THE SCHUR MULTIPLIER, PROFINITE COMPLETIONS
AND DECIDABILITY

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ABSTRACT. We fix a finitely presented group $Q$ and consider short exact sequences $1 \rightarrow N \rightarrow \Gamma \rightarrow Q \rightarrow 1$. The inclusion $N \hookrightarrow \Gamma$ induces a morphism of profinite completions $\hat{N} \rightarrow \hat{\Gamma}$. We prove that this is an isomorphism for all $N$ and $\Gamma$ if and only if $Q$ is super-perfect and has no proper subgroups of finite index.

We prove that there is no algorithm that, given a finitely presented, residually finite group $\Gamma$ and a finitely presentable subgroup $P \hookrightarrow \Gamma$, can determine whether or not $\hat{P} \rightarrow \hat{\Gamma}$ is an isomorphism.

INTRODUCTION

The profinite completion of a group $\Gamma$ is the inverse system of the directed system of finite quotients of $\Gamma$; it is denoted $\hat{\Gamma}$. The natural map $\Gamma \rightarrow \hat{\Gamma}$ is injective if and only if $\Gamma$ is residually finite. In [4] Bridson and Grunewald settled a question of Grothendieck [6] by constructing pairs of finitely presented, residually finite groups $u : P \hookrightarrow \Gamma$ such that $\hat{u} : \hat{P} \rightarrow \hat{\Gamma}$ is an isomorphism but $P$ is not isomorphic to $\Gamma$. In this article we address the following related question: given a short exact sequence of groups

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1,$$

with $G$ finitely generated, what conditions on $Q$ ensure that the induced map of profinite completions $\hat{i} : \hat{N} \rightarrow \hat{G}$ is an isomorphism, and how difficult is it to determine when these conditions hold? We shall also investigate the map $\hat{P} \rightarrow \hat{G} \times \hat{G}$ induced by the inclusion of the fibre product $P = \{(g, g') | p(g) = p(g')\}$.

It is easy to see that $\hat{i} : \hat{N} \rightarrow \hat{G}$ is surjective if and only if $Q$ has no non-trivial finite quotients. The following complementary result plays an important role in [9], [1] and [4]; for a proof see [4], p.366.

**Lemma 0.1.** If $Q$ has no non-trivial finite quotients and $H_2(Q; \mathbb{Z}) = 0$, then $\hat{i} : \hat{N} \rightarrow \hat{G}$ is injective.
The K"unneth formula assures us that if $Q$ has no non-trivial finite quotients and $H_2(Q;\mathbb{Z}) = 0$, then $Q \times Q$ has the same properties. Thus, since $P/(N \times N) \cong Q$ and $(G \times G)/(N \times N) \cong Q \times Q$, the lemma implies that the inclusions $P \hookrightarrow N \times N \rightarrow G \times G$ induce isomorphisms $\hat{P} \cong \hat{N} \times \hat{N} \cong G \times G$.

**Corollary 0.2.** If $Q$ has no non-trivial finite quotients and $H_2(Q;\mathbb{Z}) = 0$, then $P \hookrightarrow G \times G$ induces an isomorphism of profinite completions.

The first purpose of this note is to explain why one cannot dispense with the condition $H_2(Q;\mathbb{Z}) = 0$ in these results.

**Theorem A.** Let $Q$ be a finitely presented group. In order that $\hat{i} : \hat{N} \rightarrow \hat{G}$ be an isomorphism for all short exact sequences $1 \rightarrow N \xrightarrow{i} G \rightarrow Q \rightarrow 1$, it is necessary and sufficient that

1. $Q$ has no proper subgroups of finite index, and
2. $H_2(Q;\mathbb{Z}) = 0$.

In Theorem 1.2 we shall prove a similar result with $P \hookrightarrow G \times G$ in place of $N \hookrightarrow G$. These results remain valid if one quantifies over smaller classes of short exact sequences. For example, one can restrict to the case where $N$ is finitely generated and $G$ is finitely presented and residually finite.

