COMBINGS OF GROUPS AND THE GRAMMAR OF REPARAMETERIZATION

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ABSTRACT. A new construction of combings is used to distinguish between several previously indistinguishable classes of groups associated to the theory of automatic groups and non-positive curvature in group theory. We construct synchronously bounded combings for a class of groups that are neither bicombable nor automatic. The linguistic complexity of these combings is analysed: in many cases the language of words in the combing is an indexed language.

Introduction

Geometric group theory has enjoyed a period of rapid and exciting growth in the decade or so since the emergence of the theory of hyperbolic groups [20] and the theory of automatic groups [15]. However, the subject has been marred by the absence of examples to distinguish between the various classes of groups defined in terms of the geometry and linguistic complexity of the normal forms (combings) which they admit (cf. page 83 of [15] and section 6 of [21]). In particular, there have been no examples to distinguish the class of combable groups from that of bicombable groups, likewise combable groups versus automatic groups, automatic groups versus biautomatic groups, and $\mathcal{A}$-combable groups versus automatic groups, where $\mathcal{A}$ is any full abstract family of languages.

In this article we introduce a new method for constructing combings (see Theorem 3.4). We shall use the combings constructed by this method to distinguish between several previously indistinguishable classes of groups, thus responding to questions raised by Epstein et al. ([15] p.85), Gersten et al. [16], and others (e.g. [29]). In particular we shall prove:

Theorem A. *There exist combable groups that are not bicombable.*

Theorem B. *There exist combable groups whose Dehn functions are cubic.*

Corollary C. *There exist combable groups that are not automatic.*

I should emphasize that although I am using the definition of combability that has now become standard, it is not the definition first used by Epstein and Thurston, which leads to a quadratic isoperimetric inequality (see pages 83-86 of [15]). Thus, given a group Γ with finite generating set Σ, we say that Γ is combable if one can make a choice of words $(\sigma(\gamma) : \gamma \in \Gamma)$ in the letters $\Sigma^{\pm 1}$ such that $\sigma(\gamma) = \gamma$ in Γ and the *fellow-traveller property* holds: when viewed as an edge-path from 1 to γ in the Cayley graph of Γ, $\sigma(\gamma)$ remains uniformly close to $\sigma(\gamma')$ if the vertex γ' is adjacent to γ.

In Section 6 we shall analyse the linguistic complexity of the combings constructed in the proof of the above theorems. In many cases the language of words in the combing is shown to be an *indexed language*. Recall that the full abstract families consisting of regular languages, context-free languages and indexed languages form a hierarchy $\mathrm{Reg} \subset \mathrm{CF} \subset \mathrm{Ind}$ (see Section 5). Regular languages form the basis of automatic group theory (see [15]), and the work of Bridson and Gilman [10] shows that within the class of indexed languages one can construct efficient combings for the fundamental group of arbitrary compact, geometrizable 3-manifolds.

Theorem D. *There exist* Ind-*combable groups that are not* Reg-*combable (i.e. automatic).*

Here is an explicit example of a group that is Ind-combable but neither bicombable nor automatic. (Large classes of examples will be described in Sections 4 and 6 — see in particular 4.2 and 6.4.) This group has a cubic Dehn function and is the fundamental group of a compact aspherical 2-complex. It is obtained by taking two direct products of free groups $F(a_i, b_i) \times F(s_i, t_i)$ and forming their amalgamated free product along the subgroups generated by $\{s_i,\, t_i a_i\}$:

$$\langle a_1, b_1, s, t_1, a_2, b_2, t_2 \mid t_1 a_1 = t_2 a_2,\, [a_i, s] = [a_i, t_i] = [b_i, s] = [b_i, t_i] = 1,\, i = 1, 2 \rangle.$$

One shows that this group is not bicombable by proving that the centralizer of $\{a_1, a_2, b_1, b_2\}$ is not finitely generated.

Bestvina and Brady have outlined a different construction of a combable group that is not automatic. Their example has a quadratic Dehn function.

In the course of our project on subgroups of automatic groups, [3] and [4], Baumslag, Miller, Short and I shared many stimulating conversations about combings of groups, and I thank them for the seeds of thought that undoubtedly found their way through those conversations into this article. I thank Bob Gilman for the insights into the utility of formal grammars that I gained from him in the course of our joint project [10] and [9]. I thank both Lawrence Reeves and the referee for their careful reading and useful comments.

1. Combings

A choice of generating set for a group G is a surjective monoid homomorphism $\mu : \Sigma^* \to G$, where Σ^* is the free monoid on the set Σ. In the following discussion we assume that Σ is finite.

It is convenient to assume that Σ is equipped with an involution, written $s \mapsto s^{-1}$, such that $\mu(s^{-1}) = \mu(s)^{-1}$, and we shall do so without further comment. We write $|w|$ to denote the length of a word $w \in \Sigma^*$.

A *combing* of G is a map $\sigma : G \to \Sigma^*$ such that $\mu \circ \sigma(g) = g$ for all $g \in G$. It is often fruitful to regard $\sigma(g)$ as a choice of discrete path from $1 \in G$ to g: at integer times $t \leq |\sigma(g)|$ this path visits the group element $\mu(\sigma(g)_t)$, where w_t denotes the prefix of length t in a word w.

The combing is said to satisfy the *(synchronous) fellow-traveller property* if the paths to adjacent vertices stay uniformly close together. More precisely, there should exist a constant $K > 0$ such that for all $g, g' \in G$,

$$(1.1) \qquad d(\mu(\sigma(g)_t), \mu(\sigma(g')_t)) \leq K\, d(g, g')$$

for all $t \leq \max\{|\sigma(g)|, |\sigma(g')|\}$, where d is the word metric associated to our choice of generators, i.e. the unique left-invariant metric on G such that $d(1, g)$ is the length of the shortest word in $\mu^{-1}(g)$. If a finitely generated group G admits such a combing, it is said to be *combable*. It is called *bicombable* if it satisfies the following strengthening of the above inequality:

$$(1.2) \qquad d(\mu(s.\sigma(s^{-1}gs')_t), \mu(\sigma(g)_t)) \leq K$$

for all $s, s' \in \Sigma$ and $g \in G$; and in this case σ is called a *bicombing*. A bicombing is said to have the *comparable lengths property* if there is a constant C such that translated combing paths that begin and end a distance at most one apart have lengths that differ by at most C, that is

$$(1.3) \qquad \big| |\sigma(s^{-1}gs')| - |\sigma(g)| \big| \leq C$$

for all $g \in G$ and $s, s' \in \Sigma \cup \{1\}$. A combing is said to have the comparable lengths property if (1.3) holds with $s \equiv 1$.

Let $\mathcal{A}$ be a full AFL (see 5.1). One says that σ is an $\mathcal{A}$-*combing* if it satisfies the fellow-traveller property and the image of σ is a language in the class $\mathcal{A}$; if such a σ exists, one says that G is $\mathcal{A}$-*combable*. $\mathcal{A}$-*bicombings* and $\mathcal{A}$-*bicombable* groups are defined similarly.

In the special case $\mathcal{A} = \text{Reg}$, an $\mathcal{A}$-combing is called an *automatic structure* for G, and an $\mathcal{A}$-bicombing is called a *biautomatic structure* for G (see [15]).

