one just enumerates all products of conjugates of the relations $R_1^{\pm 1}$, checking to see if each is *freely* equal to $a^{-1}w_a$. □

2. A reduction of Theorem A

Given a group G and an abelian group A, a group E is called a central extension of G by A if there is a short exact sequence

$$0 \to A \to E \to G \to 1$$

and the map $G \to \mathrm{Aut}(A)$ induced by conjugation in E is trivial. We remind the reader that such central extensions are classified up to equivalence by the cohomology class $[z] \in H^2(G, A)$ of the cocycle $z : G \times G \to A$ that is defined by choosing a set-theoretic section $s : G \to E$ of the given surjection and setting $z(g, g') = s(gg')s(g')^{-1}s(g)^{-1}$ (see [7, Section IV.3]).

Suppose now that $\mathcal{G}$ and $\mathcal{Q}$ are as in the statement of Theorem A, and that we are given finite presentations $\langle \mathcal{A}_1 \mid \mathcal{R}_1 \rangle$ and $\langle \mathcal{A}_2 \mid \mathcal{R}_2 \rangle$ for groups G_1, $G_2 \in \mathcal{G}$. Denote by Z_1 the centre of G_1 and denote by Q_1 the quotient G_1/Z_1. Define Z_2 and Q_2 similarly. The results of the previous section and the hypothesis that the isomorphism problem is solvable in $\mathcal{Q}$ allow us to decide the isomorphism types of the groups Q_1, Q_2, Z_1, Z_2. If $Q_1 \not\cong Q_2$ or $Z_1 \not\cong Z_2$, then we conclude that the original groups Γ_1 and Γ_2 are not isomorphic. Thus Theorem A has been reduced to a problem of deciding whether two central extensions are equivalent.

Suppose now that $Q_1 \cong Q_2$ and $Z_1 \cong Z_2$, and refer to these groups as Q and Z respectively. In Section 3 we will construct an irredundant enumeration of the possible central extensions of Q by Z. In the light of the following observation, this enumeration allows us to determine whether G_1 and G_2 are isomorphic.

Lemma 2.1. *Let $\mathcal{G}$ be a class of finitely presented groups. Given an irredundant enumeration of (presentations for) the groups in $\mathcal{G}$, one can decide whether or not an arbitrary pair of finite presentations of groups from $\mathcal{G}$ define isomorphic groups.*

Proof. Given two finite presentations $G_1 = \langle \mathcal{A}_1 \mid \mathcal{R}_1 \rangle$ and $G_2 = \langle \mathcal{A}_2 \mid \mathcal{R}_2 \rangle$ with $G_1, G_2 \in \mathcal{G}$, one sets G_2 aside and searches the enumeration of $\mathcal{G}$ to identify which group on the list is isomorphic to G_1. One does this by using Lemma 1.7 repeatedly, as in the proof of Proposition 1.1. Repeating this procedure with G_2 in place of G_1 will determine whether or not G_1 and G_2 are isomorphic to the same element in the enumeration of $\mathcal{G}$ and hence to one another. □

3. Algorithmic construction of classifying spaces

The considerations in the previous section compel us to enumerate the possible central extensions of Q by Z, and for this we need an algorithm to calculate $H^2(Q, Z)$ starting from any finite presentation of Q. More generally, we wish to calculate $H^*(Q, A)$ and $H_*(Q, A)$, where A is a finitely generated abelian group. We shall achieve this by describing an algorithm that constructs finite skeleta of a $K(Q, 1)$. This construction (Theorem 3.3) depends on something less than the existence of a combing of Q and explicit knowledge of the fellow-traveller constant for this combing; in the case of automatic groups, this constant can be calculated as in Lemma 2.3.2 of [10] (cf. Proposition 1.1). The construction is similar to that described by S. M. Gersten [11] in proving that asynchronously automatic groups are of type F_3. See also [1] and [10, section 10.2].

Each attaching map in our $K(Q, 1)$ will be given by a subdivision map followed by a restricted form of singular combinatorial map. By definition, if $f : L \to K$ is a ***singular combinatorial*** map between CW-complexes then for every open n-cell $\sigma \in L$, either the restriction of f to σ is a homeomorphism onto an open n-cell of K, or else $f(\sigma) \subset K^{(n-1)}$. The more restrictive notion of a ***semi-combinatorial map*** is obtained by requiring that if $f(\sigma) \subset K^{(n-1)}$ then $f|_\sigma$ is a constant map. A semi-combinatorial complex is one where the attaching maps of all cells are semi-combinatorial.

The $K(Q, 1)$ that we will construct is a CW-complex in which many n-cells are standard n-cubes, combinatorially. The attaching maps of the remaining cells are defined on subdivisions of a restricted type on the boundary of a standard cube; the attaching maps themselves are a restricted kind of semi-combinatorial map.

3.1. k-Lipschitz contractions.

