ON THE SUBGROUPS OF SEMIHYPERBOLIC GROUPS

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Abstract. We show that if G is the fundamental group of a compact non-positively curved manifold of dimension ≤ 3 then so too is every finitely presented subgroup $\Gamma \subset G$. A similar result holds for complexes of dimension ≤ 2. But in higher dimensions one encounters subtle obstructions to the existence of a compact non-positively curved $K(\Gamma, 1)$: higher finiteness conditions, the complexity of decision problems in Γ, the structure of centralizers etc.

We prove some general results concerning the word and conjugacy problems for groups. Using these results we investigate families of non-positively curved manifolds M and subgroups $\Gamma \subset \pi_1 M$ that arise as the fundamental groups of closed aspherical submanifolds $N_\Gamma \subset M$; these subgroups enjoy properties reminiscent of semihyperbolic groups, but they are not semihyperbolic.

For André Haefliger, in friendship, with absolute respect

Introduction

If a group acts properly and cocompactly by isometries on a simply-connected space of non-positive curvature[1] then one can say a great deal about the structure of the group. We write $\mathcal{C}_0$ to denote the class of groups that admit such actions. This article is part of a programme to understand which groups belong to $\mathcal{C}_0$. One benefit of pursuing such an understanding is that as one does so one finds increasingly subtle obstructions (at the level of π_1) to the existence of metrics of non-positive curvature on spaces of particular interest, such as aspherical manifolds.

Some of the most significant properties of the groups $G \in \mathcal{C}_0$ are inherited by subgroups $\Gamma \subset G$, but many others are not — see Chapter III.Γ of [14]. It is therefore useful to know if an abstract group Γ can be realised as such a subgroup, but it is more useful to know if one can promote the resulting action of Γ to a cocompact action and hence deduce that $\Gamma \in \mathcal{C}_0$. Thus, given a compact non-positively curved space X, we investigate the subgroups $\Gamma \subset \pi_1 X$ and ask under what circumstances one can deduce that Γ is the fundamental group of a *compact* non-positively curved space. In order to allow groups with torsion, one should rephrase this question in terms of cocompact actions rather than fundamental groups. And recasting the

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[1]Throughout this article "non-positive curvature" will be in the sense of A.D. Alexandrov [14].

problem in terms of the intrinsic geometry of groups one asks: which groups are subgroups of semihyperbolic groups, and which of these subgroups are themselves semihyperbolic?

In low dimensions there are satisfying answers to this problem. If X is 1 dimensional then in order to obtain a cocompact action for $\Gamma \subset \pi_1 X$ it is sufficient to assume that Γ is finitely generated — this is an immediate consequence of the fact that subgroups of free groups are free. In dimension 2 it is not sufficient to assume that Γ is finitely generated, because there exist examples where Γ is finitely generated but not finitely presented, whereas the fundamental group of any compact non-positively curved space is finitely presented. But this is the only obstruction: using tower arguments one can show that if X is a compact non-positively curved 2-complex then every finitely presented subgroup of $\pi_1 X$ is also the fundamental group of a compact non-positively curved 2-complex — see page 216 of [14].

In dimension 3 a similar result holds, but only for manifolds. The manifolds may have boundary, and the boundary is not required to be locally geodesically convex in the given path metric.

Theorem 0.1. *If Γ is the fundamental group of a compact irreducible 3-manifold that admits a path-metric of non-positive curvature, then every finitely generated subgroup of Γ is also the fundamental group of such a 3-manifold.*

We shall prove this theorem in Section 4. It is a consequence of Scott's Compact Core Theorem [31] and an existence theorem for metrics of non-positive curvature that one can deduce from Thurston's Geometrization Theorem for Haken manifolds (Theorem 4.3): if a compact P^2-irreducible 3-manifold M has non-empty boundary, then M supports a metric of non-positive curvature.

If X is a non-positively curved manifold of dimension at least 4, or a non-positively curved complex of dimension at least 3, then the situation is more complicated. This point is discussed at some length in Section III.Γ.5 of my book with André Haefliger, where we concluded by saying "we have seen that the finitely presented subgroups of the fundamental groups of compact non-positively curved spaces form a much more diverse class that the fundamental groups themselves." The discussion in that section focused on higher finiteness properties and the complexity of the word and conjugacy problems, following the results in [6], [7], [4], [5] and [8]. Examples were also given in [14] of subgroups in which the centralizers of certain finite subsets were not finitely generated. Here we shall explore further aspects of the diversity to be found amongst subgroups.

Recall that the major characteristics of semihyperbolic groups include the fact that they have Eilenberg-MacLane spaces with finite skeleta, they admit efficient solutions to the word and conjugacy problems, and the centralizer of every finite subset is again a semihyperbolic group. The members of $\mathcal{C}_0$ are the proto-typical semihyperbolic groups.

Theorem 0.2. *There exist closed non-positively curved manifolds M and subgroups $\Gamma \subset \pi_1 M$ such that:*

1. *$\Gamma \hookrightarrow \pi_1 M$ is a quasi-isometric embedding;*
2. *Γ is the fundamental group of a closed aspherical submanifold of M;*
3. *Γ satisfies a polynomial isoperimetric inequality;*
4. *Γ has a solvable conjugacy problem;*
5. *the centralizer $C_\Gamma(S)$ of every subset $S \subseteq \Gamma$ is the fundamental group of a closed aspherical manifold; moreover, the word and conjugacy problems are solvable in $C_\Gamma(S)$, but*
6. *there exist $\gamma \in \Gamma$ such that $C_\Gamma(\gamma)$ does not satisfy a polynomial isoperimetric inequality, and therefore Γ is not semihyperbolic.*

The proof of this theorem is outlined in Section 1 below, where explicit families of examples are given.

Parts (3) and (4) of Theorem 0.2 will be obtained as special cases of the following results, which are of independent interest. These results are the subject of Sections 2 and 3 respectively. Section 2 also contains a number of related results concerning Dehn functions, including a discussion of the Dehn functions of the centralizers that arise in Theorem 0.2.

Theorem 0.3. *Let Γ_1, Γ_2 and Γ_3 be finitely presented groups. If $\phi : \Gamma_1 \times \Gamma_2 \times \Gamma_3 \to \mathbb{Z}$ is a homomorphism whose restriction to each Γ_i is surjective, then the kernel of ϕ is finitely presentedand its Dehn function is*

$$\delta_{\ker\ \phi}(n) \preceq n\ \delta_{\Gamma_1 \times \Gamma_2}(n)\ +\ \delta_{\Gamma_3}(n).$$

Theorem 0.4. *Let G be a bicombable group and let $N \subset G$ be a normal subgroup. If the generalized word problem for G/N is solvable, then N has a solvable conjugacy problem.*

Note that N is not assumed to be finitely generated (see 3.6). Note also that if the generalized word problem in G/N is unsolvable, then N may have an unsolvable word problem even if G is the fundamental group of a compact non-positively curved manifold — see [5].

The class of groups with a solvable generalized word problem includes free groups, surface groups and fundamental groups of Seifert fibre spaces, polycyclic-by-finite groups, and finitely generated abelian-by-nilpotent groups. See [25] for a survey and references.

Each section of this paper can be read independently.

1. A Family of Aspherical Manifolds

In this section we describe a family of aspherical manifolds M_n and subgroups $\Gamma_n \subset \pi_1 M_n$, and we explain why they satisfy Theorem 0.2.

1.1. **The Manifolds M_n and the subgroups $\Gamma_n \subset \pi_1 M_n$**

Bill Thurston has shown that there are infinitely many closed 3-manifolds of constant negative curvature that fibre over the circle [34]. Let $X_1, \dots, X_n$ be such manifolds and fix a fibration $p^i : X_i \to \mathbb{S}^1$ with compact fibre Σ_i. From the long exact sequence of the fibration we get a short exact sequence $1 \to \pi_1\Sigma_i \to \pi_1 X_i \to \mathbb{Z} \to 1$, and hence $\pi_1 X_i = S_i \rtimes_{\psi_i} \mathbb{Z}$, where $S_i = \pi_1\Sigma_i$.