Remark 1.1. (1) It is easy to check that all of the above properties are independent of the chosen finite generating set for the group. In other words, given a finite generating set for a group, if the group admits a certain type of

combing with respect to that set of generators, then it admits such a combing with respect to any other finite set of generators.

(2) Although it is not the focus of this article, for completeness I should mention that there are many situations in which it is natural to allow reparameterizations of paths in the definition of the fellow-traveller property. This relaxation of the fellow-traveller condition results in the (weaker) notion of an *asynchronously* $\mathcal{A}$-*combable* group — see [15], [10] and 6.5(3).

1.1. **Basic Properties.** We remind the reader that the *Dehn function* $\delta(n)$ of a finitely presented group $\Gamma = \langle \Sigma \mid \mathcal{R} \rangle$ measures the complexity of the word problem in Γ. More precisely, it gives the optimal bound on the number of relators $r \in \mathcal{R}$ that one must apply in order to show that a word in the free group $F(\Sigma)$ is null-homotopic, i.e. represents $1 \in \Gamma$. Thus if N is the least integer such that for every null-homotopic word w of length $\leq n$ there is an equality $w = \prod_{i=1}^{M} x_i^{-1} r_i x_i$ in $F(\Sigma)$, with $r_i \in \mathcal{R}^{\pm 1}$ and $M \leq N$, then $\delta(n) = N$. Up to a standard equivalence relation, the Dehn function of a group is independent of the chosen finite presentation.

The following standard result is proved using diagrammatic techniques (see, for example, [15] or [6]).

Lemma 1.2. *If a finitely generated group G admits a combing σ that satisfies the fellow-traveller property then:*

(1) *G is finitely presented;*

(2) *G has a solvable word problem and its Dehn function is bounded above by a constant times $n \, L_\sigma(n)$, where $L_\sigma(n)$ is the length function of σ:*

$$L_\sigma(n) := \max\{|\sigma(g)| \mid d(1,g) \leq n\}.$$

(Note that if σ has the comparable lengths property then $L_\sigma(n) \simeq n$.)

We shall also need the following standard facts.

Lemma 1.3.

(1) *Every (bi)automatic structure has the comparable lengths property.*

(2) *If G_1 and G_2 are biautomatic, then so is $G_1 \times G_2$. More generally, given any full abstract family of languages $\mathcal{A}$, if G_1 and G_2 are $\mathcal{A}$-bicombable, then so is $G_1 \times G_2$.*

Proof. The first assertion is an easy consequence of the interpretation of regular languages as precisely the languages accepted by finite state automata — see [15]. Given bicombings $\sigma^j : G_j \to \Sigma_j^*$, one obtains a bicombing of $G_1 \times G_2$ by defining $\sigma(g_1, g_2) = \sigma^1(g_1)\sigma^2(g_2) \in (\Sigma_1 \coprod \Sigma_2)^*$. Lemma 5.1 ensures that if $\sigma^j(G_j) \in \mathcal{A}$ for $j = 1, 2$, then $\sigma(G_1 \times G_2) \in \mathcal{A}$. $\square$

Given a combing $\sigma : G \to \Sigma^*$, a subgroup $H \subset G$ is said to be σ-*quasi-convex* if there is a constant $C > 0$ such that $d(\mu(\sigma(h)_t), H) \leq C$ for all $h \in H$ and all $t \leq |\sigma(h)|$.

The following lemma is taken from [10]. It is based on ideas of M. Gromov, S.M. Gersten and H. Short. Our first use of this lemma will be in the (linguistically trivial) case where $\mathcal{A}$ is the family of all subsets of finitely generated free monoids (in which case an $\mathcal{A}$-combing is simply a combing with the fellow-traveller property).

Lemma 1.4. *Let $\mathcal{A}$ be a full abstract family of languages and let $\sigma : G \to \Sigma^*$ be an $\mathcal{A}$-combing.*

(1) If $H \subset G$ is σ-quasi-convex, then it is $\mathcal{A}$-combable; in particular it is finitely presented. If σ is a bicombing, then H is $\mathcal{A}$-bicombable. Moreover, if $\mathcal{A} = \mathrm{Reg}$ then $H \hookrightarrow G$ is a quasi-isometric embedding with respect to any choice of word metrics.

(2) If $H_1, H_2 \subset G$ are σ-quasi-convex, then so is $H_1 \cap H_2$.

(3) If σ is an $\mathcal{A}$-bicombing, then the centralizer of every $g \in G$ is σ-quasi-convex.

1.2. **Subgroups that cannot be quasi-convex.** It follows from part (1) of the previous lemma that if a subgroup $H \subset G$ is not itself $\mathcal{A}$-combable then it cannot be σ-quasi-convex with respect to any $\mathcal{A}$-combing of G. Also, if G is automatic and $H \subset G$ is not quasi-isometrically embedded, then H cannot be quasi-convex with respect to any automatic structure. Parts (2) and (3) of the lemma can be used to derive more subtle obstructions to quasi-convexity, as we shall now explain.

Proposition 1.5. *Let G and Q be finitely generated groups. Let $\phi : Q \to \mathrm{Aut}(G)$ be a homomorphism whose kernel is not bicombable. Then:*

(1) $Q \subset G \rtimes_\phi Q$ is not quasi-convex with respect to any bicombing of $G \rtimes_\phi Q$.

(2) If ϕ factorizes as $Q \to G \to \mathrm{Inn}(G) \subset \mathrm{Aut}(G)$, then $G \rtimes_\phi Q \cong G \times Q$, and in this case if G and Q are $\mathcal{A}$-bicombable then $G \rtimes_\phi Q$ is also.

Proof. The centralizer of G in $G \rtimes_\phi Q$ is the intersection of the centralizers of the generators of G, which by parts (2) and (3) of the above lemma is quasi-convex with respect to any bicombing of $G \rtimes_\phi Q$. The intersection of Q with this centralizer is the kernel of ϕ, and by hypothesis this is not bicombable. Therefore, by part (2) of the above lemma, Q cannot be quasi-convex with respect to any bicombing of $G \rtimes_\phi Q$.

Assume that ϕ factorizes as is (2), through $\phi_0 : Q \to G$. It is easy to check[1] that $G \rtimes_\phi Q$ is then the direct product of $G = G \times \{1\}$ and $\overline{Q} = \{\phi_0(q)^{-1}q \mid q \in Q\} \cong Q$, hence is $\mathcal{A}$-bicombable. $\qquad\square$

Remarks 1.6. (1) Similar reasoning shows that if the kernel of ϕ is not quasi-isometrically embedded in Q, then Q cannot be quasi-convex with respect to any biautomatic structure on $G \rtimes_\phi Q$.

(2) Note that the subgroup $Q \subset G \rtimes Q$ is a retract and hence is quasi-isometrically embedded.

Example 1.7. Let F be a finitely generated free group of rank n. Let G be a bicombable group and let $H \subset \mathrm{Aut}(G)$ be an infinite group that is not free of rank n. Let $\phi : F \to H$ be a surjection. Then, by part (1) of the proposition, F is not quasi-convex with respect to any bicombing of $G \rtimes_\phi F$, because the kernel of ϕ is not finitely generated.

In particular, if G has no centre (so $G = \mathrm{Inn}(G)$), taking $H \subset \mathrm{Inn}(G)$ and appealing to 1.6(2), we obtain examples of quasi-isometrically embedded free subgroups of $G \times F \cong G \rtimes_\phi F$ that cannot be quasi-convex with respect to any bicombing (cf. 2.5).