We remind the reader that a group G with finite generating set $\mathcal{A} = \mathcal{A}^{-1}$ is said to be ***combable*** if there is a constant k and a (not necessarily regular) sublanguage $\{\sigma_g : g \in G\} \subset \mathcal{A}^*$ mapping bijectively to G under the homomorphism $\mathcal{A}^* \to G$ such that $\rho(\sigma_g(t), \sigma_{ga}(t)) \leq k$ for all $g \in G$, $a \in \mathcal{A}$ and all integers $t > 0$, where ρ is the word metric associated to $\mathcal{A}$ and $\sigma_g(t)$ is the image in G of the prefix of length t in σ_g (this prefix is taken to be equal to the whole word if t is greater than the length of the word). Such a constant k is called a ***fellow-traveller constant***.

One says that a finitely generated group G ***admits k-Lipschitz contractions*** if, given every finite subset $S \subset G$, there is a map $H_S : S \times \mathbb{N} \to G$ such that for all $s, t \in S$ and $n \in \mathbb{N}$ we have $\rho(H_S(s,n), H_S(t,n)) \leq k\,\rho(s,t)$ and $\rho(H_S(s,n), H_S(s,n+1)) \leq 1$, with H_S constant on $S \times \{0\}$ and $H_S(*, n) = \mathrm{id}_S$ for n sufficiently large (where ρ denotes the word metric).

If G is combable with combing σ_g and fellow-traveller constant k, then one obtains k-Lipschitz contractions by defining $H_S(s,n) = \sigma_s(n)$, regardless of S. More generally, groups that admit a coning of finite (asynchronous) width in the sense of [3] admit k-Lipschitz contractions.

A well-known argument that has appeared in many forms uses van Kampen diagrams to show that combable groups are finitely presented and satisfy an exponential isoperimetric inequality. This argument originated in [10] (pages 52 and 152); cf. [3] page 600. We record a version of it here because it provides the template for the homotopies described in Section 3.4. We remind the reader that if a word w in the letters $\mathcal{A}$ represents the identity in $G = \langle \mathcal{A} \mid R \rangle$, then $\mathrm{area}(w)$ is the least integer N such that w can be expressed in the free group $F(\mathcal{A})$ as a product of N conjugates of the defining relators and their inverses.

Proposition 3.1. *If G admits k-Lipschitz contractions then $G = \langle \mathcal{A} \mid R \rangle$ where R consists of all words of length at most $2(k+1)$ that represent the identity in G. Moreover, if $w = 1$ in G, then* $\mathrm{area}(w) \leq |w|.(|\mathcal{A}| + 1)^{k|w|}$.

Proof. We must prove that the 2-complex X obtained from the Cayley graph $\mathcal{C}_{\mathcal{A}}(G)$ by attaching 2-cells along all loops of length at most $2(k+1)$ is simply connected. For this it is enough to explain how to contract any edge-loop in X. Let $l : L \to X^{(1)}$ be an edge loop labelled by $w \in \mathcal{A}^*$ (with $L = [0, |w|] \subset \mathbb{R}$). Let S be the set of vertices in the image of h and let $H_S : S \times \mathbb{N} \to G$ be as in the definition of k-Lipschitz contractions. Let $N \in \mathbb{N}$ be the least integer such that $H_S(*, N) = \mathrm{id}_S$. We cellulate $D = L \times [0, N]$ as a squared complex in the obvious manner. Then $h_S(x,n) := H_S(l(x), n)$ is a map from the 0-skeleton of D to G. Given a directed 1-cell in D with initial vertex u and terminus v, we label it by a shortest word in $\mathcal{A}^*$ that equals $h(u)^{-1}h(v)$. Note that this word (which may be empty) has length at most 1 if $u = (x,n)$ and $v = (x, n+1)$, and length at most k if $u = (x,n)$ and $v = (y,n)$. By construction, there is a 2-cell in X whose attaching map describes the loop in $\mathcal{C}_{\mathcal{A}}(G)$ labelled by the word that one reads around the boundary of each 2-cell in D. (Edges labelled by the empty word are collapsed as are 2-cells whose entire boundary is collapsed.) Thus the map $D^{(1)} \to X^{(1)}$ that extends h_S and is described by the labelling of 1-cells, extends to a map from D to X. This map gives a contraction of the original loop l.