It is an open question of considerable interest to determine if there exists an algorithm that can determine, given a finite presentation, whether the group presented has any non-trivial finite quotients or not. On the other hand, it is known that there is no algorithm that, given a finite presentation, can determine whether or not the second homology of the group is trivial [7], [5]. By exploiting a carefully crafted instance of this phenomenon we prove:

**Theorem B.** There exists a finite set $\mathcal{X}$ and recursive sequences $(R_n)$ and $(S_n)$ of finite sets of words of a fixed finite cardinality in the letters $\mathcal{X} \pm 1$, such that:

1. for all $n \in \mathbb{N}$, the group $\Gamma_n = \langle \mathcal{X} \mid R_n \rangle$ is residually finite;
2. for all $n \in \mathbb{N}$, the subgroup $P_n \subset \Gamma_n$ generated by the image of $S_n$ is finitely presentable;
3. there is no algorithm that can determine for which $n$ the map $\hat{P}_n \rightarrow \hat{\Gamma}_n$ induced by inclusion is an isomorphism.

1. **The proof of theorem A**

We are considering short exact sequences $1 \rightarrow N \xrightarrow{i} G \xrightarrow{p} Q \rightarrow 1$ with $G$ finitely generated.

It is clear that $i : \hat{N} \rightarrow \hat{G}$ is onto if and only if $Q$ has no non-trivial finite quotients. Therefore, in the light of Lemma 0.1, the following proposition completes the proof of theorem A.
Proposition 1.1. Let $Q$ be a finitely presented group that has no non-trivial finite quotients. If $H_2(Q;\mathbb{Z})$ is non-trivial, then there is a short exact sequence $1 \to N \to G \to Q \to 1$ of residually finite groups, with $G$ finitely presented and $N$ finitely generated, such that $i : N \to G$ is not injective. There is also such a sequence with $G$ finitely generated and free (but $N$ not finitely generated).

Proof. In the course of the proof we shall construct not only $1 \to N \to G \to Q \to 1$ but also a short exact sequence $1 \to I \to F \to \tilde{Q} \to 1$, where $I$ is contained in $N$ and $\tilde{u} : I \to \tilde{G}$ is surjective ($\tilde{Q}$ has no non-trivial finite quotients). We will show that $N/I$ maps onto some non-trivial finite group, which implies that the morphism $\tilde{u}_N : I \to \tilde{N}$ induced by inclusion is not surjective. Since $\tilde{u} = \tilde{i} \circ \tilde{u}_N$ is surjective, it will follow that $\tilde{i}$ is not injective.

The group $\tilde{Q}$ that we use in this construction is the universal central extension of $Q$. We remind the reader that a central extension of a group $Q$ is a group $\tilde{Q}$ equipped with a homomorphism $\pi : \tilde{Q} \to Q$ whose kernel is central in $\tilde{Q}$. Such an extension is universal if given any other central extension $\pi' : E \to Q$ of $Q$, there is a unique homomorphism $f : \tilde{Q} \to E$ such that $\pi' \circ f = \pi$.

The standard reference for universal central extensions is [8], pp. 43-47. The properties that we need here are these: $Q$ has a universal central extension $\tilde{Q}$ if (and only if) $H_1(Q;\mathbb{Z}) = 0$; there is a short exact sequence $1 \to H_2(Q;\mathbb{Z}) \to \tilde{Q} \xrightarrow{\pi} Q \to 1$; and if $Q$ has no non-trivial finite quotients, then neither does $\tilde{Q}$.

Since $Q$ and $H_2(Q;\mathbb{Z})$ are finitely presented, so is $\tilde{Q}$. We fix a finite presentation $\tilde{Q} = \langle A \mid R \rangle$ and consider the associated short exact sequence $1 \to I \to F \to \tilde{Q} \to 1$, where $F$ is the free group on $A$ and $I$ is the normal closure of $R$. We fix a finite set $Z \subset F$ whose image in $\tilde{Q}$ generates the kernel of $\tilde{\pi} : \tilde{Q} \to Q$ and define $N \subset F$ to be the normal closure of $R \cup Z$. Thus we obtain a short exact sequence $1 \to N \to F \to Q \to 1$.

By construction, $N/I$ is isomorphic to $H_2(Q;\mathbb{Z})$. Thus it is a non-trivial finitely generated abelian group, and hence has a finite quotient.