2. Groups that are not bicombable

We shall use the properties of centralizers described in Lemma 1.4 to show that certain groups are not bicombable. The following general observations are useful in this regard.

Notation: Given a group G and a subgroup H, we write $G\dot{*}_H$ to denote the trivial HNN extension $(G, t \mid [t, h] = 1 \,\forall h \in H)$. We write $C_G(S)$ to denote the centralizer of $S \subset G$, and $Z(G)$ to denote the centre of G.

Lemma 2.1. *For $i = 1, 2$, let $\phi_i : Q \to \mathrm{Aut}(N_i)$ be a homomorphism. Let $\phi : Q \to \mathrm{Aut}(N_1 * N_2)$ be the induced homomorphism (written $q \mapsto \phi_q$), and let $P = (N_1 * N_2) \rtimes_\phi Q$. If the groups N_1 and N_2 are both non-trivial, then $C_P(N_1 * N_2) = \ker \phi = \ker \phi_1 \cap \ker \phi_2$.*

Proof. Each $\gamma \in P$ can be written uniquely in the form $\gamma = wq$ with $q \in Q$ and $w \in N_1 * N_2$. If $\gamma \in C_P(N_1 * N_2)$ then for $i = 1, 2$ and all $n \in N_i$ we have $n = \gamma n \gamma^{-1} = w\phi_q(n)w^{-1}$, hence $w^{-1}nw = \phi_q(n) \in N_i$. But $w^{-1}nw \in N_i$ implies $w \in N_i$. Hence $w \in N_1 \cap N_2 = \{1\}$. Therefore $\phi_q(n) = n$ for all $n \in N_1 \cup N_2$, and $\gamma = q \in \ker \phi$. $\qquad\square$

Lemma 2.2. *In any HNN extension of the form $\Gamma = (N \rtimes_\psi H, t \mid [t, h] = 1 \,\forall h \in H)$, one has $C_\Gamma(N) \cap C_\Gamma(t) = \ker \psi$.*

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[1]By definition, in $G \rtimes_\phi Q$ we have $qgq^{-1} = \phi_0(q)g\phi_0(q)^{-1}$ for all $g \in G$ and $q \in Q$.

Proof. It follows immediately from Britton's Lemma that $C_\Gamma(t) = H \times \langle t \rangle$, and that $t^i N t^{-i} \cap N = \{1\}$ if $i \neq 0$. $\square$

Proposition 2.3. *For $i = 1, 2$, let $\phi_i : Q \to \mathrm{Aut}(G_i)$ be a homomorphism, with G_i non-trivial. Let $\Gamma_i = G_i \rtimes_{\phi_i} Q$.*

(1) If $\ker \phi_1 \cap \ker \phi_2$ is not bicombable (e.g. if it is not finitely presented), then $\Gamma_1 *_Q \Gamma_2$ is not bicombable.

(2) If $\ker \phi_i$ is not bicombable, then the trivial HNN extension $\Gamma_i \dot{*}_Q$ is not bicombable.

Proof. $\Gamma_1 *_Q \Gamma_2 = (G_1 * G_2) \rtimes_\phi Q$ where $\phi : Q \to \mathrm{Aut}(G_1 * G_2)$ is the map induced by ϕ_1 and ϕ_2. Thus $C(G_1 * G_2)$ is not bicombable, by Lemma 2.1, and hence $\Gamma_1 *_Q \Gamma_2$ is not bicombable, by Lemma 1.4.

The second part of the proposition follows from Lemmas 2.2 and 1.4. $\square$

Corollary 2.4. *Let G and Q be groups and let $g \in G$ be such that no power of g is central. Let $h : Q \to \mathbb{Z}$ be a homomorphism whose kernel is not bicombable. Let $\hat{Q} \subset G \times Q$ be the subgroup $\{\hat{q} = (g^{-h(q)}, q) \mid q \in Q\}$. (Note $\hat{Q} \cong Q$.) Then,*

(1) the double $(G \times Q) *_{\hat{Q}} (G \times Q)$ is not bicombable, and

(2) the trivial HNN extension $(G \times Q) \dot{*}_{\hat{Q}}$ is not bicombable.

Proof. As in the proof of Proposition 1.5, we note that $G \times Q \cong G \rtimes_\phi \hat{Q}$, where $\phi : \hat{Q} \to \mathrm{Aut}(G)$ takes $\hat{q}$ to the inner automorphism by $g^{-h(q)}$. Since g^n is not central in G if $n \neq 0$, we have $\ker \phi = \ker h$, and we are in the setting of the proposition, with Q replaced by $\hat{Q}$, with $G = G_1 = G_2$, and with $\phi = \phi_1 = \phi_2$. $\square$

In order to animate the preceding result with examples, we seek biautomatic groups that admit surjections $h : Q \to \mathbb{Z}$ whose kernels are not bicombable. Fortunately, we do not have to look far. For example, if Q is a finitely generated non-abelian free group, or surface group, then the kernel of an epimorphism $h : Q \to \mathbb{Z}$ will never be finitely generated. And if Q is a direct product of finitely many such free and surface groups, then the kernel of h will not be of type FP_∞ (and hence will not be bicombable). Other interesting examples come from the study of Artin groups [5] and the Rips construction [28].

In the case where Q is free, as a special case of the preceding corollary we have:

Corollary 2.5. *If F is a non-abelian free group and G is a group that contains an element g no power of which is central, then $\Gamma = G \times F$ contains a subgroup $\hat{F} \cong F$ with $\Gamma = G \rtimes \hat{F}$ such that neither $\Gamma *_{\hat{F}} \Gamma$ nor $\Gamma \dot{*}_{\hat{F}}$ is bicombable.*

Example 2.6. The presentation displayed in the Introduction describes the group obtained by applying Corollary 2.4(1) to the free groups $G = F(a,b)$ and $Q = F(s,t)$, taking $g = a$ and $h(s) = 0, h(t) = 1$.

3. Combining combings and dealing with laggards

We shall be considering groups of the form $\Gamma = (N_1 * N_2) \rtimes Q$ and combings of the form

$$\sigma(\gamma) = \sigma^1(q)\sigma^1(n_{1,1})\sigma^2(n_{2,1})\cdots\sigma^1(n_{1,m})\sigma^2(n_{2,m}),$$

where $\gamma = qn_{1,1}n_{2,1}\cdots n_{1,m}n_{2,m}$ with $q \in Q$ and $n_{i,j} \in N_i$, and where σ^i is a given combing on $N_i \rtimes Q$ (see Theorem 3.4). When comparing combing lines to adjacent vertices γ and $\gamma' = \gamma s$ in Γ, where $s \in Q$ is a generator, we are faced with the picture shown in figure 1.
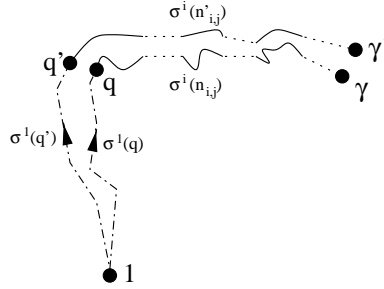

Figure 1: A combing that may suffer from laggards.