Let M_n denote the closed non-positively curved manifold $X_1 \times \cdots \times X_n$, let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, equipped with the product metric, and consider the obvious fibration $M_n \to \mathbb{T}^n$ with fibre $F = \Sigma_1 \times \cdots \times \Sigma_n$. Let N_n denote the inverse image in M_n of an $(n-1)$-dimensional sub-torus $B \subset \mathbb{T}^n$ orthogonal to the diagonal. The $(3n-1)$-dimensional manifold N_n is the total space of a bundle whose base B and fibre F are aspherical, so the long exact sequence in homotopy tells us that N_n is aspherical. One may also view N_n as the fibre of the bundle $M_n \to \mathbb{S}^1$ defined by $(x_1, \dots, x_n) \mapsto p^1(x_1) + \cdots + p^n(x_n)$. From the long exact sequence of this fibration one sees that $\Gamma_n := \pi_1 N_n$ is the kernel of the homomorphism from $\pi_1 M_n = \pi_1 X_1 \times \cdots \times \pi_1 X_n$ to $\mathbb{Z}$ obtained by taking the pointwise sum of the maps p^i_*.

1.2. **A Summary of the Proof of Theorem 0.2**

We shall prove that if $n \geq 3$ then M_n and Γ_n satisfy Theorem 0.2.

The following arguments are numbered in accordance with the items in Theorem 0.2.

(1) Lemma 1.4 gives a general criterion that shows $\Gamma_n \hookrightarrow \pi_1 M_n$ to be a quasi-isometric embedding if $n \geq 2$.

(2) We have constructed Γ_n as the fundamental group of a closed aspherical submanifold $N_n \subset M_n$ (for all $n \geq 1$).

(3) The fundamental group of any compact negatively curved space is word hyperbolic, and the Dehn function of a direct product of word hyperbolic groups is at most quadratic. Thus item (3) of Theorem 0.2 is a special case[2] of Thereom 2.2, which we shall prove in Section 2. Here we need $n \neq 2$ — see (2.7).

(4) In Section 3 we shall prove Thereom 0.4. Fundamental groups of compact non-positively curved spaces are bicombable [2] and the generalized word problem is solvable in $\mathbb{Z}$, so this theorem implies that Γ_n has a solvable conjugacy problem for all $n \geq 1$.

(5) Corollary 1.6 shows that the centralizer of every subset of Γ_n is isomorphic to a group of the form $\Gamma_p \times \mathbb{Z}^r$ with $0 \leq r + p \leq n$, or $\pi_1 W \times \mathbb{Z}^{r'}$, where W is

[2] I am not sure if the cubic bound yielded by Theorem 2.2 can be improved.

a finite-sheeted covering of a manifold of the form $M_{p'}$ with $1 \leq p' \leq n-2$ and $r'+p' \leq n-1$. Among the centralizers of individual elements one finds only the possibilities $p+r=n$ and $p'+r'=n-1$, but all possible values of (p,r) and (p',r') occur subject to these conditions.

In the light of (2) and (4), we know that Γ_p is the fundamental group of an aspherical manifold and that it has a solvable conjugacy problem. Moreover $\pi_1 W$ also has these properties, as does $\mathbb{Z}^r$, and both properties are preserved under the formation of finite direct products.

(6) Following the discussion in (3), we know that the Dehn function of each of the groups in (5) is polynomial, except for $\Gamma_2 \times \mathbb{Z}^r$. Theorem 2.7 shows that the Dehn function of Γ_2, and hence that of $\Gamma_2 \times \mathbb{Z}^r$, is exponential. The Dehn function of a semihyperbolic group is either linear or quadratic, therefore $\Gamma_2 \times \mathbb{Z}^r$ is not semihyperbolic.

If a group Γ is semihyperbolic, then so are the centralizers of all finite subsets in Γ (see [2]). Since $\Gamma_2 \times \mathbb{Z}^{n-2}$ is a centralizer of an element of Γ_n, we see that Γ_n is not semihyperbolic if $n \geq 2$, and therefore N_n is not homotopy equivalent to any compact space of non-positive curvature. □

Remarks 1.3. (1) By construction, the groups Γ_n are subgroups of a direct product of copies of $\mathrm{PSL}(2,\mathbb{C})$. A theorem of Farrell and Jones [17] states that such groups satisfy the Borel conjecture. Thus, up to homeomorphism, N_n is the only closed aspherical manifold with fundamental group Γ_n.

(2) As an alternative way of seeing that Γ_n is not semihyperbolic when $n \geq 3$, one can note that the centralizers of certain finite subsets are not quasi-isometrically embedded in Γ_n, whereas such centralizers are always quasi-isometrically embedded in semihyperbolic groups [2]. Specifically, the centralizer in Γ_n of $\Gamma_n \cap (1 \times \pi_1 X_2 \times \cdots \times \pi_1 X_n)$ is S_1, the fundamental group of the surface fibre $\Sigma_1 \subset X_1$. The distortion of S_1 in $\pi_1 X_1$ is exponential, so in the light of part (1) of Theorem 0.2, its distortion in Γ_n is also exponential.

Lemma 1.4. *Let Γ_1 and Γ_2 be finitely generated groups. If $\phi : \Gamma_1 \times \Gamma_2 \to \mathbb{Z}$ is a homomorphism whose restriction to each Γ_i is surjective, then* $\ker \phi$ *is finitely generated and* $\ker \phi \hookrightarrow \Gamma_1 \times \Gamma_2$ *is a quasi-isometric embedding.*

Proof. Fix a generator t for $\mathbb{Z}$. For $j=1,2$, choose an element $a_j \in \Gamma_j$ with $\phi(a_j)=t$ and choose a finite generating set $\mathcal{A}_j = \mathcal{B}_j \cup \{a_j\}$ for Γ_j, where $\mathcal{B}_j \subset \ker \phi$. Equip $\Gamma_1 \times \Gamma_2$ with the word metric associated to the generating set $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$.

Let $\theta = a_1 a_2^{-1}$. We claim that $\ker \phi$ is generated by $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \{\theta\}$, and that with respect to this choice of word metrics $d_{\mathcal{B}}(1,g) \leq 2\, d_{\mathcal{A}}(1,g)$ for all $g \in \ker \phi$.

To see this, first note that given any $g \in \Gamma_1 \times \Gamma_2$ one can write $g = w_1 w_2$, where w_j is a geodesic word in the letters $\mathcal{A}_j^{\pm 1}$ (hence $d_{\mathcal{A}}(1,g) = |w_1| + |w_2|$). Let e_j denote the sum of the exponents of all occurences of a_j in w_j. If $g \in \ker \phi$ then $e_1 = -e_2$ and $g = w_1' \theta^{e_2} w_2'$, where w_1' is obtained from w_1 by replacing each occurrence of a_1^{ε} with θ^{ε} and w_2' is obtained from w_2 by replacing each occurrence of a_2^{ε} with $\theta^{-\varepsilon}$. □

Centralizers

We now turn our attention to the structure of centralizers in subgroups of direct products. We write $C_H(S)$ to denote the centralizer of a subset S in a group H.

If a torsion-free group H is word hyperbolic, then $C_H(h) \cong \mathbb{Z}$ for every $h \in H \smallsetminus \{1\}$. Thus the following proposition will apply with $D = \pi_1 M_n$ and $K = \Gamma_n$ (notation of (1.1)).

Proposition 1.5. *Let $G_1, \dots, G_n$ be torsion-free groups in which the centralizer of every non-trivial element is cyclic. Let $D = G_1 \times \cdots \times G_n$. Let $\phi : D \to \mathbb{Z}$ be a homomorphism whose restriction to each of the G_i is surjective and let $K = \ker \phi$. Let $S \subset K$ be a subset. Let S_i be the image of S under the coordinate projection $D \to G_i$. Assume that $\langle S_i \rangle \cong \mathbb{Z}$ if $i \le r$, that $\langle S_i \rangle$ is not cyclic if $r+1 \le i \le q-1$, and that $S_i = \{1\}$ if $q \le i \le n$.* Then:

- *for $r \ge 1$ and $q - 1 \ne n$* **either** $C_K(S) \cong D^0_{q,n} \times \mathbb{Z}^{r-1}$*, where $D^0_{q,n}$ is a subgroup of finite index in $D_{q,n} := G_q \times \cdots \times G_n$,* **or else** $C_K(S) \cong K_{q,n} \times \mathbb{Z}^r$*, where $K_{q,n} = \ker \phi|_{D_{q,n}}$. (If $r = 1$ and $q = 2$ then the only possibility that arises is $C_K(S) \cong K_{2,n} \times \mathbb{Z}$.)*
- *If $r = 0$ and $q - 1 \ne n$, then $C_K(S) \cong K_{q,n}$.*
- *If $r \ge 1$ and $q - 1 = n$, then $C_K(S) \cong \mathbb{Z}^r$.*
- *If $r = 0$ and $q - 1 = n$, then $C_K(S)$ is trivial.*

Proof. The centralizer of S in $D = G_1 \times \cdots \times G_n$ is $C_{G_1}(S_1) \times \cdots \times C_{G_n}(S_n)$, so what we must calculate is the intersection of K with this product.

