# THE SCHUR MULTIPLIER, PROFINITE COMPLETIONS AND DECIDABILITY

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Abstract. We fix a finitely presented group $Q$ and consider short exact sequences $1 \to N \to \Gamma \to Q \to 1$. The inclusion $N \hookrightarrow \Gamma$ induces a morphism of profinite completions $\hat{N} \to \hat{\Gamma}$. We prove that this is an isomorphism for all $N$ and $\Gamma$ if and only if $Q$ is super-perfect and has no proper subgroups of finite index.

We prove that there is no algorithm that, given a finitely presented, residually finite group $\Gamma$ and a finitely presentable subgroup $P \hookrightarrow \Gamma$, can determine whether or not $\hat{P} \to \hat{\Gamma}$ is an isomorphism.

## Introduction

The profinite completion of a group $\Gamma$ is the inverse system of the directed system of finite quotients of $\Gamma$; it is denoted $\hat{\Gamma}$. The natural map $\Gamma \to \hat{\Gamma}$ is injective if and only if $\Gamma$ is residually finite. In [4] Bridson and Grunewald settled a question of Grothendieck [6] by constructing pairs of finitely presented, residually finite groups $u : P \hookrightarrow \Gamma$ such that $\hat{u} : \hat{P} \to \hat{\Gamma}$ is an isomorphism but $P$ is not isomorphic to $\Gamma$. In this article we address the following related question: given a short exact sequence of groups

$$1 \to N \xrightarrow{\iota} G \xrightarrow{p} Q \to 1,$$

with $G$ finitely generated, what conditions on $Q$ ensure that the induced map of profinite completions $\hat{\iota} : \hat{N} \to \hat{G}$ is an isomorphism, and how difficult is it to determine when these conditions hold? We shall also investigate the map $\hat{P} \to \hat{G} \times \hat{G}$ induced by the inclusion of the fibre product $P = \{(g, g') \mid p(g) = p(g')\}$.

It is easy to see that $\hat{\iota} : \hat{N} \to \hat{G}$ is surjective if and only if $Q$ has no non-trivial finite quotients. The following complementary result plays an important role in [9], [1] and [4]; for a proof see [4], p.366.

**Lemma 0.1.** *If $Q$ has no non-trivial finite quotients and $H_2(Q; \mathbb{Z}) = 0$, then $\hat{\iota} : \hat{N} \to \hat{G}$ is injective.*

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The Künneth formula assures us that if $Q$ has no non-trivial finite quotients and $H_2(Q;\mathbb{Z}) = 0$, then $Q \times Q$ has the same properties. Thus, since $P/(N \times N) \cong Q$ and $(G\times G)/(N\times N) \cong Q\times Q$, the lemma implies that the inclusions $P \leftarrow N \times N \to G \times G$ induce isomorphisms $\hat{P} \cong \hat{N} \times \hat{N} \cong G \times G$.

**Corollary 0.2.** *If $Q$ has no non-trivial finite quotients and $H_2(Q;\mathbb{Z}) = 0$, then $P \hookrightarrow G \times G$ induces an isomorphism of profinite completions.*

The first purpose of this note is to explain why one cannot dispense with the condition $H_2(Q;\mathbb{Z}) = 0$ in these results.

**Theorem A.** *Let $Q$ be a finitely presented group. In order that $\hat{\iota} : \hat{N} \to \hat{G}$ be an isomorphism for all short exact sequences $1 \to N \xrightarrow{\iota} G \to Q \to 1$, it is necessary and sufficient that*

1. *$Q$ has no proper subgroups of finite index, and*
2. *$H_2(Q;\mathbb{Z}) = 0$.*

In Theorem 1.2 we shall prove a similar result with $P \hookrightarrow G \times G$ in place of $N \hookrightarrow G$. These results remain valid if one quantifies over smaller classes of short exact sequences. For example, one can restrict to the case where $N$ is finitely generated and $G$ is finitely presented and residually finite.

It is an open question of considerable interest to determine if there exists an algorithm that can determine, given a finite presentation, whether the group presented has any non-trivial finite quotients or not. On the other hand, it is known that there is no algorithm that, given a finite presentation, can determine whether or not the second homology of the group is trivial [7], [5]. By exploiting a carefully crafted instance of this phenomenon we prove:

**Theorem B.** *There exists a finite set $\mathcal{X}$ and recursive sequences $(R_n)$ and $(S_n)$ of finite sets of words of a fixed finite cardinality in the letters $\mathcal{X}^{\pm 1}$, such that:*

1. *for all $n \in \mathbb{N}$, the group $\Gamma_n = \langle \mathcal{X} \mid R_n \rangle$ is residually finite;*
2. *for all $n \in \mathbb{N}$, the subgroup $P_n \subset \Gamma_n$ generated by the image of $S_n$ is finitely presentable;*
3. *there is no algorithm that can determine for which $n$ the map $\hat{P}_n \to \hat{\Gamma}_n$ induced by inclusion is an isomorphism.*

## 1. The proof of theorem A

We are considering short exact sequences $1 \to N \xrightarrow{\iota} G \xrightarrow{p} Q \to 1$ with $G$ finitely generated.