If we assume that σ^1 and σ^2 are bicombings, then the basic subpaths of $\sigma(\gamma)$ and $\sigma(\gamma')$ labelled $\sigma^i(n_{i,j})$ and $\sigma^i(n'_{i,j})$ synchronously fellow-travel — condition (1.2). However, the lengths of this pair of basic subpaths need not be the same, and this can result in "time-lags" that may accumulate as a result of concatenation, causing σ_γ and $\sigma_{\gamma'}$ to be only *asynchronous* fellow-travellers.

One can solve this time-lag problem by slowing the paths concerned in a controlled manner. Since the time-lag problem and its solution are essentially metric phenomena, we shall phrase them in the language of metric spaces. In order to do so, we need some definitions that describe those paths in arbitrary metric spaces that behave like the paths in discrete groups defined by words in a finite generating set.

By a *path* in a metric space X we shall mean a (not necessarily continuous) map $p : [0, T_p] \to X$. It is convenient to define $p(t) = p(T_p)$ for $t \geq T_p$ and to write $p(\infty)$ for the terminus $p(T_p)$. The *synchronous distance* between two paths can then be defined as $D(p_1, p_2) := \sup_t d(p_1(t), p_2(t))$.

Let $k > 0$ be a constant. We say that p has *speed* $\leq k$ if $d(p(t), p(t')) \leq k|t - t'|$ for all t, t'. Given a constant $\mu \geq 1$ we write $\frac{1}{\mu}p$ to denote the path $[0, \mu T_p] \to X$ defined by $\frac{1}{\mu}p(t) := p(t/\mu)$. In particular, $T_{\frac{1}{\mu}p} = \mu T_p$.

The concatenation of two paths $p_1 : [0, T_{p_1}] \to X$ and $p_2 : [0, T_{p_2}] \to X$ is defined if $p_1(\infty) = p_2(0)$; it is the path $p_1 \cdot p_2 = [0, T_{p_1} + T_{p_2}] \to X$ defined by $p_1 \cdot p_2(t) = p_1(t)$ if $t \le T_{p_1}$ and $p_1 \cdot p_2(t) := p_2(t - T_{p_1})$ if $t \ge T_{p_1}$.

Lemma 3.1. *Fix a sequence of positive numbers (k_i) such that $1 + \sum_{j=1}^{i} k_j \le k_{i+1}$ for all $i \ge 1$. Let X be a metric space and consider two finite sequences of paths $(p_0, \ldots, p_n)$ and $(p'_0, \ldots, p'_n)$, all of speed ≤ 1, where $p_{i-1}(\infty) = p_i(0)$ and $p'_{i-1}(\infty) = p'_i(0)$ for $i = 1, \ldots, n$. Suppose that there exist constants $K > 0$ and $C \ge 2$ such that $D(p_i, p'_i) \le K$ and $|T_{p_i} - T_{p'_i}| \le C$ for $i = 0, \ldots, n$. Let $\pi = p_0 \cdot \frac{1}{k_1} p_1 \cdot \cdots \cdot \frac{1}{k_n} p_n$ and let $\pi' = p'_0 \cdot \frac{1}{k_1} p'_1 \cdot \cdots \cdot \frac{1}{k_n} p'_n$. Then*

$$D(\pi, \pi') \le K + C.$$

Proof. For $i = 0, \ldots, n$ the path π reaches the terminal point of its initial segment $P_i := p_0 \cdot \cdots \cdot \frac{1}{k_i} p_i$ at time $T(i) := T_{p_0} + k_1 T_{p_1} + \cdots + k_i T_{p_i}$, whereas π' reaches the terminal point of its initial segment $P'_i := p'_0 \cdot \cdots \cdot \frac{1}{k_i} p'_i$ at time $T(i)' := T_{p'_0} + k_1 T_{p'_1} + \cdots + k_i T_{p'_i}$. From the definitions of C and (k_i) we have

$$\delta_{i+1} := |T(i) - T(i)'| \le C(1 + k_1 + \cdots + k_i) < C\, k_{i+1}.$$

The idea of the proof is as follows. Suppose π reached the terminus of p_{i-1} before π' reached the terminus of p'_{i-1} and suppose that we reparameterized π so that it then waited until π' caught-up before embarking on p_i, at the same time as π' embarked on p'_i. If we adjusted the parameterizations of π and π' in this manner at the termini of all of the subpaths p_j and p'_j, then it would be clear from the definitions that π and π' (thus parameterized) would remain K-close. However, we are interested in the *synchronous* fellow-traveller property, not the asynchronous one, so we cannot allow π to wait for π', or vice versa. It is to circumvent this problem that we have introduced the weighting factors k_i: the inequality displayed above ensures that the difference in time between the moment when π embarks on p_i and the moment when π' embarks on p'_i is less than $C\, k_i$, and because we have slowed the speed of p_i by a factor of k_i, the subpath that π traverses in the interval before π' embarks on p'_i will have length less than C.

To obtain a precise estimate, we define $\tau_i = \max\{T(i-1), T(i-1)'\}$ and consider $d(\pi(t), \pi'(t))$ with $t \in [\tau_i, \tau_{i+1}]$. We may assume without loss of generality that $\delta_i = T(i-1)' - T(i-1) \ge 0$. Let $\bar{t} := \frac{1}{k_i}(t - T(i-1)')$ and note that $\pi'(t) = p'_i(\bar{t})$ for $t \in [\tau_i, T(i)']$ and $\pi(t - \delta_i) = p_i(\bar{t})$ for $t \in [\tau_i, T(i) + \delta_i]$. There are three cases to consider.

Case 1, where $t \le \min\{T(i), T(i)'\}$: In this case $\pi'(t) = p'_i(\bar{t})$ and $\pi(t - \delta_i) = p_i(\bar{t})$, so

$$d(\pi(t), \pi'(t)) \le d(\pi(t), \pi(t - \delta_i)) + d(p_i(\bar{t}), p'_i(\bar{t})) \le C + K,$$

where the estimate $d(\pi(t), \pi(t - \delta_i)) \leq C$ is obtained by noting that the path $s \mapsto \pi(t - \delta_i + s)$, with $s \in [0, \delta_i]$, has speed at most $1/k_i$ and hence has length at most δ_i/k_i, which is less than C.

Case 2, where $\tau_{i+1} = T(i)'$ and $t \in [T(i), T(i)']$: If $t \leq T(i) + \delta_i$ then the estimate from Case 1 still pertains. If $t \geq T(i) + \delta_i$, then $\bar{t} \geq T_{p_i}$ and hence $d(p_i(\infty), p'_i(\bar{t})) \leq K$. Therefore, noting that $\pi'(t) = p'_i(\bar{t})$ and $p_i(\infty) = \pi(T(i))$, we have

$$d(\pi(t), \pi'(t)) \leq d(\pi(t), \pi(T(i)) + d(p_i(\infty), p'_i(\bar{t})) \leq C + K,$$

where the estimate $d(\pi(t), \pi(T(i)) \leq C$ is obtained by noting that the path $s \mapsto \pi(T(i) + s)$, with $s \in [0, t - T(i)] \subset [0, T(i)' - T(i)]$, has speed bounded above by $1/k_{i+1}$ and hence length at most δ_{i+1}/k_{i+1}, which is less than C.