We shall assume that $r \ge 1$ and $q - 1 \ne n$, leaving the reader to furnish the easy remarks needed in the degenerate cases.

Because centralizers in the G_i are assumed to be cyclic, for $i = r+1, \dots, q-1$ we have $C_{G_i}(S_i) = \{1\}$. For $i = q, \dots, n$ we have $C_{G_i}(S_i) = G_i$. And for $i = 1, \dots, r$ we know that $C_{G_i}(S_i)$ is cyclic, equal to $\langle g_i \rangle$ say.

We shall need the semidirect product decompositon of $D_{q,n}$ that is obtained by splitting the short exact sequence associated to the restriction of ϕ,

$$1 \to K_{q,n} \to D_{q,n} \to \mathbb{Z} \to 1. \tag{1.5.1}$$

We split this sequence by choosing $t \in G_n$ such that $\phi(t)$ generates $\mathbb{Z}$, thus obtaining

$$D_{q,n} \cong K_{q,n} \rtimes_\tau \mathbb{Z}, \tag{1.5.2}$$

where $\tau \in \text{Aut } K_{q,n}$ acts as conjugation by $t \in G_n \subset D_{q,n}$.

The coordinate projection $p : D \to G_1 \times \cdots \times G_{q-1}$ gives us a short exact sequence

$$1 \to D_{q,n} \to C_D(S) \to \langle g_1, \dots, g_r \rangle \to 1.$$

We claim that each g_i actually lies in the image of $C_K(S)$. Indeed $\phi(g_i) = \phi(t)^{\mu_i}$ for some integer μ_i, and $g_i t^{-\mu_i} \in K$ commutes with S. Thus, restricting p, we have a short exact sequence

$$1 \to K_{q,n} \to C_K(S) \to \langle g_1, \dots, g_r \rangle \to 1. \tag{1.5.3}$$

Moreover, since $\langle g_1, \dots, g_r\rangle$ is *free* abelian and $g_i t^{-\mu_i}$ commutes with $g_j t^{-\mu_j}$ for $i, j = 1, \dots, r$, the map $g_i \mapsto g_i t^{-\mu_i}$ is a splitting homomorphism for this sequence, giving us a semidirect product decomposition

$$C_K(S) \cong K_{q,n} \rtimes_T \mathbb{Z}^r$$

where the image of $T : \mathbb{Z}^r \to \text{Aut } K_{q,n}$ is a subgroup of $\langle \tau \rangle$.

If the image of T is trivial (the case where all the g_i lie in the kernel of ϕ), then this semidirect product is actually direct. (If $r = 1$ and $q = 2$ then $\langle S\rangle = \langle g_1 \rangle$ so we are necessarily in this case.)

If the image of T is $\langle \tau^\rho \rangle$ where $\rho > 0$, then

$$K_{q,n} \rtimes_T \mathbb{Z}^r \cong \big(K_{q,n} \rtimes_{\tau^\rho} \mathbb{Z}\big) \times \ker T.$$

And since $\ker T \cong \mathbb{Z}^{r-1}$, we deduce from (1.5.2) that

$$C_K(S) \cong D^0_{q,n} \times \mathbb{Z}^{r-1},$$

where $D^0_{q,n} := \phi^{-1}(\rho\mathbb{Z}) \cap D_{q,n}$ has index ρ in $D_{q,n}$. □

Let $\mathcal{M}_n$ denote the set of closed $3n$-dimensional manifolds M_n that are obtained by taking the product of n compact hyperbolic 3-manifolds that fibre over the circle. Let $\mathcal{G}_n$ denote the class of groups $\Gamma_n \subset \pi_1 M_n$ that arise when one applies the construction at the beginning of this section to each $M_n \in \mathcal{M}_n$. Define $\mathcal{G}_0$ to consist only of the trivial group and $\mathcal{M}_0$ to consist only of the manifold with one point.

Corollary 1.6. *If $\Gamma \in \mathcal{G}_n$, then the centralizer of every subset $S \subset \Gamma$ is isomorphic either to a group of the form $G \times \mathbb{Z}^r$ with $G \in \mathcal{G}_p$ and $0 \le r + p \le n$, or else to a group of the form $\pi_1 W \times \mathbb{Z}^{r'}$, where W is a finite-sheeted covering of some manifold $M \in \mathcal{M}_{p'}$, with $1 \le p' \le n-2$ and $r' + p' \le n-1$.*

The centralizers of individual elements run through all of the possible forms allowed with $p + r = n$ and $p' + r' = n - 1$.

Proof. The fundamental group of a closed hyperbolic 3-manifold is torsion-free and word hyperbolic. All of the centralizers in such a group are infinite cyclic, so we are in the setting of the proposition with $\Gamma = K$. In the notation of the proposition we have $K_{n,q} \in \mathcal{G}_{n-q+1}$ and $D_{q,n} = \pi_1 M$ with $M \in \mathcal{M}_{n-q+1}$.

Among the centralizers of individual elements only the possibilities $p+r = n$ and $p' + r' = n - 1$ can arise because the projections S_i are necessarily cyclic, therefore $q = r + 1$. Moreover, the proof of the proposition shows that in order to obtain an element whose centralizer is of the form $G \times \mathbb{Z}^r$ with $G \in \mathcal{G}_{n-r}$, it suffices to take $\gamma = (g_1, \dots, g_r, 1, \dots, 1)$, where each $g_i \in G_i \smallsetminus \{1\}$ lies in $\ker \phi$. In order to obtain a centralizer of the form $\pi_1 W \times \mathbb{Z}^{r-1}$ with W a finite-sheeted covering of some manifold $M \in \mathcal{M}_{n-r}$, one adjusts γ by replacing g_1 and g_2 with elements g_1' and g_2' such that $\phi(g_1')$ is non-trivial and $\phi(g_2') = \phi(g_1')^{-1}$. (One can control the index of the finite-sheeted covering by specifying the value of $\phi(g_1')$.)

To reduce the dimension of W by 1, one considers the centralizer of $\{\gamma_1, \gamma_2\}$, where $\gamma_i = (g_1', g_2', \dots, g_r, h_i, 1, \dots)$ and $\text{gp}\{h_1, h_2\}$ is a non-abelian free group in the kernel of $\phi|_{G_{r+1}}$. To reduce the dimension of W further, one introduces such free groups in other coordinates. $\square$

Remark 1.7. Finite direct products of word hyperbolic groups are biautomatic [15], so the above results contribute to the study of the diverse nature of subgroups of (bi)automatic groups. Theorem 0.2 shows that the groups Γ_n are not biautomatic.

2. Isoperimetric Inequalities

The main purpose of this section is to prove Theorem 0.3 (=2.2). In order to make the proof self-contained, we begin by recalling the definition of a Dehn function and relate it to the idea of "applying a relation".

2.1. **Dehn functions**

One may think of a choice of generators for a group Γ as being a set $\mathcal{A}$ and an epimorphism $\pi : F(\mathcal{A}) \twoheadrightarrow \Gamma$ from the free group on $\mathcal{A}$. If $\mathcal{A}$ is finite and $\ker \pi$ is the normal closure of $\mathcal{R} \subset F(\mathcal{A})$ then $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is called a finite presentation for Γ. One says that the *word problem* for Γ is solvable if there is an algorithm that decides which words w in the letters $\mathcal{A}^{\pm 1}$ map to the identity in Γ. This is equivalent to deciding if there exists an equality

$$w = \prod_{i=1}^{N} x_i^{-1} r_i x_i$$

in the free group, where $r_i \in \mathcal{R}^{\pm 1}$ and each $x_i \in F(\mathcal{A})$ is a word of length at most $cN + |w|$, where c is the length of the longest $r \in \mathcal{R}$. Because checking the validity of an equality in a free group is a trivial matter, solving the word problem in Γ is equivalent to identifying a computable function f such that for any word w one knows that $w = 1$ in Γ if and only if there is an equality of the above form with $N \leq f(|w|)$, where $|w|$ is the length of w. The smallest function $f : \mathbb{N} \to \mathbb{N}$ with this property is called the *Dehn function* of Γ and is denoted δ_Γ.

