ASYMPTOTICS BEYOND ALL ORDERS AND STOKES LINES IN NONLINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

J.R. KING* AND S.J. CHAPMAN†

Abstract. A technique for calculating exponentially small terms beyond all orders in singularly perturbed difference equations is presented. The approach is based on the application of a WKBJ-type ansatz to the late terms in the naive asymptotic expansion and the identification of Stokes lines, and is closely related to the well-known Stokes line smoothing phenomenon in linear ordinary differential equations. The method is illustrated by application to examples and the results extended to time-dependent differential-difference problems.

1. Introduction. This paper is concerned with an aysmptotic study of monotonic heteroclinic connections for nonlinear differential-difference equations of the form

$$\epsilon^2 \frac{\partial \phi}{\partial t}(x,t) = \phi(x+1,t) - 2\phi(x,t) + \phi(x-1,t) - \epsilon^2(F(\phi(x,t)) + \lambda), \quad x \in \mathbb{Z}, \tag{1.1}$$

with $\epsilon > 0$. For the types of $F(\phi)$ with which we are concerned the system (1.1) provides a discrete analogue of the bistable reaction-diffusion equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial s^2} - (F(\phi) + \lambda) \tag{1.2}$$

(where $s = \epsilon x$), and has applications as a model problem in dislocation dynamics and in mathematical biology (notably in the propagation of action potentials in nerve cells—see Bell & Cosner [2] and Keener [15], for example), among others. Other pertinent mathematical studies of systems of the form (1.1) include those of Zinner [24] and of Erneux & Nicolis [9]; Keener [16, 17] has recently addressed a more general class of problems using a partially ad hoc method which identifies the crucial role of exponentially small terms but does not account for the Stokes phenomenon.

Our original motivation came chiefly from the application to the motion of dislocations, and the terminology we henceforth adopt is focused on this case. In the dislocation setting, (1.1) is a variant of the Frenkel-Kontorova model [10, 5]. There $\phi(x,t)$ represents the horizontal displacement of the x-th atom in a particular chain of atoms in an elastic crystal, $\phi(x+1,t) - 2\phi(x,t) + \phi(x-1,t)$ represents the elastic force on the atom due to its neighbours in this chain, and $F(\phi(x,t))$ represents the force on it due to the atoms in a neighbouring chain (see Figure 1.1); $F(\phi)$ is periodic with period equal to the equilibrium atomic spacing of the crystal, d, say. The solution in which $\phi \equiv 0$ corresponds to an undislocated perfect lattice, while one in which ϕ varies from 0 to d corresponds to an edge dislocation (see, for example, [14, 5]). Since F arises as the derivative of a potential, its integral over a period is zero (see (2.4) below). Including a non-zero forcing term λ corresponds to applying a uniform shear to the material. Elastic waves in the crystal would be modelled by the addition of a second-order time derivative. Such waves radiate out from a moving dislocation,

*Dept. of Theoretical Mechanics, University of Nottingham, Nottingham NG7 2RD. Supported by a Leverhulme Research Fellowship.
†Mathematical Institute, 24-29 St. Giles', Oxford OX1 3LB. Supported by a Royal Society Research Fellowship.