We define $\Lambda(w)$ to be the set of positive integers M for which there is a map $h : (L \times [0, M])^{(0)} \to G$ with the following properties:

- $\rho(h(u, n), h(u, n+1)) \leq 1$ for all $u \in L$ and $n < M$
- $\rho(h(u, n), h(v, n)) \leq k$ for all adjacent vertices $u, v \in L$
- $h|_{L^{(0)} \times \{M\}}$ agrees with l and $h|_{L^{(0)} \times \{0\}}$ is a constant map

The preceding argument shows that $\Lambda(w)$ is non-empty. It also shows that h extends to a map $L \times [0, M] \to X$ that sends each open 2-cell of $L \times [0, M]$ homeomorphically to an open 2-cell of X, or else collapses it. The easier implication in van Kampen's Lemma (see [5] p.49) implies that $\text{area}(w) \leq M\,|w|$ for all $M \in \Lambda(w)$. Thus the lemma will be proved if we can argue that $\min\{M \mid M \in \Lambda(w)\} \leq (|\mathcal{A}|+1)^{k|w|}$. Since $\mathcal{A} = \mathcal{A}^{-1}$ generates G, the shortest words representing each of the word differences $h(x, n)^{-1}h(y, n)$ may be taken to be positive (or empty). It follows that as n varies there are at most $(|\mathcal{A}| + 1)^{k|w|}$ possibilities for the $|w|$-tuple of words labelling the $|w|$-tuple of edges in $L \times \{n\}$. If the $|w|$-tuple of words labelling $L \times \{n\}$ and $L \times \{n'\}$ coincide for some $n < n'$, then we can delete $L \times [n, n']$ to obtain a map showing that $M - n' + n$ is in $\Lambda(w)$. So in particular, if M is minimal then there is no repetition, and hence $M < (|\mathcal{A}| + 1)^{k|w|}$. □

A recursive upper bound on the Dehn function of a finitely presented group leads in an obvious way to a solution to the word problem.

Corollary 3.2. *If $G = \langle \mathcal{A} \rangle$ admits k-Lipschitz contractions and one can list the words in the letters $\mathcal{A}$ of length at most $2(k+1)$ that represent the identity in G, then one can solve the word problem in G.*

Theorem 3.3. *There exists an algorithm that takes as input the following data:*

(0) *a positive integer d;*
(1) *a finite set of generators $\mathcal{A}$ for a group G;*
(2) *a constant k such that G admits k-Lipschitz contractions;*
(3) *a list of the words in the letters $\mathcal{A}$ that are of length at most $2(k+1)$ and that equal 1 in G;*

and which constructs as output a finite connected semi-combinatorial cell complex K with $\pi_1 K \cong G$ and $\pi_i K = 0$ for $2 \leq i \leq d$.

Corollary 3.4. *There exists an algorithm that, given an integer d and a finite presentation of an automatic group G, will construct an explicit model for the compact d-skeleton of a $K(G, 1)$.*

Proof of Corollary. One implements the algorithm in Section 5.2 of [10] to find the automatic structure. This gives both an explicit fellow-traveller constant k and a solution to the word problem. One uses the solution to the word problem to list the words of length at most at most $2(k+1)$ in the letters $\mathcal{A}$ that equal 1 in G. Theorem 3.3 then applies (cf. Proposition 1.2 and Lemma 1.3). □

3.2. **Template of the construction for K.** Let G and d be as in Theorem 3.3. The complex K will have ***vital n-cells***, the larger collection of ***inflated n-cells***, and ***translation cells***. The first two types of cells form nested subcomplexes

$$K^{(0)} \subset K_v^{(1)} \subset K_I^{(1)} \subset K_v^{(2)} \subset K_I^{(2)} \subset \cdots \subset K_v^{(d+1)} \subset K_I^{(d+1)}$$

The translation cells up to dimension n form a subcomplex T_n, and $K^{(n)} = K_I^{(n)} \cup T_n$. By definition $K = K^{(d+1)}$. Let $p : \tilde{K} \to K$ be the universal covering.

The key properties of the construction are that, for each $n \leq d$:

(i) Each finite subcomplex of $p^{-1}(K_v^{(n)}) \subset \tilde{K}$ is contractible in $p^{-1}(K_I^{(n+1)})$;

(ii) $K_I^{(n)} \cup T_{n+1}$ strong deformation retracts to $K_v^{(n)}$.

The following property plays an important role in an induction on dimension that we use to define translation cells.

(iii) There is an algorithm that, given a finite subcomplex as in (i), will construct an explicit contraction of it.

The complex K will have fundamental group G and the construction will be entirely algorithmic. We claim that Theorem 3.3 follows. Indeed, given a map of an n-sphere into K, with $2 \leq n \leq d$, by simplicial approximation we may assume that the image lies in $K^{(n)} \subset K_I^{(n)} \cup T_{n+1}$, which contracts to $K_v^{(n)}$, and by (i) any n-sphere in $K_v^{(n)}$ is contractible in $K_I^{(n+1)}$.

3.3. **Sensible labellings.** We have a fixed generating set $\mathcal{A}$ for G.