The proof thus far deals with the case where $G$ is free; it remains to prove that we can instead arrange for $G$ to be finitely presented and $N$ to be finitely generated. For this, one replaces the short exact sequence $1 \to I \to F \to \tilde{Q} \to 1$ in the above argument by the short exact sequence $1 \to I \to \Gamma \to \tilde{Q} \to 1$ furnished by Wise’s version of the Rips construction [10], [11]; cf. the proof of theorem B below. □
In the next section we shall need the following variation on theorem A. The fibre product \( P \subset \Gamma \times \Gamma \) associated to a map \( \pi : \Gamma \to Q \) is the subgroup \( \{ (\gamma, \gamma') \mid \pi(\gamma) = \pi(\gamma') \} \). In the following proof we shall need the (easy) observation that \( P \) is a semi-direct product \( K \rtimes \Gamma \Delta \), where \( K = \ker \pi \times \{ 1 \} \) and \( \Gamma \Delta \) is the diagonal subgroup; if \( K \) is central, this is a direct product.

**Theorem 1.2.** Let \( Q \) be a finitely presented group. In order that, for all short exact sequences \( 1 \to N \xrightarrow{\iota} G \to Q \to 1 \), the inclusion of the associated fibre product induce an isomorphism \( \hat{P} \to \hat{G} \times \hat{G} \), it is necessary and sufficient that

1. \( Q \) has no proper subgroups of finite index, and
2. \( H_2(Q; \mathbb{Z}) = 0 \).

**Proof.** Corollary 0.2 establishes the sufficiency of the given conditions. To establish their necessity we argue, using the notation of the preceding proof, that if \( Q \) has no non-trivial finite quotients and \( H_2(Q; \mathbb{Z}) \neq 0 \), then the map \( \hat{I} \times \hat{I} \to \hat{P} \) induced by inclusion is not surjective. (This is enough because corollary 0.2 tells us that \( \hat{P} \to \hat{G} \times \hat{G} \) and \( \hat{I} \times \hat{I} \to \hat{G} \times \hat{G} \) are surjective.)

\( I \times I \) is the kernel of the map \( G \times G \to \hat{Q} \times \hat{Q} \), which sends \( P \) to the fibre product of \( \pi : \hat{Q} \to Q \). The semi-direct product decomposition described prior to this theorem shows that the fibre product of \( \pi \) is isomorphic to \( H_2(Q; \mathbb{Z}) \times \hat{Q} \). Hence \( P/(I \times I) \) maps onto a non-trivial finite (abelian) group. \( \square \)

2. **An inability to recognise profinite isomorphisms**

Recall that a group presentation is termed *aspherical* if the universal cover of the standard 2-complex of the presentation is contractible. In section 3.2 of [3] an argument of Collins and Miller [5] is used to construct a recursive sequence of finite presentations \( (\Pi'_n) \equiv \langle X \mid \Sigma_n \rangle \) with the following properties; see [3] lemma 3.4.

**Lemma 2.1.** Let \( \Lambda_n \) be the group with presentation \( \Pi'_n \equiv \langle X \mid \Sigma_n \rangle \).

1. For all \( n \in \mathbb{N} \), the group \( \Lambda_n \) has no non-trivial finite quotients.
2. The cardinality of \( \Sigma_n \) is independent of \( n \), and \( |\Sigma_n| > |X| \).
3. If \( \Lambda_n \) is non-trivial then the presentation \( \Pi'_n \) is aspherical.
4. There does not exist an algorithm that determine for which \( n \) the group \( \Lambda_n \) is trivial.

**Corollary 2.2.** If \( \Lambda_n \neq 1 \) then \( H_2(\Lambda_n; \mathbb{Z}) \neq 0 \).

**Proof.** If \( \Lambda_n \neq 1 \) then the presentation \( \Pi'_n \) is aspherical and hence \( H_2(\Lambda_n; \mathbb{Z}) \) may be calculated as the second homology group of the cellular chain complex of the standard 2-complex of \( \Pi'_n \). This complex has no 3-cells and has more 2-cells than 1-cells, so its second homology is a non-trivial free-abelian group. \( \square \)

The following result is Corollary 3.6 of [3]; the proof relies on an argument due to Chuck Miller.
**Proposition 2.3.** There is an algorithm that, given a finite presentation \( \langle A \mid B \rangle \) of a perfect group \( H \), will output a finite presentation \( \langle A \mid B \rangle \) for the universal central extension \( \tilde{H} \) (with \( \tilde{H} \to H \) the map induced by the identity on \( A \)). Furthermore, \( |B| = |X|(1 + |B|) \).