It is clear that $\hat{\iota} : \hat{N} \to \hat{G}$ is onto if and only if $Q$ has no non-trivial finite quotients. Therefore, in the light of Lemma 0.1, the following proposition completes the proof of theorem A.

**Proposition 1.1.** *Let $Q$ be a finitely presented group that has no non-trivial finite quotients. If $H_2(Q;\mathbb{Z})$ is non-trivial, then there is a short exact sequence $1 \to N \xrightarrow{\iota} G \xrightarrow{p} Q \to 1$ of residually finite groups, with $G$ finitely presented and $N$ finitely generated, such that $\hat{\iota} : \hat{N} \to \hat{G}$ is not injective. There is also such a sequence with $G$ finitely generated and free (but $N$ not finitely generated).*

*Proof.* In the course of the proof we shall construct not only $1 \to N \xrightarrow{\iota} G \xrightarrow{p} Q \to 1$ but also a short exact sequence $1 \xrightarrow{u} I \to G \to \tilde{Q} \to 1$, where $I$ is contained in $N$ and $\hat{u} : \hat{I} \to \hat{G}$ is surjective ($\tilde{Q}$ has no non-trivial finite quotients). We will show that $N/I$ maps onto some non-trivial finite group, which implies that the morphism $\hat{u}_N : \hat{I} \to \hat{N}$ induced by inclusion is not surjective. Since $\hat{u} = \hat{\iota} \circ \hat{u}_N$ is surjective, it will follow that $\hat{\iota}$ is not injective.

The group $\tilde{Q}$ that we use in this construction is the *universal central extension* of $Q$. We remind the reader that a central extension of a group $Q$ is a group $\tilde{Q}$ equipped with a homomorphism $\pi : \tilde{Q} \to Q$ whose kernel is central in $\tilde{Q}$. Such an extension is universal if given any other central extension $\pi' : E \to Q$ of $Q$, there is a unique homomorphism $f : \tilde{Q} \to E$ such that $\pi' \circ f = \pi$.

The standard reference for universal central extensions is [8], pp. 43-47. The properties that we need here are these: $Q$ has a universal central extension $\tilde{Q}$ if (and only if) $H_1(Q;\mathbb{Z}) = 0$; there is a short exact sequence

$$1 \to H_2(Q;\mathbb{Z}) \to \tilde{Q} \xrightarrow{\pi} Q \to 1;$$

and if $Q$ has no non-trivial finite quotients, then neither does $\tilde{Q}$.

Since $Q$ and $H_2(Q;\mathbb{Z})$ are finitely presented, so is $\tilde{Q}$. We fix a finite presentation $\tilde{Q} = \langle A \mid R \rangle$ and consider the associated short exact sequence

$$1 \to I \to F \to \tilde{Q} \to 1,$$

where $F$ is the free group on $A$ and $I$ is the normal closure of $R$. We fix a finite set $Z \subset F$ whose image in $\tilde{Q}$ generates the kernel of $\pi : \tilde{Q} \to Q$ and define $N \subset F$ to be the normal closure of $R \cup Z$. Thus we obtain a short exact sequence

$$1 \to N \to F \to Q \to 1.$$

By construction, $N/I$ is isomorphic to $H_2(Q;\mathbb{Z})$. Thus it is a non-trivial finitely generated abelian group, and hence has a finite quotient.

The proof thus far deals with the case where $G$ is free; it remains to prove that we can instead arrange for $G$ to be finitely presented and $N$ to be finitely generated. For this, one replaces the short exact sequence $1 \to I \to F \to \tilde{Q} \to 1$ in the above argument by the short exact sequence $1 \to I \to \Gamma \to \tilde{Q} \to 1$ furnished by Wise's version of the Rips construction [10], [11]; cf. the proof of theorem B below. ∎

In the next section we shall need the following variation on theorem A. The *fibre product* $P \subset \Gamma \times \Gamma$ associated to a map $\pi : \Gamma \to Q$ is the subgroup $\{(\gamma, \gamma') \mid \pi(\gamma) = \pi(\gamma')\}$. In the following proof we shall need the (easy) observation that $P$ is a semi-direct product $K \rtimes \Gamma^\Delta$, where $K = \ker \pi \times \{1\}$ and $\Gamma^\Delta$ is the diagonal subgroup; if $K$ is central, this is a direct product.