One says that a word w' in the letters $\mathcal{A}^{\pm 1}$ is obtained from a word w *by applying the relation* r if w and w' can be written in the free group $F(\mathcal{A})$ as $w = \alpha u \beta$ and $w' = \alpha v \beta$, where $uv^{-1} = r^{\pm 1}$. For example, $a^n bab^m$ can be obtained from $a^{n+1}b^{m+1}$ by applying the relation $[a, b]$. It is easy to check that the Dehn function $\delta(n)$ of $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$ calculates the least number of defining relations $r \in \mathcal{R}$ that one must apply in order to reduce to the empty word any w with $|w| \leq n$ and $w = 1$ in Γ.

The Dehn functions associated to different finite presentations of a group are $\simeq$ in the following sense. Given functions $f, g : [0, \infty) \to [0, \infty)$, one writes $f \preceq g$ if there exists a constant $c > 0$ such that

$$f(n) \leq c\, g(cn + c) + cn + c$$

for all $n \in \mathbb{N}$; one writes $f \simeq g$ if $f \preceq g$ and $g \preceq f$.

The modern theory of Dehn functions began with the work of Gromov [20]. He focused attention on the fact that Dehn functions provide a bridge between word problems for finitely presented groups and isoperimetric properties of Riemannian manifolds: if M is a closed, smooth Riemannian manifold with fundamental group Γ then the Dehn function of Γ is $\simeq$ equivalent to the least function $\Phi(l)$ such that each null-homotopic loop $S^1 \to M$ of length $\leq l$ bounds a (singular) disc $D^2 \to M$ of area $\leq \Phi(l)$. See [9] for a detailed proof of this equivalence and an overview of what is known about Dehn functions.

Theorem 2.2. *Let Γ_1, Γ_2 and Γ_3 be finitely presented groups. If $\phi : \Gamma_1 \times \Gamma_2 \times \Gamma_3 \to \mathbb{Z}$ is a homomorphism whose restriction to each Γ_i is surjective, then the kernel of ϕ is finitely presented and its Dehn function is*

$$\delta_{\ker\,\phi}(n) \preceq n\ \delta_{\Gamma_1 \times \Gamma_2}(n)\ +\ \delta_{\Gamma_3}(n).$$

Proof. We begin by proving that ker ϕ is finite presentable. The key point to observe here is that ker ϕ can be decomposed as an (internal) semidirect product ker $\phi = K \rtimes \hat{\Gamma}_3$, where $K = \ker\phi \cap (\Gamma_1 \times \Gamma_2)$ and $\hat{\Gamma}_3 \cong \Gamma_3$. This decomposition is determined by any choice of element $t \in \Gamma_1 \times \Gamma_2$ such that $\phi(t)$ generates $\mathbb{Z}$: having made such a choice one defines $\hat{\Gamma}_3 = \{gt^{-\phi(g)} \mid g \in \Gamma_3\}$ and notes that the restriction to ker ϕ of the projection $\Gamma_1 \times \Gamma_2 \times \Gamma_3 \to \Gamma_3$ is a surjection with left-inverse $g \mapsto gt^{-\phi(g)}$.

Note that the action of $\hat{\Gamma}_3$ on K via conjugation defines a map $\hat{\Gamma}_3 \to \mathrm{Aut}\ K$ whose image is generated by τ, where τ is conjugation by t.

We know from (1.4) that K is finitely generated. And $K \rtimes_\tau \mathbb{Z} \cong \mathrm{gp}\{K, t\} = \Gamma_1 \times \Gamma_2$ is finitely presented. Thus we may apply the following criterion for finite presentability.

Lemma 2.3. *Let K be a finitely generated group, let H be a finitely presented group, and let $K \rtimes H$ be the semidirect product associated to a homomorphism $H \to \mathrm{Aut}\ (K)$ that has cyclic image $\langle\psi\rangle$. If K is finitely generated and $K \rtimes_\psi \mathbb{Z}$ is finitely presented, then $K \rtimes H$ is finitely presented.*

Proof. We choose a finite presentation $\langle \mathcal{B}, t \mid S\rangle$ for $K \rtimes_\psi \langle t\rangle$, where $\mathcal{B}$ generates K and S contains a relation $t^{-1}bt\beta$ for each $b \in \mathcal{B}$, where β is a word in the letters $\mathcal{B}^{\pm 1}$ that equals $\psi(b)$ in K. We choose a finite presentation $\langle \mathcal{A} \mid \mathcal{R}\rangle$ for H, where all the elements of $\mathcal{A}$ except one, a_1 say, act trivially on K and a_1 acts as ψ.

We claim that $\langle \mathcal{B}, \mathcal{A} \mid S', \mathcal{R}, C\rangle$ is a presentation for $K \rtimes H$, where the relations S' are obtained from S by substituting a_1 for each occurrence of t, and C consists of the commutators $\{[b, a] \mid b \in \mathcal{B}, a \in \mathcal{A}, a \neq a_1\}$. It is clear that $\mathcal{B} \cup \mathcal{A}$ generates and that the relations $S', \mathcal{R}$ and C are valid in $K \rtimes H$; what we must show is that these relations are sufficient. Thus we consider a word w in the given generators and suppose that $w = 1$ in $K \rtimes H$. By applying relations of the form $a_1^{-1}ba_1\beta$ and $[b, a]$ we may rewrite w in the form uv, where u is a word in the letters $\mathcal{B}^{\pm 1}$ and v is a word in the letters $\mathcal{A}^{\pm 1}$, with $u = 1$ in K and $v = 1$ in H. We may then reduce v to the empty word using relations from $\mathcal{R}$.

Since $u = 1$ in K, there is an equality in the free group $F(\mathcal{B} \cup \{t\})$ expressing u as a product of conjugates of the relations S. This equality remains valid (now in

$F(\mathcal{B} \cup \{a_1\}))$ if we replace each letter t by a_1. Thus u is freely equal to a product of conjugates of the relations S'. □

We shall now refine the proof of the above lemma in the case where $K = \ker \phi \cap (\Gamma_1 \times \Gamma_2)$ and the semidirect product is $\ker \phi = K \rtimes \hat{\Gamma}_3$; we retain the notation of the preceding proof but assume that we are talking only of this special case.

Consider the explanation in the above proof of how one can reduce w to the empty word. If one follows this procedure, counting the number of relations applied, then one obtains an estimate on the Dehn function of $K \rtimes H$. But this estimate will involve the growth of the automorphism ψ and is therefore too crude for the purposes of Theorem 2.2. In order to obtain the required bound we must refine the reduction process for w, circumventing the effect of the growth of ψ.

Towards this end, we note that in our setting $K \hookrightarrow \langle K, a_1 \rangle \cong \Gamma_1 \times \Gamma_2$ is a quasi-isometric embedding (Lemma 1.4), and hence there is a positive constant λ such that if a word u in the free group on $\mathcal{B} \cup \{a_1\}$ represents an element of K then there exists a word v in the free group on $\mathcal{B}$ such that $v = u$ in K, where $|v| \leq \lambda |u|$.

Lemma 2.4. *Let $\delta_{1,2}$ be the Dehn function of $\langle \mathcal{B}, a_1 \mid S' \rangle \cong \Gamma_1 \times \Gamma_2$. Let λ be as above. Let u be a word in the letters $(\mathcal{B} \cup \{a_1\})^{\pm 1}$.*

If the exponent sum of a_1 in u is zero (i.e. the number of occurences of a_1 is the same as the number of occurences of a_1^{-1}), then for each $i > 1$ one can transform $a_i u$ into $u a_i$ by applying at most $\lambda\, |u| + 2\delta_{1,2}(|u|(\lambda + 1))$ relations from S' and $\{[a_i, b] \mid b \in \mathcal{B}\}$.

Proof. Given u, we choose $v \in (\mathcal{B}^{\pm 1})^*$ with $v = u$ in K and $|v| \leq \lambda |u|$. By definition, one can reduce uv^{-1} to the empty word by applying at most $\delta_{1,2}(|uv^{-1}|)$ relations from S'. And since $|uv^{-1}| \leq |u| + \lambda |u|$, this means that we can transform $a_i u = a_i(uv^{-1})v$ into $a_i v$ by applying at most $\delta_{1,2}(|u|(\lambda + 1))$ relations. We can then transform $a_i v$ into $v a_i$ by applying $|v|$ commutators of the form $[a_i, b]$. And finally we can apply $\leq \delta_{1,2}(|u|(\lambda+1))$ relations to transform $v a_i = (uv^{-1})^{-1} u a_i$ into $u a_i$. □

We are now in a position to complete the proof of Theorem 2.2. We work with the presentation

$$\ker \phi = \langle \mathcal{B}, \mathcal{A} \mid S', \mathcal{R}, C \rangle$$

constructed in the proof of Lemma 2.3.