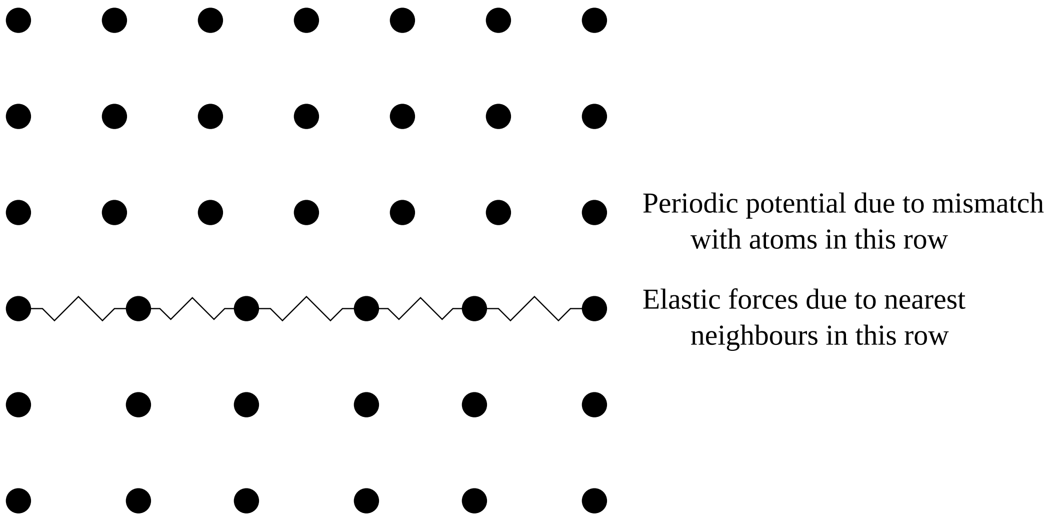


FIG. 1.1. *Diagram illustrating the Frenkel-Kontorova model.*

thereby dissipating energy to infinity. Since the timescale of dislocation motion is much longer than that for elastic wave propagation, and since the model is already very simplistic, we choose to model this dissipation through the first-order time derivative in (1.1).

The key question when using (1.1) to model dislocation dynamics is to determine the speed of a dislocation as a function of the applied stress λ. In particular it is found experimentally that there is a critical value of this stress (known as the Peierls stress) below which the dislocation will not move (see, for example, [14]), although this is typically quite small. We shall see that (1.1) is able to reproduce this phenomenon and shall show that the Peierls stress is exponentially small in the limit $\epsilon \to 0$, its calculation requiring the use of asymptotics beyond all orders.

Existing beyond-all-orders investigations of relevance to (1.1) concern the limit $\epsilon \to 0$ of its steady state form with $\lambda = 0$ (the significance of λ arises from the constraint (2.4) below on $F(\phi)$); see, for example, Lazutkin et al. [21], Amick et al. [1], Hakim & Mallick [13], Gelfreich et al. [11], Nakamura & Hamada [22] and Delshams & Ramírez-Ros [7]. It would be inappropriate to review these studies here; the optimal truncation approach we shall adopt is somewhat different and is, we hope, transparent and widely applicable (cf. Chapman et al. [6]). We touch on such matters again in Section 10.

We now outline the contents of the remainder of the paper. In Sections 2 to 5 we analyse the limit $\epsilon \to 0$ of the steady state problem with $\lambda = 0$ by the methods of [6] for general analytic nonlinearities $F(\phi)$ satisfying (2.4). Steady states and travelling waves for $\lambda \neq 0$ (these being the two possible large-time outcomes) are studied for $\epsilon \to 0$ in Sections 6 and 8 respectively, with the time-dependent problem (which turns out to be instructive to analyse before the travelling waves) being addressed in Section 7. Interactions between dislocations are studied in Section 9 and conclusions presented in Section 10. In the Appendix we briefly analyse the complementary limit $\epsilon \to \infty$, which provides useful supplementary information.

2. The late terms in the expansion for small ϵ. In the next four sections we consider the limit $\epsilon \to 0$ of the nonlinear difference equation

$$\phi(x+1) - 2\phi(x) + \phi(x-1) = \epsilon^2 F(\phi(x)), \tag{2.1}$$

$$\phi(x) \to \phi_- \quad \text{as } x \to -\infty, \tag{2.2}$$

$$\phi(x) \to \phi_+ \quad \text{as } x \to +\infty, \tag{2.3}$$

where $F(\phi_-) = F(\phi_+) = 0$, with $F'(\phi_\pm) > 0$ and

$$\int_{\phi_-}^{\phi_+} F(\phi)\, d\phi = 0. \tag{2.4}$$

We take (without loss of generality) $\phi_- < \phi_+$, with ϕ_- and ϕ_+ being such that there is a heteroclinic orbit connecting them for

$$\begin{cases} \phi_0'' &= F(\phi_0), \\ \phi_0 &\to \phi_- \quad \text{as } s \to -\infty, \\ \phi_0 &\to \phi_+ \quad \text{as } s \to +\infty, \end{cases} \tag{2.5}$$

where $\phi_0 = \phi_0(s)$ and $'$ denotes d/ds; in the dislocation case described above we have $\phi_- = 0$, $\phi_+ = d$. We assume the function F to be real for real ϕ; its precise form is not important for our exposition, but a representative example (which we consider later) is $\sin\phi$. Note that

$$\phi \sim \phi_+ - a \exp(-F'(\phi_+)^{1/2} s) \text{ as } s \to +\infty$$

for some constant a (which can without loss of generality be set equal to one by choice of the origin of s), so that f_2 in (2.17) below grows exponentially as $s \to +\infty$.