Case 3, where $\tau_{i+1} = T(i)$ and $t \in [T(i)', T(i)]$: In this case $\pi(t) = p_i(\tilde{t})$ where $\tilde{t} \geq T_{p'_i}$ and hence $d(p_i(\tilde{t}), p'_i(\infty)) \leq K$. Noting that $p'_i(\infty) = \pi'(T(i)')$, we have

$$d(\pi(t), \pi'(t)) \leq d(p_i(\tilde{t}), p'_i(\infty)) + d(\pi'(T(i)'), \pi'(t)) \leq K + C,$$

where the estimate $d(\pi'(T(i)'), \pi'(t)) \leq C$ is obtained by noting that the path $s \mapsto \pi'(T(i)' + s)$, with $s \in [0, t - T(i)']$, has speed bounded above by $1/k_{i+1}$ and hence length at most δ_{i+1}/k_{i+1}, which is less than C. $\square$

Remark 3.2. In order to enhance the clarify of the exposition, we placed a rather simple condition on the scaling factors (k_i) in the above lemma; the reader should have no difficulty in seeing that the condition can easily be weakened in a number of ways. For example, one can weaken the given condition by requiring only that there exist a constant $m > 0$ such that $1 + \sum_{j=1}^{i} k_j \leq m \, k_{i+1}$, and one can weaken it further by requiring only that this inequality hold for sufficiently large values of i. In each case the conclusion of the lemma has to be weakened slightly but its essence remains intact, namely $D(\pi, \pi')$ is uniformly bounded. In each case, the proof requires only the most minor of modifications.

We explained earlier how one can regard words in the generators Σ of a finitely generated group as discrete paths in the group. If the group is equipped with the word metric associated to Σ then these paths have speed ≤ 1. The following device allows one to follow the preceding reparameterization arguments in this algebraic setting.

Definition 3.3. Let $k > 0$ be an integer. Given a finite set Σ, a symbol $o \notin \Sigma$ and a word $w = s_1 \ldots s_n \in \Sigma^*$, we define the word $\frac{1}{k}w \in (\Sigma \cup \{o\})^*$ to be

$$\frac{1}{k}w = s_1 o \ldots o s_2 o \ldots o s_3 o \ldots o s_n o \ldots o,$$

where there are $(k - 1)$ letters o following each letter s_i.

Note that the word metric d on Γ associated to the finite generating set Σ is the same as that associated to $\Sigma \cup \{o\}$ if we define o to represent $1 \in \Gamma$. Moreover, the path in the metric space (Γ, d) defined by the word $\frac{1}{k}w$ is precisely $\frac{1}{k}\hat{w}$ (in the metric space notation defined prior to Lemma 3.1), where $\hat{w}$ is the path in Γ defined by w.

Theorem 3.4. *If the groups $\Gamma_1 = N_1 \rtimes Q$ and $\Gamma_2 = N_2 \rtimes Q$ admit bicombings that have the comparable lengths property, then $\Gamma = (N_1 * N_2) \rtimes Q = \Gamma_1 *_Q \Gamma_2$ is combable.*

Proof. Fix a finite generating set $\mathcal{B}$ for Q and extend this to finite generating sets $\Sigma_1 = \Sigma_1^{-1}$ for Γ_1 and $\Sigma_2 = \Sigma_2^{-1}$ for Γ_2. We may assume that no $s \in \Sigma_i \smallsetminus \mathcal{B}$ evaluates to an element of Q. For $i = 1, 2$, choose bicombings $\sigma^i : \Gamma_i \to \Sigma_i^*$ and fix positive integers C and K so that conditions (1.2) and (1.3) hold with $\sigma = \sigma^i$. Note in particular that

$$D(s.\sigma^i(s^{-1}gs), \sigma^i(g)) \leq K$$

for all $s \in \mathcal{B}$ and $g \in \Gamma_i$, for $i = 1, 2$.

Using the semi-direct and free product structures, each element $\gamma \in \Gamma$ can be written uniquely in the form

$$\gamma = qn_{1,1}n_{2,1} \ldots n_{1,m}n_{2,m}$$

where $q \in Q$ and where the elements $n_{i,j} \in \Gamma_i$ are non-trivial except possibly for $n_{1,1}$ and $n_{2,m}$. We define a combing $\sigma : \Gamma \to (\Sigma_1 \cup \Sigma_2)^*$ by:

$$\sigma(\gamma) = \sigma^1(q) \cdot \frac{1}{k}\sigma^1(n_{1,1}) \cdot \frac{1}{k^2}\sigma^2(n_{2,1}) \cdots \frac{1}{k^{2m-1}}\sigma^1(n_{1,m}) \cdot \frac{1}{k^{2m}}\sigma^2(n_{2,m}),$$

where $k \geq 2$ is an integer.

We must show that σ satisfies the fellow-traveller property (1.1). Viewing words as paths from $1 \in \Gamma$ in the usual way, it suffices to show that $D(\sigma(\gamma s), \sigma(\gamma)) \leq K + C$ for all $\gamma \in \Gamma$ and $s \in \Sigma_1 \cup \Sigma_2$.

Case 1: $s \in \mathcal{B}$. In this case

$$\gamma s = q \left(\prod_{(i,j)=(1,1),\ldots,(2,m)} n_{i,j} \right) s = (qs) \prod_{(i,j)=(1,1),\ldots,(2,m)} s^{-1}n_{i,j}s.$$

Therefore

$$\sigma(\gamma s) = \sigma^1(qs) \cdot \frac{1}{k}\sigma^1(s^{-1}n_{1,1}s) \cdot \frac{1}{k^2}\sigma^2(s^{-1}n_{2,1}s) \cdots$$
$$\cdots \frac{1}{k^{2m-1}}\sigma^1(s^{-1}n_{1,m}s)\frac{1}{k^{2m}}\sigma^2(s^{-1}n_{2,m}s).$$

If we write $p_{i,j}$ for the subpath of $\sigma(\gamma)$ corresponding to $\sigma^i(n_{i,j})$, and $p'_{i,j}$ for the subpath of $\sigma(\gamma s)$ corresponding to $\sigma^i(s^{-1}n_{i,j}s)$, and $P(i, j)$ and $P(i, j)'$ for

the prefixes before these subwords, then as elements of Γ we have the equality $P(i,j)' = P(i,j)\,s$, and therefore

$$D(p_{i,j}, p'_{i,j}) = D(\sigma^i(n_{i,j}), s.\sigma^i(s^{-1}n_{i,j}s)) \leq K$$

(by the first inequality in this proof). We also have $D(\sigma^1(q), \sigma^1(qs)) \leq K$. Thus we are in precisely the setting of Lemma 3.1, and hence $D(\sigma(\gamma s), \sigma(\gamma)) \leq K + C$.

Case 2: $s \in \Sigma_1 \setminus \mathcal{B}$. If $n_{2,m} \neq 1$, then $\sigma(\gamma s)$ is obtained from $\sigma(\gamma)$ simply by appending $\frac{1}{k^{2m+1}}s$, and thus $D(\sigma(\gamma s), \sigma(\gamma)) = 1$. If $n_{2,m} = 1$, then $\sigma(\gamma s)$ is obtained from $\sigma(\gamma)$ by replacing the terminal segment $\frac{1}{k^{2m-1}}\sigma^1(n_{1,m})$ with $\frac{1}{k^{2m-1}}\sigma^1(n_{1,m}s)$, and hence $D(\sigma(\gamma s), \sigma(\gamma)) \leq K$.