Let w_0 be a word in the given generators that represents the identity. As in the proof of Lemma 2.3, we shall move all occurences of the letters $\mathcal{A}^{\pm 1}$ to the right in w_0, but this time we shall use Lemma 2.4 to do it more efficiently.

Suppose

$$w_0 \equiv u_0 \alpha_1 u_1 \ldots \alpha_n u_n$$

where $\alpha_i \in \mathcal{A}^{\pm 1}$ and the u_i are (possibly empty) words in the letters $\mathcal{B}^{\pm 1}$.

If $\alpha_n \neq a_1^{\pm 1}$, define $w_1 \equiv u_0 \alpha_1 u_1 \ldots u_{n-1} u_n \alpha_n$ and note that one can transform w_0 into w_1 by applying $|u_n|$ relations $[\alpha_n, b]$ from C.

If $\alpha_n = a_1^{\pm 1}$, define $w_1 \equiv u_0\alpha_1 u_1 \dots u_{n-1}(\alpha_n u_n \alpha_n^{-1})\alpha_n$ and note that w_0 is freely equal to w_1.

In the first case define $U_{n-1} \equiv u_{n-1}u_n$ and in the second case define $U_{n-1} \equiv u_{n-1}\alpha_n u_n \alpha_n^{-1}$.

We now wish to iterate the above moves. Thus we assume inductively that we have transported $\alpha_{i+1}, \dots, \alpha_n$ to the right to obtain a word of the form

$$w_{n-i} \equiv u_0\alpha_1 u_1 \alpha_2 u_2 \dots \alpha_i U_i \alpha_{i+1}\alpha_{i+2}\dots\alpha_n$$

where U_i is a word in the letters $(\mathcal{B} \cup \{a_1\})^{\pm 1}$ that has exponent sum zero in a_1, and U_i has length at most $|u_i| + \dots + |u_n| + e(i)$ where $e(i)$ is the number of occurences of a_1 and a_1^{-1} among the letters $\alpha_{i+1}, \dots, \alpha_n$.

If $\alpha_i \neq \alpha_1$ then using Lemma 2.4 we can transform w_{n-i} into

$$w_{n-i+1} \equiv u_0\alpha_1 u_1 \dots \alpha_{i-1}u_{i-1}U_i\alpha_i \dots \alpha_n$$

by applying at most $[\lambda\,|U_i| + 2\delta_{1,2}(|U_i|(\lambda+1))]$ relations.

If $\alpha_i = a_1^{\pm 1}$, define

$$w_{n-i+1} \equiv u_0\alpha_1 u_1 \dots u_{i-1}(\alpha_i U_i \alpha_i^{-1})\alpha_i \dots \alpha_n$$

and note that w_{n-i+1} is freely equal to w_{n-i}.

In the first case define $U_{i-1} \equiv u_{i-1}U_i$ and in the second case define $U_{i-1} \equiv u_{i-1}\alpha_i U_n \alpha_i^{-1}$. Then note that

$$w_{n-i+1} \equiv u_0\alpha_1 u_1 \alpha_2 u_2 \dots \alpha_{i-1}U_{i-1}\alpha_i \dots \alpha_n$$

satisfies the inductive hypothesis.

If $\alpha_i = a_1^{\pm 1}$ then w_{n-i+1} is freely equal to w_{n-i}, and if not then we had to apply at most $[\lambda\,|U_i| + 2\delta_{1,2}(|U_i|(\lambda+1))]$ relations to transform w_{n-i} into w_{n-i+1}. Since $|U_i| \leq |u_i| + \dots |u_n| + e(i) \leq 2|w_0|$, we conclude that at most $[2\lambda\,|w_0| + 2\delta_{1,2}(2|w_0|(\lambda+1))]$ relations have been applied to perform this transformation.

At the end of this procedure we are left with a word

$$w_n = U_0\alpha_1 \dots \alpha_n$$

where U_0 is a word of length $\leq 2|w_0|$ in the letters $(\mathcal{B} \cup \{a_1\})^{\pm 1}$ that has exponent sum zero in a_1. Note that U_0 and $\alpha_1 \dots \alpha_n$ both represent the identity in $\ker \phi$. Thus we may reduce U_0 to the empty word by applying at most $\delta_{1,2}(2|w_0|)$ relations in $\langle \mathcal{B}, a_1 \mid S' \rangle$, and we may reduce $\alpha_1 \dots \alpha_n$ to the empty word by applying at most $\delta_{\Gamma_3}(n)$ relations in $\hat{\Gamma}_3 = \langle \mathcal{A} \mid R \rangle$.

Our inductive procedure had n steps and we applied at most

$$2\lambda\,|w_0| + 2\delta_{1,2}(2|w_0|(\lambda+1))$$

relations at each stage. Thus the total number of relations applied in reducing w_0 to the empty word is at most

$$2n\lambda\,|w_0| + 2n\delta_{1,2}(2|w_0|(\lambda+1)) + \delta_{1,2}(|U_0|) + \delta_{\Gamma_3}(n).$$

Since $|U_0| \leq 2|w_0|$ and $n \leq |w_0|$, we have established the upper bound required for Theorem 2.2. $\square$

I thank Danny Groves for correcting an error in an earlier version of the preceding proof.

2.5. **The Dehn functions of groups in $\mathcal{G}_2$**

We now turn our attention to the Dehn functions of the centralizers computed in Section 1. In the light of Theorem 2.2 we know that, in the notation of (1.6), if $n \geq 3$ and $\Gamma \in \mathcal{G}_n$ then the Dehn function of Γ is $\preceq n^3$. Moreover, in the light of Corollary 1.6 one sees that the Dehn functions of all centralizers are also $\preceq n^3$, with the possible exception of those that have a direct factor $K \in \mathcal{G}_2$. Recall that $K \in \mathcal{G}_2$ if there exist closed hyperbolic 3-manifolds that fibre over the circle $p_1 : M_1 \to \mathbb{S}^1$, $p_2 : M_2 \to \mathbb{S}^1$ so that the fibres are compact and K is the fundamental group of the compact 5-manifold that is the fibre of the bundle obtained by taking the pointwise sum $p_1 + p_2 : M_1 \times M_2 \to \mathbb{S}^1$.

From the long-exact sequence of the fibration $p_i : M_1 \to \mathbb{S}^1$ we see that $\pi_1 M_i \cong S_i \rtimes_{\psi_i} \langle t_i \rangle$ where S_i is the fundamental group of the surface fibre and $\psi_i \in \text{Aut } S_i$ is the map induced by the holonomy of p_i. The holonomy is pseudo-Anosov [33], and ψ is an automorphism of exponential growth ($n \mapsto \|\psi_i^n\|$ and $n \mapsto \|\psi^{-n}\|$, as defined below, are both $\simeq 2^n$).

We know from Lemma 1.4 that K is the subgroup of $\pi_1 M_1 \times \pi_1 M_2$ generated by S_1, S_2 and $t_1 t_2^{-1}$, which is a semidirect product

$$K \cong (S_1 \times S_2) \rtimes_{(\psi_1, \psi_2^{-1})} \langle t_1 t_2^{-1} \rangle.$$

Thus we are led to examine the Dehn functions of such semidirect products.

In what follows I shall assume that the reader is familiar with the interpretation of Dehn functions in terms of van Kampen diagrams (see [14] page 155, for example). We shall also need t-corridors, as described in [13].

Let G be a group with finite generating set $\mathcal{A}$ and associated word metric d. There are various natural ways of measuring the *growth* of an automorphism $\phi \in \text{Aut }(G)$ (see [10]). The most convenient definition for our present purposes is the function $n \mapsto \|\phi^n\|$ defined for $n \geq 0$ by

$$\|\phi^n\| := \frac{1}{n^2} \max_{d(1,w) \leq n} \sum_{i=0}^{n} d(1, \phi^i(w)).$$

The $\simeq$ equivalence class of $\|\phi\|$ does not depend on the choice of word metric.

Note that the introduction of the normalizing factor $1/n^2$ means that the growth of any periodic automorphism is bounded by a constant and the growth of any inner automorphism is at most linear (it can be a lot less). Note also that $\|\phi^n\|$ is always bounded by a simple exponential function.

In the case $G = \mathbb{Z}^n$, the function $n \mapsto \|\phi^n\|$ is Lipschitz equivalent to $n \mapsto |\phi^n|$, where $|\cdot|$ is any norm on the space of n-by-n real matrices. In the case where G is the fundamental group of a closed surface, $n \mapsto \|\phi^n\|$ is either linear or exponential; it is linear if and only if some power of the corresponding surface automorphism is a product of Dehn twists in disjoint curves.