We take a set of symbols in bijection with the freely reduced words in the free group on $\mathcal{A}$ that have length at most k^r (including the empty word); we call these ***magnitude k^r labels***. For $r' > r$ we make the obvious identification of the magnitude k^r labels with the corresponding subset of the magnitude $k^{r'}$ labels. These labels will be attached to the oriented edges of the 1-skeleton of our complex, so that if a directed edge e has label $w \neq \emptyset$, then[a] $\bar{e}$ has label w^{-1}. A labelling of the edges around the boundary of a square is said to be ***sensible*** if the product

[a] $\bar{e}$ is the edge e with reversed orientation.

of labels, read with consistent orientation, is equal to the identity in our group G. (Here we evaluate the label as the corresponding product of generators $a \in \mathcal{A}$, of course.) A ***sensible labelling*** of magnitude k^r on $\mathbb{D}^n$ (the n-cube) is a labelling of its directed 1-cells by labels of magnitude k^r that is sensible on each 2-dimensional face. Note that the restriction of a labelling of any magnitude to any face of $\mathbb{D}^n$ is a sensible labelling of the same magnitude.

Two labellings of $\mathbb{D}^n$ are said to be ***equivalent*** if one is carried to the other by a symmetry of $\mathbb{D}^n$.

We highlight a trivial but important observation which explains why we articulated Corollary 3.2.

Lemma 3.5. *If the word problem is solvable in G then there is an algorithm that, given $d \in \mathbb{N}$, will list the finitely many (equivalence classes of) sensible labellings of any given magnitude for cubes up to dimension d (and then halt).* □

3.4. **The complex K.** There is a single 0-cell in K.

The 1-cells are in bijection with and are labelled by reduced word over $\mathcal{A}$ of length at most k^d (including the empty word $\emptyset$). Choose (arbitrarily) an orientation on each edge. Denote by e_w the edge labelled by $w \in \mathcal{A}^*$. Identify the edges $\overline{e}_w$ and $e_{w^{-1}}$ (where w^{-1} is the inverse of w in the free group on A, and $\overline{e}$ is the edge e with opposite orientation). Those edges labelled by words of length at most one will be called ***vital 1-cells***.

There is one 2-cell for each equivalence class of labelled squares as in Figure 1.

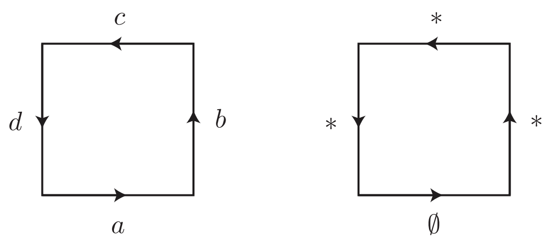

Figure 1. Labelled squares. Edges are labelled by words $a, b, c, d \in \{w \in \mathcal{A}^* \mid |w| \leq k^d\}$ satisfying $abcd = 1$ in G. The square on the right maps via semi-combinatorial gluing maps, the edges labelled $*$ mapping to the 0-cell of K.

The complex $K_I^{(n)}$ is defined to have one n-cell for each equivalence class of labellings of $\mathbb{D}^n$ of magnitude k^d. In addition K_I has one ***degenerate*** n-cell for each equivalence class of labellings of the 1-cells of $\mathbb{D}^n$ by elements of the set $\{\emptyset, *\}$, where at least one 1-cell of $\mathbb{D}^n$ is labelled by $\emptyset$. The attaching map of a degenerate cell sends the directed 1-cells labelled $\emptyset$ to the directed 1-cell of K labelled $\emptyset$, it collapses the 1-cells labelled $*$ to the 0-cell of K, and it collapses any k-dimensional face whose entire 1-skeleton is labelled $*$. The cells in $K_I^{(n)}$ are called ***inflated***.

For each positive integer $n \leq d+1$, the subcomplex $K_v^{(n)} \subset K_I^{(n)}$ consists of the i-cells, with $i \leq n$, that have labellings of magnitude k^{i-1} together with the degenerate i-cells. The cells in $K_v^{(n)}$ are called ***vital***.

All we need to know about the translation cells for the moment is that there will be no translation cells of dimension less than 2 and the translation 2-cells have attaching maps given by sensible labels (see Figure 2).

Remark 3.6. A noteworthy feature of the above construction is that it depends heavily on the integer d fixed at the beginning of the procedure. Since one knows that the n-skeleton of a complex K with d-connected universal cover can serve as the n-skeleton of a classifying space for $\pi_1 K$, one would prefer an algorithmic construction of $K(G,1)^{(d)}$ that avoids this dependence. But the dependence on d is difficult to avoid in an explicit construction. It emerges from the fact that during a k-Lipschitz contraction, the diameter of the 1-skeleton of any n-cell can expand by a factor of k: crudely speaking, this means that one has to have 2-cells whose attaching maps cover all possibilities up to scale k (cf. Proposition 3.1); one then has to contract the 2-skeleton within the 3-skeleton of the universal cover, and the natural construction of this contraction requires 3-cells whose attaching maps are larger (in an appropriate sense) by a further factor of k, and so on.