Case 3: $s \in \Sigma_2 \setminus \mathcal{B}$. Entirely similar to case 2. $\square$

Remarks 3.5. (1) In the above theorem there is no assumption that either Q or N_i is quasi-convex. Indeed there are many interesting examples where the N_i are not even finitely generated.

(2) In general $\Gamma_1 *_Q \Gamma_2$ will not be bicombable (see below).

(3) In the above construction one could replace $\sigma|_Q = \sigma^1|_Q$ by any combing of Q that has the fellow-traveller and comparable lengths properties — the proof would work without alteration. Thus the combing on Q need not be related to σ^1 or σ^2 in any way.

(4) There is a good deal of flexibility in how one parameterizes the combing lines in the above proof: for convenience we used scaling factors k^i, but our appeal to Lemma 3.1 would have remained valid is we had used any sequence (k_i) for which there is a constant $m > 0$ such that $1 + \sum_{j=1}^{i} k_j \leq mk_{i+1}$ (cf. Remark 3.2). In particular, given an integer $k \geq 2$, we could have taken $k_i = k^i + 1$. This observation will be useful in Section 6.

In the light of Lemma 1.3(1), as special cases of the above theorem we have:

Corollary 3.6.   *(1) Let Q be a finitely generated group. If $\Gamma_1 = N_1 \rtimes Q$ and $\Gamma_2 = N_2 \rtimes Q$ are biautomatic, then $\Gamma_1 *_Q \Gamma_2$ is combable.*

 (2) If Q and $G = N \rtimes Q$ are biautomatic, then the trivial HNN extension $G\,{\dot{*}}_Q$ is combable.

Proof. For (2), note that $G\,{\dot{*}}_Q = (N \rtimes Q) *_Q (\mathbb{Z} \times Q)$. $\square$

4. Proof of Theorems A and B

By combining 3.4 and 2.3 we obtain the main technical result behind Theorems A and B:

Main Lemma. *If the groups $\Gamma_1 = N_1 \rtimes_{\phi_1} Q$ and $\Gamma_2 = N_2 \rtimes_{\phi_2} Q$ admit bicombings that have the comparable lengths property, then $\Gamma = \Gamma_1 *_Q \Gamma_2$ is combable. But if $\ker \phi_1 \cap \ker \phi_2$ is not bicombable, then neither is Γ.*

We claim that there are many groups that satisfy the hypotheses of both parts of this lemma, and hence Theorem A follows from it. In order to obtain a large class of examples, we combine Corollaries 2.5 and 3.6 and refine them in the following manner:

Theorem 4.1. *Let G be a biautomatic group and let F be a finitely generated non-abelian free group. If G modulo its centre is infinite, then $\Gamma_0 := G \times F$ contains a subgroup $\hat{F} \cong F$ with $\Gamma_0 = G \rtimes \hat{F}$ such that:*
(1) $\Gamma_0 *_{\hat{F}} \Gamma_0$ and $\Gamma_0 \dot{*}_{\hat{F}}$ are combable, but
(2) neither $\Gamma_0 *_{\hat{F}} \Gamma_0$ nor $\Gamma_0 \dot{*}_{\hat{F}}$ is bicombable.

Proof. If we assume that G contains an element g no power of which is central, then the theorem follows immediately from 2.5 and 3.6. If G does not contain such an element g, then $G/Z(G)$ would be an infinite torsion group. But if a biautomatic group is infinite then it contains an element of infinite order [18], and Mosher proves that $G/Z(G)$ is biautomatic [23]. $\square$

We recalled the definition of the Dehn function of a finite presentation before stating Lemma 1.2.

Proof of Theorem B. Let F be a finitely generated non-abelian free group. By taking $G = F$ in the preceding theorem we obtain combable groups $(F \times F) \dot{*}_{\hat{F}}$ and $(F \times F) *_{\hat{F}} (F \times F)$. The Dehn functions of both of these groups are cubic — this is proved on page 504 of [12] by a direct analysis of van Kampen diagrams. Moreover, a simple adaptation of the arguments presented there shows that all of the groups described in Theorem 4.1 have cubic Dehn functions. $\square$

Automatic groups satisfy a quadratic isoperimetric inequality (see Lemma 1.2), so the preceding proof yields:

Corollary 4.2. *The combable groups described in Theorem 4.1 are not automatic.*

Remark 4.3. It is still not known whether all automatic groups are biautomatic. Nor is it known whether all groups that admit a synchronous combing by quasi-geodesics are bicombable.

5. Full abstract families of languages

In the next section we shall analyse the grammatical complexity of the combings constructed in Theorem 4.1. In preparation for this, in this section we

recall some basic facts about formal languages, i.e. subsets of finitely generated free monoids. Our discussion follows that of [10]. We refer the reader to Hopcroft and Ullman [22] for an introduction to formal language theory, and to Epstein *et al.* [15] for a thorough account of how the study of such languages came to play an important role in group theory and geometry/topology.

Given a set (alphabet) Σ, we write Σ^* to denote the free monoid over Σ. As in Section 1, we shall implicitly assume that Σ is finite. The identity element of Σ^* is the empty word ϵ, and multiplication of words is by concatenation. Given $L \subseteq \Sigma^*$, we let L^* denote the submonoid of Σ^* generated by L. If $L, M \subseteq \Sigma^*$, then $LM = \{ w \mid w = xy, x \in L, y \in M \}$ is the product of L and M. If $L \subseteq \Sigma^*$, then Σ is called an alphabet for L, and L is called a language over Σ.

We will use regular, context–free, and indexed languages. These classes of languages form an increasing hierarchy. We consider these classes simultaneously as instances of a *full abstract family of languages*. A full AFL is a class of languages which contains a non-empty language and is closed under the operations listed in part (1) of the following lemma. See [22] Chapter 11 for more details.

Lemma 5.1. *Let $\mathcal{A}$ be a full AFL.*

(1) *$\mathcal{A}$ is closed under homomorphism (i.e., image under homomorphism of the ambient free monoid to another finitely generated free monoid), inverse homomorphism (likewise), intersection with regular sets, union, product, and generation of submonoid.*

(2) *$\mathcal{A}$ contains all regular languages.*

(3) *If L_1 and L_2 are both in $\mathcal{A}$, then the language $L_1 \bullet L_2$ consisting of ϵ and all words $w = u_1 \ldots u_n$ with successive u_i chosen alternately from L_1 and L_2 is also in the $\mathcal{A}$. (The notation $L_1 \bullet L_2$ is not standard, but we shall use it consistently.)*

(4) *The classes of regular, context–free, and indexed languages are each a full AFL.*

Proof. Part (1) is true by definition; for (2) see [22] and for (4) see [22] and [1]. Finally the language $L_1 \bullet L_2$ in (3) can be constructed from L_1 and L_2 by the operations given in (i): it is $(L_1 L_2)^* \cup (L_1 L_2)^* L_1 \cup (L_2 L_1)^* \cup (L_2 L_1)^* L_2$. $\square$

Grammars.

Regular, context–free and indexed languages are generated by corresponding types of *grammars* (see [22]). Briefly, an indexed grammar has disjoint finite sets N, Σ, F of nonterminals, terminals, and indices respectively. The language (over Σ) generated by the grammar is obtained by beginning with a designated

start symbol $S \in N$ and performing substitutions. The allowed substitutions are determined by a finite set of *productions*, and the grammar is completely described by N, Σ, F, S, and the set of productions. Productions of three types are allowed: (1) $A \to \alpha$ (2) $A \to Bf$ (3) $Af \to \alpha$, where A and B are nonterminals, $\alpha \in (N \cup \Sigma)^*$, and f is an index. Roughly speaking (we shall explain more precisely in a moment) the corresponding substitutions consist of replacing the left-hand side of a production by the right-hand side. This procedure generates words in $(N \cup \Sigma \cup F)^*$. The words in Σ^* which can be derived in this way form the language generated by the grammar.