We write $s = \epsilon x$ in (2.1), giving

$$\phi(s+\epsilon) - 2\phi(s) + \phi(s-\epsilon) = \epsilon^2 F(\phi(s)), \tag{2.6}$$

$$\phi \to \phi_- \quad \text{as } s \to -\infty, \tag{2.7}$$

$$\phi \to \phi_+ \quad \text{as } s \to +\infty. \tag{2.8}$$

In what follows we impose the condition (2.7) for real s and then seek solutions which also satisfy (2.8) and are monotonic in s; we shall see that the latter can be achieved only by appropriately "sampling" at successive real values of s separated by a distance ϵ. We begin in the usual way by expanding naively in powers of ϵ in the form

$$\phi(s) \sim \sum_{n=0}^{\infty} \epsilon^{2n} \phi_n(s), \tag{2.9}$$

so that ϕ_0 satisfies (2.5) and for $n \geq 1$

$$2 \sum_{m=1}^{n+1} \frac{\phi_{n-m+1}^{(2m)}}{(2m)!} = F'(\phi_0)\phi_n + \cdots, \tag{2.10}$$

$$\phi_n \to 0 \quad \text{as } s \to \pm\infty, \tag{2.11}$$

which determines $\phi_n(s)$ up to translations in s. It can be shown *a posteriori* that successive terms on the right-hand side of (2.10) are $O(1/n)$ relative to the previous one. If we are to truncate the series (2.9) optimally we need to know the behaviour of ϕ_n as $n \to \infty$. It follows from equation (2.10) that the set of singular points of ϕ_n will be the same as that of ϕ_0 for all n. However, since each time we calculate a new term in the asymptotic expansion we differentiate the previous term four times and integrate twice, if ϕ_{n-1} has a singularity of strength p then ϕ_n will have one of strength $p+2$. In consequence, whenever the leading-order solution has a singularity somewhere in the complex plane, the expansion (2.9) will in general be divergent in the form of factorial/power (cf. [8]; see the end of Section 6, however).

Suppose the leading-order solution ϕ_0 has singularities at the points $s = \sigma_i$, $i = 1, 2, \cdots$, with $\sigma_{2j} = \bar{\sigma}_{2j-1}$, where the overbar denotes complex conjugation. Motivated by [6] and the comments above, we seek a solution of (2.10) as $n \to \infty$ of the form

$$\phi_n \sim \frac{(-1)^n \Gamma(2n+\gamma+1)}{u^{2n+\gamma+1}} \left(f(s) + \frac{1}{n} g(s) + \frac{1}{n^2} h(s) + \cdots \right), \tag{2.12}$$

where u and γ may be functions of s but are independent of n (the factor $u^{\gamma+1}$ in the denominator could be absorbed into the series beginning with f, but its inclusion here simplifies the subsequent algebra somewhat). Substituting this ansatz into equation (2.10) we find that the leading-order behaviour as $n \to \infty$ is given by

$$\sum_{m=1}^{\infty} \frac{(-1)^m (u')^{2m}}{(2m)!} = 0. \tag{2.13}$$

Hence $\cos u' = 1$, giving $u' = 2M\pi$ where $M \in \mathbb{Z}$. Since changing the sign of u corresponds simply to modifying the functions f, g etc., we may take $M > 0$ without loss of generality. Since the singular points of ϕ_n correspond to those of ϕ_0, u will be zero at $s = \sigma_i$, for some i, giving $u = 2M\pi(s - \sigma_i)$. In general, the behaviour of ϕ_n will be a sum of expressions of the form (2.12), one for each σ_i and M. However, this sum is dominated by the terms with $M = 1$ and, for real s, by the singularities σ_i which are closest to the real axis, since these have the smallest value of $|u|$. We let σ denote one such nearest singularity in the upper half plane.