The following result is distilled from the argument used to establish the lower bound in the Main Theorem of [13].

Proposition 2.6. *Let G_1 and G_2 be finitely generated groups and for $i = 1, 2$ let $\phi_i \in \mathrm{Aut}\ (G_i)$ be an automorphism. Let $\phi = (\phi_1, \phi_2^{-1}) \in \mathrm{Aut}\ (G_1 \times G_2)$. If $\Gamma = (G_1 \times G_2) \rtimes_\phi \mathbb{Z}$ is finitely presented, then its Dehn function is*

$$\succeq n^2 \min\{\|\phi_1^n\|, \|\phi_2^n\|\}.$$

Proof. For $i = 1, 2$, let $\mathcal{A}_i$ be a finite generating set for G_i. We choose a finite presentation $\langle \mathcal{A}_1, \mathcal{A}_2, t \mid \mathcal{R}\rangle$ for Γ where the only relations involving t are of the form $tat^{-1} = u_a$, where $a \in \mathcal{A}_1 \cup \mathcal{A}_2$ and u_a is a shortest word in the free group on the appropriate $\mathcal{A}_i$ representing $\phi_i^{\varepsilon_i}(a) \in G_i$ where $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$.

For each $i \in \{1, 2\}$ and $n > 0$ we choose the shortest word $w(i, n)$ of length $\leq n$ that attains the maximum in the definition of $\|\phi_i^n\|$. Consider a minimal van Kampen diagram Δ for the word $W_n := [t^{2n}w(1, n)t^{-2n},\ w(2, n)]$. No proper subword of W_n represents $1 \in \Gamma$ and therefore Δ is a topological disc. The pattern of t-corridors in this diagram will be as described on page 12 of [13], and the total number of 2-cells in these corridors is at least the minimum of $\sum_{i=0}^n d_{G_1}(1, \phi_1^i(w(1, n)))$ and $\sum_{i=0}^n d_{G_2}(1, \phi_2^i(w(2, n)))$. Thus Δ has area at least $n^2 \min\{\|\phi_1^n\|, \|\phi_2^n\|\}$ 2-cells. And since $\partial\Delta$ has length $|W_n| \leq 12n$, the proposition is proved. $\square$

The class of groups $\mathcal{G}_n$ was defined immediately prior to (1.6).

Theorem 2.7. *Let $\Gamma \in \mathcal{G}_n$. If $n \geq 3$ then the Dehn function of Γ is $\preceq n^3$. If $n = 2$ the Dehn function of Γ is exponential. And if $n = 1$ then the Dehn function of Γ is linear.*

Proof. The case $n \geq 3$ is an immediate application of Theorem 2.2. In the light of the discussion in (2.5), Proposition 2.6 provides the exponential lower bound needed in the case $n = 2$; the complementary upper bound comes from the fact that the group is asynchronously combable, see [11]. The class $\mathcal{G}_1$ consists of the fundamental groups of closed hyperbolic surfaces. $\square$

We note one other consequence of Proposition 2.6.

Corollary 2.8. *Let G be a finitely generated group and let $\phi \in \mathrm{Aut}\ (G)$. Let $\Phi = (\phi, \phi^{-1}) \in \mathrm{Aut}\ (G \times G)$ and let $\Gamma = (G \times G) \rtimes_\Phi \mathbb{Z}$.*

1. *If $G \rtimes_\phi \mathbb{Z}$ is finitely presented then Γ is finitely presented, and*
2. *the Dehn function of Γ is $\succeq n^2\|\phi^n\|$.*

Proof. (1) can be seen as a special case of (2.3) by noting that $\Gamma \cong G \rtimes H$ where $H = G \rtimes \mathbb{Z}$ acts on G via the obvious map $G \rtimes \mathbb{Z} \to \mathbb{Z} \to \langle\phi\rangle$. And (2) follows from (2.6). $\square$

3. The conjugacy problem for normal subgroups

This section is entirely devoted to proving the following theorem.

Theorem 3.1. *Let G be a bicombable group and let $N \subset G$ be a normal subgroup. If the generalized word problem for G/N is solvable, then N has a solvable conjugacy problem.*

In this theorem we do not assume that N is finitely generated. The set of words in the generators of G that represent elements of N is a recursive set, and we shall describe an algorithm for deciding when two such words represent elements that are conjugate in N. If N is finitely generated (hence recursively presented) then this is equivalent to showing that for each choice of finite generating set for N there is an algorithm to decide which pairs of words in those generators represent conjugate elements of N. Examples to which the theorem applies are discussed at the end of the section. For the sake of clarity, we begin by recalling:

Definition 3.2. Let Γ be a group with a fixed finite generating set. One says that the *generalized word problem* for Γ is solvable if there is an algorithm that takes as input any finite list of words $(x_1, \dots, x_n, y)$ in the given generators and decides whether or not y belongs to the subgroup generated by $\{x_1, \dots, x_n\}$.

One can show that the solvability of the generalized word problem is a property of Γ and not the specific choice of finite generating set. Interesting examples of groups with this property are provided by *lerf* groups, i.e. groups all of whose finitely generated subgroups are closed in the profinite topology. Specific classes of examples were mentioned in the introduction, and we refer to [25] for a thorough survey.

The nature of what we are doing in this section obviously demands that we be careful about our use of language. Thus we define a *choice of generators* for a group G to be a monoid homomorphism $\mu : \mathcal{A}^* \to G$, where $\mathcal{A}^*$ is the free monoid on the set $\mathcal{A}$. We shall always assume that $\mathcal{A}$ is equipped with an involution, written $a \mapsto a^{-1}$, such that $\mu(a^{-1}) = \mu(a)^{-1}$. (This involution extends uniquely to an involutary monoid homomorphism of $\mathcal{A}^*$ with the same property.) When there is no danger of ambiguity one suppresses mention of the map μ and uses phrases such as "$ab = g$ in G" to mean $\mu(ab) = g$, or "u represents g" to mean "$\mu(u) = g$".

For the convenience of the reader we recall what it means for a group to be bicombable [32], [2].

Definition 3.3. Let $\mu : \mathcal{A}^* \to G$ be a choice of generators for the group G; assume that $\mathcal{A}$ is finite and let d be the associated word metric. Associated to each word $w \in \mathcal{A}^*$ one has the discrete path $t \mapsto w_t$, where $t \in \mathbb{N}$ and w_t is the image in G of the prefix of w that has length t (if $t \geq |w|$ then $w_t := \mu(w)$). The group G is said to be *bicombable* if there exists a map of sets $\sigma : G \to \mathcal{A}^*$ and a constant $\lambda > 0$ such that $d(\sigma(g)_t, a.\sigma(a^{-1}ga')_t) \leq \lambda$ for all $a, a' \in \mathcal{A}$, all $g \in G$, and all $t \in \mathbb{N}$.

A subgroup $H \subset G$ is said to be *k-quasiconvex* (with respect to σ) if for every $h \in H$ and every $t \in \mathbb{N}$ one has $d(\sigma(h)_t, H) \leq k$.

It is straightforward to show that the above definition does not depend on the choice of finite generating set (see [32], [2]). Note that there is no requirement that the paths $t \mapsto \sigma(g)_t$ be quasi-geodesic; imposing such a constraint leads to the smaller class of semihyperbolic groups. The class of semihyperbolic groups includes all biautomatic groups [15] and all groups that act properly and cocompactly by isometries on CAT(0) spaces — see [2] and Section III.Γ.4 of [14].