Lemma 3.7. *The maps $K_v \hookrightarrow K_I \hookrightarrow K$ induce isomorphisms of fundamental groups, and $\pi_1 K \cong G$.*

Proof. The 2-skeleton of K_v is obtained from that of the standard 2-complex P of the presentation for G given in Proposition 3.1 by adding an additional 1-cell labelled $\emptyset$, a 1-cell labelled by each word $w \in \mathcal{A}^*$ with $2 \leq |w| \leq k^d$, and many extra 2-cells. The 1-cell labelled $\emptyset$ is null-homotopic in K_I because we have the degenerate 2-cell with boundary label $(*, *, *, \emptyset)$. If $|w| \geq 2$, then $w \cong w_0 a$ for some $a \in \mathcal{A}$ and $|w_0| < |w|$. Thus the edge labelled w can be homotoped into the subcomplex with 1-cells labelled by words of lesser length. An obvious induction on $|w|$ now implies that $P \hookrightarrow K_v^{(2)}$ induces an epimorphism of fundamental groups. To see that this is actually an isomorphism, it suffices to note that the definition of *sensible* is framed so that the additional 2-cells impose on the generators $\mathcal{A}$ only relations that are valid in G. The same considerations apply to $P \hookrightarrow K_I^{(2)} \cup T_2 = K^{(2)}$. □

With this lemma in hand, we can identify G with the 0-skeleton of $\tilde{K}$ and regard the Cayley graph $C_{\mathcal{A}}(G)$ as a subcomplex of $\tilde{K}^{(1)}$. It is also justifies abbreviating $p^{-1}(K_v^{(n)})$ to $\tilde{K}_v^{(n)}$ and $p^{-1}(\tilde{K}_I^{(n)})$ to $\tilde{K}_I^{(n)}$. We do so in the following proposition.

Proposition 3.8. *If G admits k-Lipschitz contractions, then any finite subcomplex $S \subset \tilde{K}_v^{(n)}$ is contractible in $\tilde{K}_I^{(n+1)}$.*

Proof. We follow the proof of Proposition 3.1. Let S_0 be the vertex set of S, let $H_{S_0} : S_0 \times \mathbb{N} \to G = \tilde{K}^{(0)}$ be as in the definition of k-Lipschitz contractions and let $N \in \mathbb{N}$ be the least integer such that $H_{S_0}(s, N) = s$ for all $s \in S_0$. We cellulate $Y = S \times [-1, N]$ in the obvious manner: there are (horizontal) m-cells of the form $e \times [t, t+1]$, with e an $(m-1)$-cell of S, and (vertical) m-cells of the form $e' \times \{t\}$, with e' an m-cell of S. The attaching maps of the cells of S determine the attaching maps of the cells in Y.

We restrict H_{S_0} to $S_0 \times [0, N]$, then extend this to a map $h_S : Y^{(0)} \to G = \tilde{K}^{(0)}$ by sending $S \times \{-1\}$ to the same vertex as $S \times \{0\}$. If E is an m-cell in Y with attaching map $\phi_E : \mathbb{D}^m \to Y$ and ε is a 1-cell of $\mathbb{D}^m$ whose endpoints map to u and v, then we label ε by a shortest word in the letters $\mathcal{A}$ that equals $h_S(u)^{-1} h_S(v)$. The 1-cells mapping to $S_0 \times [0, 1)$ are labelled $*$ (where $*$ is the special label introduced above for degenerate cells).

The two key points to observe are: first, since the attaching maps of the m-cells of S send the 1-cells of $\mathbb{D}^m$ to edge-paths of length at most k^n, the labels we have assigned to the 1-cells of cubical cells for Y are words of length at most k^{n+1} (because H_{S_0} is a k-Lipschitz contraction); secondly, the labellings of these cells of Y are sensible, by construction.

Since we added an m-cell to $K_I^{(m)}$ for each sensible labelling of $\mathbb{D}^m$ of magnitude k^{n+1}, the labelling of the cells in Y determines a natural map $Y \to K_I^{(n+1)}$ that extends h_S, maps (x, N) to x for all $x \in S$, and is constant on $S \times \{-1\}$. Thus we have constructed the desired contraction of S in $K_I^{(n+1)}$. □

3.5. **Algorithmic construction of contractions and translation cells.** We follow the second part of the proof of Proposition 3.1 to prove:

Addendum 3.9. *Given the data described in Theorem 3.3, there is an algorithm that will construct the contractions in Proposition 3.8.*

Proof. Given S, one fixes a positive integer N and tries to attach a label $h(u) \in G$ to each 0-cell in $S \times [-1, N]$, and a word w_ε to the directed edges ε of the domains of the characteristic maps $\phi_E : \mathbb{D}^m \to S\times[-1, N]$ so that

- if $\phi(\varepsilon)$ joins u to v then $w_\varepsilon = h_S(u)^{-1}h_S(v)$ in G,
- the labels w_ε form a sensible labelling of magnitude k^{n+1} on $\mathbb{D}^m$ for each m-cell in $S \times [0, N]$,
- the labelling of each cell mapping to $S \times \{N\}$ coincides with the labelling determining the characteristic map of the corresponding cell of $S \subset \tilde{K}^{(n)}$,
- the cells mapping to $S\times[-1, 0)$ have degenerate labellings, with the 1-cells mapping to $S \times \{-1\}$ all labelled $*$.