Words in $(N \cup \Sigma \cup F)^*$ are called sentential forms. To apply a production of type (1) to a sentential form β one finds an occurrence of A in β and substitutes α for A. The string of indices δ (possibly the empty string) following the occurrence of A in β is repeated after each nonterminal in α (there is never a string of indices following a terminal). In case (2) one simply substitutes Bf for A. In case (3) one finds an occurrence of Af followed by a string of indices δ, substitutes α for Af, and repeats δ after each nonterminal in α.

If a sentential form β can be obtained from a sentential form α by a finite sequence of productions, then one writes $\alpha \xrightarrow{*} \beta$.

A context–free grammar is the same as an indexed grammar except that there are no indices, and a regular grammar is a context–free grammar in which all productions have a right-hand side consisting of an element of $\Sigma \cup \{\epsilon\}$ possibly followed by a single nonterminal.

Example 5.2. An example of a context-free language that is not regular is $\{a^n b^n \mid n \in \mathbb{N}\}$ and an example of an indexed language that is not context-free is $\{a^n b^{n^2} \mid n \in \mathbb{N}\}$ (see [10], [19] and the next section for more interesting examples). An example of a context sensitive language that is not indexed is $\{(ab^n)^n \mid n \in \mathbb{N}\}$.

Remark 5.3. Indexed languages enjoy the following "context-freeness": if α and β are sentential forms for an indexed grammar and α does not end with an index, then $\{\gamma \mid \alpha\beta \xrightarrow{*} \gamma\} = \{\gamma_1\gamma_2 \mid \alpha \xrightarrow{*} \gamma_1, \beta \xrightarrow{*} \gamma_2\}$. Moreover, one may factorize $\alpha\beta \xrightarrow{*} \gamma_1\gamma_2$ as either $\alpha\beta \xrightarrow{*} \gamma_1\beta \xrightarrow{*} \gamma_1\gamma_2$ or $\alpha\beta \xrightarrow{*} \alpha\gamma_2 \xrightarrow{*} \gamma_1\gamma_2$.

A celebrated theorem of Muller and Schupp [24] (which relies on Dunwoody's Accessibility Theorem [14]) states that, given a finite generating set for a group G, the set of words representing the identity in G is a context-free language if and only if G contains a free subgroup of finite index. We need a special case of (the easier direction of) this result:

Lemma 5.4. *If F is a free group with finite generating set $\Sigma = \Sigma^{-1}$, then the set of words in Σ^* that represent the identity in F is a context-free language.*

Corollary 5.5. *Let G be a group with a finite set of generators $\mu : \Sigma^* \to G$. Let $L \subset \Sigma^*$ be a regular language. Let $\phi : G \to F$ be an epimorphism to a finitely generated free group. Then $\{w \in L \mid \mu(w) \in \ker \phi\}$ is a context-free language.*

Proof. According to the lemma, the language $\{w \in \Sigma^* \mid \phi \circ \mu(w) = 1\}$ is context-free. The intersection of a context-free language and a regular language is always context-free (see 5.1). □

6. The grammar of reparameterization

In this section we shall prove:

Theorem D. *There exist* Ind*-combable groups that are not* Reg*-combable (i.e. automatic).*

Indexed languages enter our discussion naturally for the following reason: in order to construct the combings that we needed in our proof of Theorems A and B, we had to "slow-down" certain combings that lacked the synchronous fellow-traveller property. In other words, we redefined the combing in such a way that the discrete path that each combing line traced through the group was left unaltered but the speed at which it traversed this path was reduced (see 3.3). We shall now explore the effect that this slowing-down has on the grammatical complexity of the language of words in the combing. In particular, assuming that the original language was generated by a certain grammar, we shall explain how to modify the grammar so as to slow the combing in the desired manner. This modification is achieved by mingling the original grammar with a certain indexed grammar; if the original grammar was regular or context-free, then the final grammar will be indexed, and hence the image of the final combing will an indexed language.

Definition 6.1. Let $k > 0$ be an integer. Given a finite set Σ, a symbol $o \notin \Sigma$ and languages $L_1, L_2 \subset \Sigma^*$, we define $[\frac{1}{k}]L_1 \bullet L_2$ to be the sub-language of $(\Sigma \cup \{o\})^*$ consisting of the empty word and

$$\{w_0 \frac{1}{k_1} w_1 \ldots \frac{1}{k_m} w_m \mid m \in \mathbb{N}, \ w_i \text{ taken alternately from } L_1 \text{ and } L_2\}$$

where $k_i = k^i + 1$ (notation of 3.3). Thus a word in $[\frac{1}{k}]L_1 \bullet L_2$ is obtained from a word $w_0 \ldots w_m \in L_1 \bullet L_2$ by introducing a string of k^i letters o after each letter in the subword w_i for $i = 1, \ldots, m$.

The Indexed Grammar of Reparameterization.

Consider the indexed grammar with initial state S that has non-terminals $A_1, A_2, X_1, X_2, Y_1, Y_2, S_1, S_2, I$, terminals x_1, x_2, i, two indices f, g, and productions

$$S \to X_1 g \ \text{ or } \ X_2 g \ \text{ or } \ X_1 g A_2 g \text{ or } X_2 g A_1 g$$
$$A_1 \to Y_1 f; \quad A_2 \to Y_2 f$$
$$Y_1 \to X_1 \text{ or } X_1 A_2; \quad Y_2 \to X_2 \text{ or } X_2 A_1$$
$$X_1 g \to x_1; \quad X_2 g \to x_2;$$
$$X_1 f \to x_1 I; \quad X_2 f \to x_2 I; \quad I f \to i I \quad I g \to i.$$

If we were to delete from this grammar the indices f and g and the symbol I, then we would be left with a context-free grammar generating the language of all words of the form $x_1 x_2 x_1 x_2 \ldots$ and $x_2 x_1 x_2 x_1 \ldots$. The effect of introducing the indices and I is to place a string of letters i after each letter of each word in the language just described: the string after the m-th letter of each word is $(m-1)$ letters long. Thus the shortest words in the language are:

$$x_1, \ x_2, \ x_1 x_2 i, \ x_2 x_1 i, \ x_1 x_2 i x_1 ii, \ x_2 x_1 i x_2 ii, x_1 x_2 i x_1 ii x_2 iii, \ x_2 x_1 i x_2 ii x_1 iii, \ldots$$

For example,

$$S \to X_1 g A_2 g \to x_1 A_2 g \to x_1 Y_2 fg \to x_1 X_2 fg A_1 fg \to x_1 X_2 fg Y_1 f^2 g \to$$
$$\to x_1 X_2 fg X_1 f^2 g \to x_1 x_2 I g X_1 f^2 g \to x_1 x_2 I g x_1 I fg \to$$
$$\to x_1 x_2 I g x_1 i I g \to x_1 x_2 I g x_1 ii \to x_1 x_2 i x_1 ii.$$

Remark 6.2. There is an obvious simplification of the above grammar: one could replace the $X_j g$ in the top row by x_j and remove the productions in the penultimate row. We have chosen the more clumsy structure because it lends itself readily to the following application.