Proposition 3.4. *Let $G, \mathcal{A}, d, \lambda$ and $\sigma : G \to \mathcal{A}^*$ be as in the above definition. Let $B(n)$ denote the ball of radius n about $1 \in G$ and let $V(n)$ denote the cardinality of $B(n)$. Given $S \subset \mathcal{A}^*$, we write $C_G(S)$ to denote the centralizer in G of $\mu(S)$.*

1. *If $d(1, g) \leq n$ then $C_G(g)$ is $V(\lambda n)$-quasiconvex.*
2. *If $H \subset G$ is k-quasiconvex then it is generated by $H \cap B(2k+1)$.*
3. *There is an algorithm which, given any finite set of words $S \subset \mathcal{A}^*$, will compute a finite set of generators for $C_G(S)$.*

Proof. The first two items are proved in [2] and [32] (also [14]) using variations on the techniques developed by Gersten and Short in [19]. These items tell us that, given a word $w \in \mathcal{A}^*$, in order to obtain a set of generators for $C_G(w)$ we need only decide which words u of length at most $2V(\lambda|w|)+1$ are such that $u^{-1}wuw = 1$ in G. And the validity of such an equality can be decided because G has a solvable word problem. To pass to the case of arbitrary finite subsets $S \subset \mathcal{A}^*$, one simply notes that $C_G(S) = \bigcap\{C_G(w) \mid w \in S\}$, and the intersection of any two k-quasiconvex subgroups is $V(2k)$-quasiconvex (see any of the above references). □

In [32] and [2] it is proved that bicombable groups have the q.m.c. property ([14] page 445), and hence:

Proposition 3.5. *If a group G with finite generating set $\mathcal{A}$ is bicombable, then there exists a constant θ such that words $u, v \in \mathcal{A}^*$ represent conjugate elements of G iff there is a word $w \in \mathcal{A}^*$ of length less than $\theta^{\max\{|u|,|v|\}}$ with $w^{-1}uw = v$ in G.*

3.6. The Proof of Theorem 3.1

Let G be a bicombable group with finite generating set $\mathcal{A}$. Let $N \subset G$ be a normal subgroup and suppose that the generalized word problem for G/N is solvable. We may regard $\mathcal{A}$ as a choice of generators for G/N by composing the natural maps $\mathcal{A}^* \to G$ and $G \to G/N$. Because G/N is assumed to have a solvable (generalized) word problem, the words $w \in \mathcal{A}^*$ that represent the identity in G/N form a recursive subset. We shall describe an algorithm that, given two words u_1 and u_2 in this subset, will determine if the elements of N that they represent are conjugate in N.

The first part of the desired algorithm is a straightforward application of (3.5): one checks if u_1 and u_2 are conjugate in G; if they are not then we can stop because they are *a fortiori* not conjugate in N; if they are conjugate in G, then using (3.5) and the fact that G has a solvable word problem, we can algorithmically find a word $w \in \mathcal{A}^*$ such that $w^{-1}u_1w = u_2$ in G. We then ask: how unique is w?

If v is such that $v^{-1}u_1v = u_2$ in G, then $wv^{-1} \in C_G(u_1)$. Conversely, if $z \in C_G(u_1)$ then zw conjugates u_1 to u_2. Thus there exists $v \in N$ with $v^{-1}u_1v = u_2$ if and only if N intersects the coset $C_G(u_1)w \subseteq G$. This intersection is non-empty if and only if the image of w in G/N lies in the image of $C_G(u_1)$. Part 3 of Proposition 3.4 provides an algorithm for computing words whose images in G/N generate the image of $C_G(u_1)$. And since the generalized word problem for G/N is assumed to be solvable, we therefore have an algorithm to decide if w lies in the image of $C_G(u_1)$. $\square$

In the above proof we did not use the full force of the hypothesis that the generalized word problem was solvable in G/N, we only used the fact that one could decide membership of the images of centralizers $C_G(u)$. Thus if one places a hypothesis on G that restricts the nature of these images, then one need only require that G/N admit an algorithm to decide membership of subgroups of the given type. For example, in a hyperbolic group one knows that the centralizer of any non-trivial element is virtually cyclic, and there is a uniform algorithm to decide membership of virtually cyclic subgroups (because there is an algorithm that gives a bound of the quasi-convexity constant), thus we have:

Proposition 3.7. *Let $1 \to N \to \Gamma \to Q \to 1$ be an exact sequence of groups. If Γ and Q are word hyperbolic, then N has a solvable conjugacy problem.*

Examples 3.8. (1) In [6] Bestvina and Brady clarified the relationship between various higher finiteness properties of groups, in particular they constructed finitely generated groups that are not finitely presented but are of type FP_2. Each of their examples is constructed as the kernel of a map $\Gamma \to \mathbb{Z}$ where Γ is the fundamental group of a compact non-positively curved space. According to Theorem 3.1, all of these groups have a solvable conjugacy problem. In particular, the Stallings-Bieri groups (see [7] and page 483 of [14]) all have a solvable conjugacy problem.

(2) There exist hyperbolic groups Γ and finitely generated normal subgroups $N \subset \Gamma$ such that N has an unsolvable conjugacy problem (see [14] III.Γ.5.20). On the other hand, all known examples of finitely presented subgroups of hyperbolic groups have a solvable conjugacy problem. Indeed, the known examples are either hyperbolic or else arise as the kernel of a map to $\mathbb{Z}$ (see [8]), in which case Theorem 3.1 applies.

4. 3-Manifolds of Non-Positive Curvature

Throughout this section, we use the phrase "3-manifold" to mean a 3-dimensional manifold whose boundary may be non-empty.

Theorem 4.1. *If Γ is the fundamental group of a compact irreducible 3-manifold that admits a path-metric of non-positive curvature, then every finitely generated subgroup of Γ is also the fundamental group of such a 3-manifold.*

According to Scott's Compact Core Theorem [31], if M is a compact, connected 3-manifold and $H \subset \pi_1 M$ is finitely generated, then the corresponding covering space $\widetilde{M}/H$ has a compact core. More precisely, there exists a compact, connected 3-manifold N and an embedding $N \hookrightarrow \widetilde{M}/H$ that induces an isomorphism of fundamental groups. If $\pi_2 M = 0$, then $N \hookrightarrow \widetilde{M}/H$ is a homotopy equivalence.

Let X be a compact 3-manifold. If $\pi_1 X$ is torsion-free then X does not contain a copy of $\mathbb{R}\mathrm{P}^2$, and therefore X will be irreducible (and P^2-irreducible) if and only if $\pi_2 X = 0$ and X does not contain a fake 3-cell (i.e. a compact, contractible submanifold with boundary $\mathbb{S}^2$ that is not homeomorphic to the standard 3-ball), see [21]. If X does not contain a fake 3-cell, then neither does any covering space of X (nor any subspace of such a covering). And if $\pi_2 X = 0$ then π_2 of any covering space is trivial. By applying these considerations to M and the compact core of $\widetilde{M}/H$ we get:

Lemma 4.2. *If M is a compact, irreducible 3-manifold and $H \subset \pi_1 M$ is a finitely generated, torsion-free subgroup, then the compact core of $\widetilde{M}/H$ is* P^2*-irreducible.*

Theorem 4.1 is trivial in the case where the given subgroup has finite index in $\pi_1 M$. If the subgroup has infinite index, then the compact core of the corresponding covering will have non-empty boundary. And since the fundamental group of any compact space of non-positive curvature is torsion-free, Lemma 4.2 shows that this core will be P^2-irreducible. Thus Theorem 4.1 follows from:

Theorem 4.3. *Let M be a compact P^2-irreducible 3-manifold. If the boundary of M is non-empty, then M supports a path-metric of non-positive curvature.*

We shall deduce this result from Thurston's Geometrization Theorem for Haken manifolds [26]. The case where the boundary of M has Euler characteristic zero is due to Leeb [24] (see also [23]), and we shall use his result.

Remarks 4.4. (1) The proofs in this section are taken from the notes of my lecture at the workshop "Aspects of Modern Geometry" at the ETH in Zürich in 1995. I also gave a brief sketch of the proof at the Groups St Andrews in Bath meeting (and wrote an overview in my notes [12]). I would like to take this opportunity to thank the organisers of both of these stimulating meetings.

(2) Throughout this article we use the term *non-positive curvature* in the sense of A.D. Alexandrov. In the setting of the above theorem, one can arrange for the metric to be given by a Riemannian metric on the interior of the manifold. On the other hand, one cannot in general construct a metric of non-positive curvature in which the boundary is locally geodesically convex (there may exist arbitrarily small neighbourhoods in which the shortest curve connecting two points in ∂M is longer than the shortest curve connecting them in M). Indeed if the boundary is locally geodesically convex then the fundamental of each component injects into $\pi_1 M$ (see [14] page 201), and this is impossible if $\pi_1 M$ is free for example.

(3) The irreducibiliy hypothesis in (4.3) is probably unnecessary: if M is closed and non-positively curved then one can prove that its universal covering is homeomorphic to $\mathbb{R}^3$, so in particular M is P^2-irreducible; but if M has boundary, then I cannot see a direct argument to rule out the possibility that M might contain a fake 3-cell.