In the preceding proposition we proved that for some N such a choice of labels exists, so one can algorithmically run all over all possible choices, picking labels arbitrarily and using the solution to the word problem in G (Corollary 3.2) to check if the choices satisfy the above conditions. (In fact, as in Proposition 3.1, one has an *a priori* bound on N that is an exponential function of the number of cells in S.) □

We now define, inductively, the translation cells.

There is one translation 2-cell for each word $w \in F(\mathcal{A})$ of length between 2 and k^d, that is, for each 1-cell in K_I that is not a vital 1-cell. Each translation 2-cell is of the form shown in Figure 2 with attaching maps determined by the indicated (sensible) labelling.

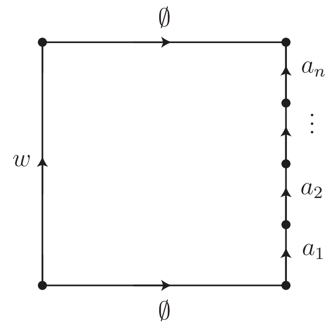

Figure 2. A translation 2-cell. The labels on the edges are shown. There is such a 2-cell for each word $w = a_1a_2 \dots a_n \in F(\mathcal{A})$ of length between 2 and k^d.

Let $n \geq 1$ be an integer no greater than $d + 1$. There is one ***translation cell*** of dimension $n + 1$ for each n-cell of K_I that is not vital. The attaching map of one codimension-1 face is the characteristic map of the given inflated n-cell; write $\mathbb{D}^n = \mathbb{D}^{n-1} \times [0, 1]$ and assume this is the face $\mathbb{D}^{n-1} \times \{0\}$. On the faces of the form $F \times [0,1]$ with $F < \mathbb{D}^{n-1}$ the attaching map is the translation cell corresponding to F (which is well-defined, by induction). We now have the attaching map defined on the boundary of $\mathbb{D}^{n-1} \times \{1\}$, yielding a subcomplex of $K_v^{(n-1)}$ which is contractible in $K_I^{(n)}$ (again these are inductive assumptions). The addendum above yields an explicit contraction, defined as a map from a cellulation of $\partial\mathbb{D}^{n-1} \times [0,1]$ to $K_I^{(n)}$. We identify this last complex with a cylinder joining the boundary of $\mathbb{D}^{n-1} \times \{1\}$ to a concentric $(n-1)$-cube near the centre of $\mathbb{D}^{n-1} \times \{1\}$, and we then complete the description of the attaching map of our translation cell by sending the interior of this $(n-1)$ to the vertex of K_I. Figure 3 depicts the case of a translation 3-cell.

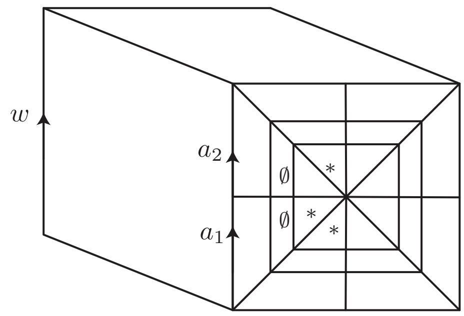

Figure 3. A translation 3-cell. The back face is labelled by a single inflated 2-cell. The side faces are translation 2-cells. The front face is subdivided into conical wedges which are shown in Figure 4

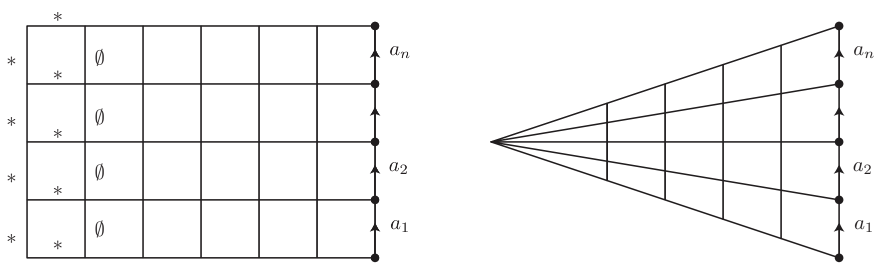

Figure 4. The figure on the left shows a standard homotopy of an edge path. The map from this to the complex K factors through the conical quotient shown on the right.

Remark 3.10. It is important to note that this construction is entirely algorithmic.