**Remarks 2.4.** (1) The images in \( \tilde{H} \) of the words \( b \in B \) will be central, and together they generate the kernel of \( \tilde{H} \to H \).

(2) If \( H \) has a compact classifying space (as the groups \( \Lambda_n \) do), then \( \tilde{H} \) does also; cf. [3] proposition 1.3.

**The Proof of Theorem B.** Let \( (\tilde{\Pi}_n) \) be the recursive sequence of finite presentations obtained by applying the algorithm of Proposition 2.3 to \( (\Pi'_n) \).

We follow the main construction of [4] with the presentations \( \tilde{\Pi}_n \equiv \langle X \mid \Sigma_n \rangle \) of the groups \( \tilde{\Lambda}_n \) as input. As in [4], Wise’s version of the Rips construction [11] can be used to construct an algorithm that associates to each finite group-presentation \( Q \equiv \langle Y \mid \Sigma \rangle \) a finite presentation \( \mathcal{G} \equiv \langle Y, a_1, a_2, a_3 \mid \Sigma \rangle \) of a residually finite group \( G \) such that \( |\Sigma| = |\Sigma| + 6|Y| \). There is an exact sequence

\[ 1 \to I \to G \to Q \to 1, \]

with \( I = \langle a_1, a_2, a_3 \rangle \), where \( Q \) is the group presented by \( Q \).

We use this algorithm to convert the recursive sequence of presentations \( (\tilde{\Pi}_n) \) from lemma 2.1 into a recursive sequence of presentations \( \langle \mathcal{X} \mid R_n \rangle \) for the groups \( \Gamma_n := G_n \times G_n \), where \( \mathcal{X} \) is the disjoint union of two copies of \( X \cup \{a_1, a_2, a_3\} \) and \( R_n \) is the obvious set of relations arising from the presentation \( G_n = \langle X, a_1, a_2, a_3 \mid \Sigma_n \rangle \). Note that \( |R_n| = 2(|\Sigma_n| + 6|X|) + \frac{1}{2}(|X| + 3)(|X| + 2) \) is independent of \( n \), by lemma 2.1(2) and proposition 2.3.

Let \( N_n \) denote the kernel of the composition \( G_n \to \tilde{\Lambda}_n \to \tilde{\Lambda}_n \). This is generated by \( \{a_1, a_2, a_3\} \cup \{\sigma \mid \sigma \in \Sigma_n\} \); see remark 2.4(1).

Let \( P_n \subset G_n \times G_n \) be the fibre product associated to the short exact sequence \( 1 \to N_n \to G_n \to \tilde{\Lambda}_n \to 1 \). Note that \( P_n \) is generated by the set \( S_n := \{(a_i, 1), (1, a_i) \mid i = 1, 2, 3\} \cup \{ (\sigma, 1) \mid \sigma \in \Sigma_n \} \cup \{ (y, y) \mid y \in Y \} \), whose cardinality does not depend on \( n \). The 1-2-3 theorem of [2] applies in this situation because \( N_n \) is finitely generated, \( G_n \) is finitely presented and \( \tilde{\Lambda}_n \) has a compact classifying space (remark 2.4(2)). The theorem states that under these circumstances \( P \) is finitely presentable.

In the proof of theorem 1.2, we showed that \( \hat{P}_n \to \hat{G}_n \times \hat{G}_n \) is not injective if \( H_2(\Lambda_n; \mathbb{Z}) \neq 0 \), and corollary 2.2 assures us that this is the case if \( \Lambda_n \neq 1 \). On the other hand, if \( \Lambda_n = 1 \) then \( P = G_n \times G_n \). According to lemma 2.1(4), there is no algorithm that can determine, for each \( n \), which of these alternatives holds. Thus theorem B is proved. \( \square \)
References


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