(4) One knows a great deal about the fundamental groups of compact non-positively curved manifolds, and in certain cases this may provide useful information about the fundamental groups of the 3-manifolds in Theorem 4.3. For example, the solution to the conjugacy problem for fundamental groups of compact non-positively curved spaces ([14] page 446) is more efficient than the original solution for knot groups [28]. Of course, one also knows a great deal about 3-manifold groups, so such applications are rather limited.

The Proof of Theorem 4.3.

Let M be a compact P^2-irreducible 3-manifold with non-empty boundary. We are not assuming that ∂M is incompressible. A standard finiteness arguments yields a collection of disjoint, properly embedded, closed 2-discs $D_1, \dots, D_n \subset M$ such that each connected component $M_1, \dots, M_r$ of the complement of an open neighbourhood of these discs has incompressible boundary. Since M is irreducible, if ∂M_i contains a 2-sphere then $\partial M_i = \mathbb{S}^2$ and M_i is a ball. We focus on the components that are not balls — let N be one of them.

N is Haken and therefore satisfies Thurston's Geometrization Conjecture [34]. This means that there is a collection $\mathcal{S}$ of disjoint incompressible tori and Klein bottles such that the components of the complement of an open neighbourhood of these surfaces have interiors that are quotients of the eight 3-dimensional geometries (see [30]). It is common to use the term "piece" to describe these components. The interiors of the pieces that have negative Euler characteristic are geometrically finite hyperbolic manifolds [34] (we shall explain this more precisely in a moment).

In the present setting, instead of cutting N open along $\mathcal{S}$, we cut it open only along the sub-collection $\mathcal{S}' \subset \mathcal{S}$ consisting of those surfaces in $\mathcal{S}$ that lie in the boundary of the pieces with negative Euler characteristic. Thus we cut N into two (perhaps disconnected) manifolds N_1 and N_2, where N_1 has Euler characteristic zero and N_2 is the disjoint union of the pieces with negative Euler characteristic.

Leeb [24] has shown that if a Haken 3-manifold has non-empty boundary and zero Euler characteristic, then it supports a metric of non-positive curvature in which the boundary is totally geodesic (and therefore flat). We assume that N_1 is metrized in this manner.

The following discussion of the pieces with negative Euler characteristic summarizes a number of facts about geometrically finite[3] Kleinian groups, in particular the pleated-surface structure on the components of the boundary of the convex hull of the limit set. These results are due to Bill Thurston [33]; an overview can be found

[3] I am grateful to Brian Bowditch, Lawrence Reeves and Bill Thurston for helpful comments concerning these facts.

in [26], and Chapter 19 of Misha Kapovich's book [22] gives a more complete account. The reader who is unfamiliar with pleated surfaces may also wish to consult [16] (and the other articles in that volume).

If Γ is the fundamental group of a component $W \subset N_2$, then Γ can be realised as a geometrically finite hyperbolic group that has no parabolics of rank one. W is then homeomorphic to the manifold obtained as follows: take the convex hull $\mathcal{C} \subset \mathbb{H}^3$ of the limit set of Γ in $\partial\mathbb{H}^3$, remove a Γ-equivariant system of disjoint open horoballs about the parabolic fixed points of Γ, then form the quotient of the resulting (truncated) space by the action of Γ. The path-metric on the complement of any system of disjoint horoballs in $\mathbb{H}^n$ is non-positively curved ([14] page 362), and hence the quotient metric on W is also. The torus and Klein bottle components of ∂W are the images in W of the horospheres that bound the deleted horoballs in $\mathbb{H}^3$, and the components of ∂W that have higher genus are the images of the boundary components of $\mathcal{C}$. The intrinsic path metric on each component of higher genus makes it locally isometric to the hyperbolic plane. Moreover, on a dense open set in each such component, every point has a neighbourhood in ∂W that is isometrically embedded in W (in other words, these components of ∂W have a dense open set consisting of regions that are totally geodesic in W, namely the complementary regions of the bending lamination).

Notice that in this construction the metric on each component of ∂W with higher genus is determined by the action of Γ on $\mathbb{H}^3$, but the metric on torus and Klein bottle components is only determined up to scaling. Indeed, one is free to replace the original choice of horoballs in $\mathbb{H}^3$ with a Γ-equivariant family of smaller balls centred at the same points of $\partial\mathbb{H}^3$. By doing so one can shrink the metric on the tori and Klein bottles in ∂W by any factors that one likes. From the point of the view of the quotient, this modification may be viewed as follows. $\mathcal{C}/\Gamma$ is a complete manifold of finite volume (possibly with boundary) which has one cusp $C \approx S \times [0, \infty)$ for each torus and Klein bottle component $S \subset \partial W$. We obtained the original metric on W by truncating each cusp in $\mathcal{C}/\Gamma$ with the projection of a horosphere. When we take a concentric horosphere closer to infinity we obtain a metric in which the boundary surface has the same shape but lesser area. As we consider modifications to cusps, it is convenient to give a name, W_0, to our original choice of $W \subset \mathcal{C}/\Gamma$.

We have metrized N_1 and N_2 as compact non-positively curved manifolds. Now we wish to reglue them along the surfaces in $\mathcal{S}'$ so as to obtain a metric of non-positive curvature on N. But here we face the difficulty that the (flat) metrics on the pairs of tori and Klein bottles that are to be glued may be different. To remedy this problem, one adjusts the shape of the boundary tori and Klein bottles in N_2 using warped metrics — this is explained nicely by Schroeder in [29], also [24] and elsewhere. More precisely, one can show that (in the notation of the preceding paragraph) for each cusp $C \approx S \times [0, \infty)$ and each flat Riemannian metric ds_0 on S, there is a constant c_0 such that for all $c < c_0$, one can endow C with a metric of non-positive curvature so that this metric agrees with the original metric on $C \subset \mathbb{H}^3/\Gamma$

near $S \times \{0\}$ and for some $T_0 > 0$ the metric on $S \times [T_0, \infty)$ is the product metric $ds^2 = c\,ds_0^2 + dt^2$. By truncating the cusp, retaining $C_0 = S \times [0, T]$ for some $T > T_0$, we obtain a metric on $W \approx W_0 \cup C_0$ that agrees with the original metric on W_0, has non-positive curvature, and in which the boundary component $S \times \{T\}$ is totally geodesic, and has metric $c\,ds_0^2$.

After making suitable adjustments of this type to all of the cusps of N_2 (and by scaling the metric on N_1 so as to allow for the constants c_0 above), we may assume that each of the pairs of surfaces to be glued in reassembling N from N_1 and N_2 consists of two isometric surfaces that are totally geodesic (i.e. locally geodesically convex) in the N_i. The gluing lemma for CAT(0) spaces [14] page 351 (or simple Riemannian considerations) tells us that the metric on N obtained by gluing N_1 and N_2 has non-positive curvature.

All that remains is to reassemble our original manifold M from the finitely many pieces M_i into which we cut it using discs. (Recall that the manifold N that we were considering above was a generic $M_i \neq \mathbb{B}^3$.)

At this stage we know that each M_i is either a ball (in which case we metrize it as a Euclidean cube), or else can be given a metric of non-positive curvature in which the boundary components with zero Euler characterstic are totally geodesic (hence flat), and the higher genus surfaces are metrized as pleated surfaces. The discs D_i along which we originally cut M, tell us which boundary components we must glue. (Since any two discs on a compact surface are isotopic, we are free to choose where to place the gluing discs on the surfaces that are to be glued.)

There is obviously no problem in gluing pieces with Euclidean boundary (including the cubes): one simply takes a small isometrically embedded Euclidean disc on each of the two surfaces to be identified and glues them by an isometry; [14] page 351 assures us that the resulting metric has non-positive curvature. Likewise, by gluing small isometrically-embedded hyperbolic discs, one can perform the required gluings between higher-genus boundary components.

In order to make the remaining gluings between Euclidean and higher-genus boundary components we use *gluing tubes*. Such a tube τ is the compact subset of $\mathbb{H}^3$ obtained by intersecting three sets: an ε-neighbourhood of a geodesic line ℓ, the exterior of an open horoball H about an endpoint of ℓ, and a closed half-space that contains H and is bounded by a hyperplane orthogonal to ℓ. The path metric induced on $\tau \subset \mathbb{H}^3$ is non-positively curved ([14] page 362). The end of τ on the horosphere is a Euclidean disc (of radius $> \varepsilon$) and the other end of τ is a hyperbolic disc of radius ε; both of these discs are isometrically embedded in τ. Thus if we attach the ends of τ by isometries to isometrically discs in the boundaries of two of the pieces of M, then the induced metric on the resulting manifold will have non-positive curvature.

This completes the construction of a metric of non-positive curvature on M.

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