To complete the proof of Theorem 3.3 it only remains to check that item (ii) of subsection 3.2 holds.

Lemma 3.11. $K_I^{(n)} \cup T_{n+1}$ *strong deformation retracts to* $K_v^{(n)}$.

Proof. We have constructed a translation $(n+1)$-cell t for each non-vital n-cell e in K_I. The translation cell provides a homotopy pushing e into $K_v^{(n)}$, and by construction the homotopies for distinct cells agree on faces of intersection. □

3.6. **Proof of Theorems B and C.** Theorem B follows immediately from Theorem 3.3 and the fact that the Epstein algorithm calculates a fellow-traveller constant of the automatic group from any presentation of that group (cf. Proposition 1.2). Elementary algebra allows one to compute the (co)homology of the complex K from its cellular chain complex, and this equals the (co)homology of $\pi_1 K$ in dimensions up to d, so Theorem C is an immediate consequence of Theorem B. □

4. Proof of Theorem A

In Section 2 we reduced Theorem A to the problem of enumerating the central extensions of a fixed biautomatic group Q by a given finitely generated abelian group Z. So in the light of Theorem B, the following proposition completes the proof.

Proposition 4.1. *If one has an explicit finite model for the 3-skeleton of a* $K(G,1)$*, then one can irredundantly enumerate the central extensions of* G *with given finitely generated kernel* A.

Proof. Elementary algebra allows one to explicitly calculate cellular 2-cocycles representing the elements of $H^2(G,A)$. Each such cocycle σ is an assignment of elements of A to the (oriented) 2-cells of K. We may assume that K has only one vertex (contracting a maximal tree if necessary), in which case the 2-skeleton of K corresponds to a (finite) presentation for G in the sense that $G = \langle S \mid r_1, \dots, r_m \rangle$, where the $s_i \in S$ are the oriented 1-cells (there is a chosen orientation, so that S is in bijection with the physical 1-cells) and the r_j are the attaching paths of the 2-cells — there is a choice of an oriented starting corner; different choices would lead to r being replaced by a cyclic permutation of itself or its (free) inverse. We write $|r|$ for the oriented 2-cell with boundary label r.

The extension of G by A corresponding to $[\sigma] \in H^2(G,A)$ is the group with presentation

$$\langle S \cup X \mid Y; r_i = a_i, i = 1, \dots, m \text{ and } [x,y] = 1 \; \forall\, x \in X, y \in S \rangle,$$

where $A = \langle X \mid Y \rangle$ and a_i is a word in $X^{\pm 1}$ which equals $\sigma(|r_i|)$ in A. Thus, from the 3-skeleton of the $K(G,1)$ one obtains a collection of representatives for the elements of $H^2(G,A)$, and from that an irredundant enumeration of the central extensions of G by A, in the form of finite presentations.

Note that if we had chosen a different starting corner for r_i (but kept the orientation the same) then in this presentation we would instead of $r_i = a_i$ have $r_i^* = a_i$, where $r_i^* = s_j r_i s_j^{-1}$ (freely) for some s_j. Thus the presentation obtained via this change would differ from the one above by obvious Tietze transformations exploiting the relations $[x, s_j] = 1$. And a change of choice of orientation for $|r_i|$ would simply replace $r_i = a_i$ by $r_i^{-1} = a_i^{-1}$, since $\sigma(|r_i^{-1}|) = -\sigma(|r_i|)$. □

5. Proof of Theorem D

Recall that the translation number of an element g of a finitely generated group is

$$\tau(g) = \lim_{m\to\infty} \frac{1}{m} d(1, g^m),$$

where d is a fixed word metric on the group. The translation functions $g \mapsto \tau(g)$ associated to different word metrics are Lipschitz equivalent, hence the statement 'g has non-zero translation length' is independent of generating set. Also, since $d(1, x^{-1}g^m x)$ and $d(1, g^m)$ differ by at most $2\,d(1,x)$, the number $\tau(g)$ depends only on the conjugacy class of g. And if g and h commute, then $\tau(gh) \leq \tau(g) + \tau(h)$.

Lemma 5.1. *Let G be a finitely generated group in which central elements of infinite order have positive translation numbers and in which the torsion subgroup of the centre is finite (e.g., a biautomatic group). Then $G/Z(G)$ is a group with finite centre. Moreover, if $Z(G)$ is torsion-free, then the centre of $G/Z(G)$ is trivial.*

Proof. Let a be an element of G that maps to a central element in $G/Z(G)$. Then for all $g \in G$ we have $gag^{-1} = za$, where z (which depends on a and g) is central in G. But then $g^n a g^{-n} = z^n a$ for all positive integers n. This implies that the translation number of z is zero, as for all $n \in \mathbb{N}$ we have:

$$\begin{aligned} n\tau(z) &= \tau(z^n) = \tau(z^n a a^{-1}) \\ &\leq \tau(z^n a) + \tau(a^{-1}) \text{ (since } z^n a \text{ and } a^{-1} \text{ commute)} \\ &= \tau(g^n a g^{-n}) + \tau(a) = 2\tau(a). \end{aligned}$$