**Theorem 1.2.** *Let $Q$ be a finitely presented group. In order that, for all short exact sequences $1 \to N \xrightarrow{\iota} G \to Q \to 1$, the inclusion of the associated fibre product induce an isomorphism $\hat{P} \to \hat{G} \times \hat{G}$, it is necessary and sufficient that*

(1) *$Q$ has no proper subgroups of finite index, and*
(2) *$H_2(Q; \mathbb{Z}) = 0$.*

*Proof.* Corollary 0.2 establishes the sufficiency of the given conditions. To establish their necessity we argue, using the notation of the preceding proof, that if $Q$ has no non-trivial finite quotients and $H_2(Q; \mathbb{Z}) \neq 0$, then the map $\hat{I} \times \hat{I} \to \hat{P}$ induced by inclusion is not surjective. (This is enough because corollary 0.2 tells us that $\hat{P} \to \hat{G} \times \hat{G}$ and $\hat{I} \times \hat{I} \to \hat{G} \times \hat{G}$ are surjective.)

$I \times I$ is the kernel of the map $G \times G \to \tilde{Q} \times \tilde{Q}$, which sends $P$ to the fibre product of $\pi : \tilde{Q} \to Q$. The semi-direct product decomposition described prior to this theorem shows that the fibre product of $\pi$ is isomorphic to $H_2(Q; \mathbb{Z}) \times \tilde{Q}$. Hence $P/(I \times I)$ maps onto a non-trivial finite (abelian) group. □

## 2. An inability to recognise profinite isomorphisms

Recall that a group presentation is termed *aspherical* if the universal cover of the standard 2-complex of the presentation is contractible. In section 3.2 of [3] an argument of Collins and Miller [5] is used to construct a recursive sequence of finite presentations $(\Pi'_n) \equiv \langle X \mid \underline{\Sigma}_n \rangle$ with the following properties; see [3] lemma 3.4.

**Lemma 2.1.** *Let $\Lambda_n$ be the group with presentation $\Pi'_n \equiv \langle X \mid \underline{\Sigma}_n \rangle$.*

(1) *For all $n \in \mathbb{N}$, the group $\Lambda_n$ has no non-trivial finite quotients.*
(2) *The cardinality of $\underline{\Sigma}_n$ is independent of $n$, and $|\underline{\Sigma}_n| > |X|$.*
(3) *If $\Lambda_n$ is non-trivial then the presentation $\Pi'_n$ is aspherical.*
(4) *There does not exist an algorithm that determine for which $n$ the group $\Lambda_n$ is trivial.*

**Corollary 2.2.** *If $\Lambda_n \neq 1$ then $H_2(\Lambda_n; \mathbb{Z}) \neq 0$.*

*Proof.* If $\Lambda_n \neq 1$ then the presentation $\Pi'_n$ is aspherical and hence $H_2(\Lambda_n; \mathbb{Z})$ may be calculated as the second homology group of the cellular chain complex of the standard 2-complex of $\Pi'_n$. This complex has no 3-cells and has more 2-cells than 1-cells, so its second homology is a non-trivial free-abelian group. □

The following result is Corollary 3.6 of [3]; the proof relies on an argument due to Chuck Miller.

**Proposition 2.3.** *There is an algorithm that, given a finite presentation $\langle A \mid \underline{B}\rangle$ of a perfect group $H$, will output a finite presentation $\langle A \mid B\rangle$ for the universal central extension $\tilde{H}$ (with $\tilde{H} \to H$ the map induced by the identity on $A$). Furthermore, $|B| = |X|(1 + |\underline{B}|)$.*

*Remarks* 2.4. (1) The images in $\tilde{H}$ of the words $b \in B$ will be central, and together they generate the kernel of $\tilde{H} \to H$.

(2) If $H$ has a compact classifying space (as the groups $\Lambda_n$ do), then $\tilde{H}$ does also; cf. [3] proposition 1.3.

**The Proof of Theorem B.** Let $(\tilde{\Pi}_n)$ be the recursive sequence of finite presentations obtained by applying the algorithm of Proposition 2.3 to $(\Pi'_n)$.