Theorem 6.3. *Let Σ be a finite set and let $k > 0$ be an integer. If $L_1, L_2 \subset \Sigma^*$ are context-free languages, then $L := [\frac{1}{k}] L_1 \bullet L_2$ is an indexed language.*

Proof. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be context-free grammars over the alphabet Σ, generating the languages L_1 and L_2 respectively. We shall generate L by an indexed grammar over the alphabet $\Sigma \cup \{o\}$, where the symbol o is as in the definition of $[\frac{1}{k}] L_1 \bullet L_2$.

Let S denote the start symbol of our grammar. The grammar has two indices f and g. The non-terminals of the grammar are of five kinds: the non-terminals of $\mathcal{G}_1$ and $\mathcal{G}_2$; the symbols S_1 and S_2 that were previously regarded as the start symbols of $\mathcal{G}_1$ and $\mathcal{G}_2$; one symbol B for each terminal $b \in \Sigma$; four new symbols A_1, A_2, Y_1, Y_2 (that will perform as they did in the grammar of reparameterization described prior to this theorem); and one other symbol O. We group the productions into four types: we have the productions listed in the

first three rows of the grammar of reparameterization, modified by replacing X_1 and X_2 by S_1 and S_2 respectively; we also have the productions from the grammars $\mathcal{G}_1$ and $\mathcal{G}_2$, which are unaltered except for the fact that every occurrence of $b \in \Sigma$ is replaced by B; we have two additional productions for each $b \in \Sigma$, namely

$$Bf \to bO \ldots O \quad \text{and} \quad Bg \to b,$$

where there are k letters O after the b in the first production; and finally we have the productions

$$Of \to O \ldots O \text{ and } Og \to o,$$

where there are k letters O on the right of the first production.

It follows from remark 5.3 that a word $w \in (\Sigma \cup \{o\})^*$ lies in the language generated by the above grammar if and only if w can be obtained by applying a sequence of productions from the (modified) grammar of reparameterization, followed by a sequence of productions from the (modified) grammars $\mathcal{G}_j$, followed by instances of the last two types of productions displayed above. Thus we may assume that

$$S \xrightarrow{*} \alpha \xrightarrow{*} \beta \xrightarrow{*} w,$$

where the sentential form α is a word in the symbols $\{S_1, S_2, g, f\}$, where β is a word in f, g and the various symbols B, and where $\beta \xrightarrow{*} w$ involves only the two new types of production (the $Bg \to b$ and those involving O).

The productions in the first three rows of the grammar of reparameterization generate the following sentential forms in $\{X_1, X_2, g, f\}$:

$$X_1 g X_2 f g X_1 f^2 g X_2 f^3 g \ldots X_1 f^{2r} g \quad \text{and} \quad X_1 g X_2 f g X_1 f^2 g X_2 f^3 g \ldots X_2 f^{2r+1} g$$

(r any non-negative integer) and the same forms with the roles of X_1 and X_2 reversed. The form α is obtained from such a form by making the substitutions $X_j \to S_j$.

One proceeds from α to β by applying a sequence of productions from the (modified) context-free grammars $\mathcal{G}_j$ to each of the non-terminals in α. (The sequences of productions applied to the different non-terminals in α are independent because the $\mathcal{G}_j$ are context-free.) An important point to note is that since we have replaced all of the terminals b in the productions of $\mathcal{G}_j$ with non-terminals B, the index $f^m g$ appearing after a non-terminal S_j in α appears after *every* symbol in the sentential forms derived from that S_j in the course of passing to β. Thus β is a product of sub-forms $Bf^m g$. For example, if $\alpha = S_1 g S_2 f g S_1 f^2 g \ldots S_1 f^{2r} g$, then $\beta = U_0 V_1 U_2 \ldots U_r$ where: every symbol in U_i is a non-terminal B followed by a string of $2i$ indices f, followed by g; every symbol in V_i is a non-terminal B followed by a string of $2i - 1$ indices f, followed by g; the word obtained from U_j by deleting the indices f and g

and then replacing each B by the corresponding $b \in \Sigma$ belongs to L_1; the word obtained from V_j by deleting the indices f and g and then replacing each B by the corresponding $b \in \Sigma$ belongs to L_2.

Finally, to realize $\beta \xrightarrow{*} w \in L$, to each of the sub-forms $Bf^m g$ in β one applies the production $Bf \to bO \ldots O$ (if $m \neq 0$) followed by $(m-1)$ generations of productions $Of \to O \ldots O$, then (having got rid of all of the indices f) one applies the production $Og \to o$ to each Og (and $Bg \to b$ in the case $m = 0$). For example,

$$Bf^3 g \to b(Offg)^k \xrightarrow{*} b((Ofg)^k)^k \xrightarrow{*} b(((Og)^k)^k)^k = b(Og)^{k^3} \xrightarrow{*} bo^{k^3}.$$

This description of $\beta \xrightarrow{*} w$, together with our earlier description of α and $\alpha \xrightarrow{*} \beta$, completes the proof that $S \xrightarrow{*} w$ if and only if $w \in [\frac{1}{k}]L_1 \bullet L_2$; in other words L is the indexed language generated by the indexed grammar described in the second paragraph of the proof. $\square$

Theorem D, as stated in the introduction, is an immediate consequence of 4.1, 4.2 and the following theorem.

Theorem 6.4. *Let $\Gamma_1 = N_1 \rtimes F$ and $\Gamma_2 = N_2 \rtimes F$ be biautomatic groups. If F is finitely generated and free, then $\Gamma = \Gamma_1 *_F \Gamma_2$ is* Ind-*combable.*

Proof. In the proof of 4.1 we constructed a combing $\sigma : \Gamma \to (\Sigma \cup \{o\})^*$. If we assume that the scaling factors used in the construction of σ are modified as in Remark 3.5(4), then, in the language of 6.1, the image of σ is $L_0 \cdot [\frac{1}{k}]L_1 \bullet L_2$, where L_0 can be taken to be the image of an automatic structure on F, and L_i is the image of N_i under a regular bicombing (biautomatic structure) $\sigma^i : \Gamma_i \to \Sigma^*$.

Because $\sigma^i(\Gamma_i)$ is a regular language for $i = 1, 2$, it follows from Lemma 5.5 that L_1 and L_2 are context-free languages. But then Theorem 6.3 implies that $[\frac{1}{k}]L_1 \bullet L_2$ is an indexed language. Since the class of indexed languages is closed under the formation of products, we deduce that the image of σ is an indexed language. $\square$

Remarks 6.5. (1) When I came to write the final draft of this article, it seemed most natural to arrange it so that indexed grammars appear in a secondary role at the end. However, it was my attempts to understand the class of groups that admit context-free and indexed combings that first led me to the results presented in earlier sections. Further results in this direction are given in [8].

(2) The above theorem and proof remain valid if one replaces the hypothesis that the Γ_i are biautomatic by the hypothesis that the groups have CF-bicombings with the comparable lengths property.

(3) The combings that we have been considering for the groups in Theorem 4.1 were obtained by reparameterizing a certain combing that had the

asynchronous fellow-traveller property. Arguing as in the last paragraph of the proof of the above theorem, one sees that the image of this asynchronous combing is context free.

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