Thus for all $g \in G$, we have $a^{-1}ga = zg$, with z a torsion element of $Z(G)$. Since the torsion subgroup of $Z(G)$ is finite and G is finitely generated, it follows that there are only finitely many possibilities for the inner automorphism ad(a) $\in \mathrm{Inn}(G) = G/Z(G)$. In other words, $Z(G/Z(G))$ is finite. □

Remark 5.2. The finite presentations of biautomatic groups with finite centre, and the finite presentations of biautomatic groups with torsion-free centre, both form recursively enumerable classes.

In the light of this lemma, and Theorem A, we obtain:

Theorem D. *The isomorphism problem among biautomatic groups is solvable if and only if the isomorphism problem is solvable among biautomatic groups with finite centre.*

Remark 5.3. It also follows from Lemma 5.1 that if one could solve the isomorphism problem for biautomatic groups with trivial centre, then one could solve the isomorphism problem for torsion-free biautomatic groups. A similar reduction pertains for biautomatic groups that are perfect, even in the presence of torsion.

6. Rational structures on abelian groups

In Section 1 we constructed an automatic structure for the centre $Z(\Gamma)$, from this obtained a finite presentation and then used the isomorphism problem for finitely generated abelian groups to identify the isomorphism type of $Z(\Gamma)$. In this section we present some results which derive information directly from the automatic structure.

We remind the reader that a ***rational structure*** for a group G with finite semigroup generators X is a regular language $\mathcal{L} \subset X^*$ that maps bijectively to G under the natural map $X^* \to G$.

Lemma 6.1. *Given a finitely generated abelian group A and a rational structure*[b] $\mathcal{L} \to A$*, one can decide the torsion free rank ρ of A. Specifically, it is the degree of growth of the regular language $\mathcal{L}$.*

Proof. Since words of length at most n map into the ball of radius n in A, the growth of $\mathcal{L}$ is no larger than that of A. So $\mathcal{L}$ is a regular language with polynomial growth of degree at most ρ. In particular it is a union of basic languages of the form $a_0 l_1^* a_1 \cdots l_n^* a_n$, with $n \leq \rho$. As $\mathcal{L} \to A$ is injective, each of the words l_i projects to an infinite order element in A. Therefore, the image of each of the finitely many basic sublanguages is contained in a bounded neighbourhood of a subgroup $\mathbb{Z}^n \leq A$, where $n \leq \rho$. Since this image is the whole of A, at least one of these subgroups must actually have rank $n = \rho$. Thus the degree of polynomial growth of $\mathcal{L}$ is ρ. □

[b] no fellow-traveller property is assumed

It is possible to have the same regular language mapping bijectively to different abelian groups (of the same torsion free rank). For example, let $\mathcal{L} \subseteq \{x^{\pm 1}, y\}^*$ be the language defined by $\mathcal{L} = x^{\pm} * \cup x^{\pm *} y$. Then $\mathcal{L}$ maps bijectively onto $\mathbb{Z} \cong \langle a \mid - \rangle$ via $x^{\pm 1} \mapsto a^{\pm 2}$, $y \mapsto a$, and also onto $\mathbb{Z} \times C_2 \cong \langle b, c \mid [b,c] = 1 = c^2 \rangle$ via $x^{\pm 1} \mapsto b^{\pm 1}$, $y \mapsto c$.

Continuing with the notation of Lemma 6.1, we have an abelian group A with rational structure $\mathcal{L} \to A$ where $\mathcal{L}$ is a union $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_r$ of languages of the form $\mathcal{L}_i = a_0 l_1^* a_1 \cdots l_n^* a_n$.

Lemma 6.2. *Each sublanguage $\mathcal{L}_i$ contains at most one element which projects to a finite order element of A.*

Proof. Suppose that $\gamma \in \mu(\mathcal{L}_i)$ has order p. Let $f = \mu(a_0 a_1 \cdots a_n)$ and $g_i = \mu(l_i)$. If $\gamma = f g_1^{m_1} \cdots g_n^{m_n}$, then $\gamma^p = f^p g_1^{pm_1} \cdots g_n^{pm_n} = 1$. For any $\xi \in \mu(\mathcal{L}_i)$ not equal to γ, we have $\xi = f g_1^{r_1} \cdots g_n^{r_n}$ with $(r_1, \ldots, r_n) \neq (m_1, \ldots, m_n)$. Therefore $\xi^p = f^p g_1^{pr_1} \cdots g_n^{pr_n} = g_1^{p(r_1 - m_1)} \cdots g_n^{p(r_n - m_n)}$ has infinite order. □

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