We follow the main construction of [4] with the presentations $\tilde{\Pi}_n \equiv \langle X \mid \Sigma_n\rangle$ of the groups $\tilde{\Lambda}_n$ as input. As in [4], Wise's version of the Rips construction [11] can be used to construct an algorithm that associates to each finite group-presentation $\mathcal{Q} \equiv \langle Y \mid \Sigma\rangle$ a finite presentation $\mathcal{G} \equiv \langle Y, a_1, a_2, a_3 \mid \check{\Sigma}\rangle$ of a residually finite group $G$ such that $|\check{\Sigma}| = |\Sigma| + 6|Y|$. There is an exact sequence

$$1 \to I \to G \xrightarrow{p} Q \to 1,$$

with $I = \langle a_1, a_2, a_3\rangle$, where $Q$ is the group presented by $\mathcal{Q}$.

We use this algorithm to convert the recursive sequence of presentations $(\tilde{\Pi}_n)$ from lemma 2.1 into a recursive sequence of presentations $\langle \mathcal{X} \mid R_n\rangle$ for the groups $\Gamma_n := G_n \times G_n$, where $\mathcal{X}$ is the disjoint union of two copies of $X \cup \{a_1, a_2, a_3\}$ and $R_n$ is the obvious set of relations arising from the presentation $G_n = \langle X, a_1, a_2, a_3 \mid \check{\Sigma}_n\rangle$. Note that $|R_n| = 2(|\Sigma_n| + 6|X|) + \frac{1}{2}(|X|+3)(|X|+2)$ is independent of $n$, by lemma 2.1(2) and proposition 2.3.

Let $N_n$ denote the kernel of the composition $G_n \to \tilde{\Lambda}_n \to \Lambda_n$. This is generated by $\{a_1,\, a_2,\, a_3\} \cup \{\sigma \mid \sigma \in \underline{\Sigma}_n\}$; see remark 2.4(1).

Let $P_n \subset G_n \times G_n$ be the fibre product associated to the short exact sequence $1 \to N_n \to G_n \to \Lambda_n \to 1$. Note that $P_n$ is generated by the set $S_n := \{(a_i, 1),\, (1, a_i) \mid i = 1,2,3\} \cup \{(\sigma, 1) \mid \sigma \in \underline{\Sigma}_n\} \cup \{(y, y) \mid y \in Y\}$, whose cardinality does not depend on $n$. The 1-2-3 theorem of [2] applies in this situation because $N_n$ is finitely generated, $G_n$ is finitely presented and $\tilde{\Lambda}_n$ has a compact classifying space (remark 2.4(2)). The theorem states that under these circumstances $P$ is finitely presentable.

In the proof of theorem 1.2, we showed that $\hat{P}_n \to \hat{G}_n \times \hat{G}_n$ is not injective if $H_2(\Lambda_n; \mathbb{Z}) \neq 0$, and corollary 2.2 assures us that this is the case if $\Lambda_n \neq 1$. On the other hand, if $\Lambda_n = 1$ then $P = G_n \times G_n$. According to lemma 2.1(4), there is no algorithm that can determine, for each $n$, which of these alternatives holds. Thus theorem B is proved. □

## References


[1] H. Bass and A. Lubotzky, *Nonarithmetic superrigid groups: counterexamples to Platonov's conjecture*, Annals of Math. **151** (2000), 1151–1173.

[2] G. Baumslag, M.R. Bridson, C.F. Miller III, H. Short, *Fibre products, non-positive curvature, and decision problems*, Comm. Math. Helv. **75** (2000), 457–477.

[3] M.R. Bridson, *Decision problems and profinite completions or groups*, preprint, Oxford, April 2008.

[4] M.R. Bridson and F.J. Grunewald, *Grothendieck's problems concerning profinite completions and representations of groups*, Annals of Math. **160** (2004), 359–373.

[5] D. J. Collins and C. F. Miller III, *The word problem in groups of cohomological dimension 2*, in "Groups St. Andrews in Bath", LMS Lecture Notes 270 (1998), (eds. Campbell, Robertson, Smith), pp. 211–218.

[6] A. Grothendieck, *Représentations linéaires et compactification profinie des groupes discrets*, Manuscripta Math. **2** (1970), 375–396.

[7] C. McA. Gordon, *Some embedding theorems and undecidability questions for groups*, in Combinatorial and Geometric Group Theory (Edinburgh 1993), LMS Lecture Notes **204**. Cambridge (1995), 105–110.

[8] J. Milnor, *'Introduction to Algebraic K-Theory'*, Ann. Math. Studies 72, Princeton University Press, Princeton 1971.

[9] V. Platonov and O.I. Tavgen, *Grothendieck's problem on profinite completions of groups*, Soviet Math. Doklady **33** (1986), 822-825.

[10] E. Rips, *Subgroups of small cancellation groups*, Bull. London Math Soc., **14** (1982), 45-47.

[11] D. Wise, *A residually finite version of Rips's construction*, Bull. London Math. Soc. 35 (2003), 23-29.



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