To determine γ and f we need to go to higher orders in (2.10) as $n \to \infty$. Balancing terms on the left-hand side of $O(\log n/n)$ gives $\gamma' = 0$, and γ is thus in fact a constant. With $u = 2M\pi(s - \sigma)$, we have from (2.12) that

$$\begin{aligned} \phi_{n-m+1}^{(2m)} \sim{}& \frac{(-1)^{n+m-1}\Gamma(2n+\gamma+3)(u')^{2m}}{u^{2n+\gamma+3}} \left(f + \frac{1}{n} g + \frac{1}{n^2} \left((m-1)g + h \right) \right) \\ &+ \frac{(-1)^{n+m}\Gamma(2n+\gamma+2) 2m (u')^{2m-1}}{u^{2n+\gamma+2}} \left(f' + \frac{1}{n} g' \right) \\ &+ \frac{(-1)^{n+m-1}\Gamma(2n+\gamma+1) 2m(2m-1)(u')^{2m-2}}{2u^{2n+\gamma+1}} f'' \end{aligned} \tag{2.14}$$

as $n \to \infty$; substituting (2.14) into the left-hand side of (2.10) yields

$$\frac{2(-1)^{n-1}\Gamma(2n+\gamma+3)}{u^{2n+\gamma+3}} \left(\left\{ \sum_{m=1}^{\infty} \frac{(-1)^m (u')^{2m}}{(2m)!} \right\} \left(f + \frac{1}{n} g + \frac{1}{n^2} (h - g) \right) \right. \tag{2.15}$$

$$
\begin{aligned}
&\qquad -\frac{u'}{2}\left\{\sum_{m=1}^{\infty}\frac{(-1)^{m-1}(u')^{2m-1}}{(2m-1)!}\right\}\frac{1}{n^2}g\Bigg)\\
&+\frac{2(-1)^{n-1}\Gamma(2n+\gamma+2)}{u^{2n+\gamma+2}}\left\{\sum_{m=1}^{\infty}\frac{(-1)^{m-1}(u')^{2m-1}}{(2m-1)!}\right\}\left(f'+\frac{1}{n}g'\right)\\
&+\frac{(-1)^{n}\Gamma(2n+\gamma+1)}{u^{2n+\gamma+1}}\left\{\sum_{m=0}^{\infty}\frac{(-1)^{m}(u')^{2m}}{(2m)!}\right\}f''
\end{aligned}
$$

as $n \to \infty$. In making this substitution the only terms neglected (due to replacing the upper limits of summation in (2.10) by infinity) are in fact exponentially smaller in n than those included; of course, algebraically smaller terms have been neglected from the series in (2.12) and in writing down (2.14). The dominant term in (2.15) is the one involving the undifferentiated f, and it is the requirement that this vanish which leads to (2.13). Since $\cos u' = 1$, we have that

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}(u')^{2m-1}}{(2m-1)!} = \sin u' = 0.$$

The fact that the first correction term in (2.15) automatically vanishes when the leading one does indicates a non-generic feature of the current problem, with the Stokes smoothing being described by a second-order ordinary differential equation (see (3.13) below), rather than the more usual first-order one. The leading-order balance in (2.10) as $n \to \infty$ is thus simply

$$f'' = F'(\phi_0)f, \tag{2.16}$$

with solutions

$$f_1(s) = \phi_0'(s), \qquad f_2(s;\sigma) = \phi_0'(s)\int_\sigma^s \frac{1}{(\phi_0'(s'))^2}\,ds'. \tag{2.17}$$

This completes the first step in our procedure, namely the determination of the large n behaviour of ϕ_n; the contribution for given M from the singularity at $s = \sigma$ is

$$\phi_n \sim \Lambda_1(M)\frac{\Gamma(2n+\gamma_1+1)(-1)^n}{(2M\pi(s-\sigma))^{2n+\gamma_1+1}}f_1 + \Lambda_2(M)\frac{\Gamma(2n+\gamma_2+1)(-1)^n}{(2M\pi(s-\sigma))^{2n+\gamma_2+1}}f_2 \tag{2.18}$$

as $n \to \infty$. The only unknowns in (2.18) are the constants Λ_1 and Λ_2; a procedure for calculating the dominant one of these will be illustrated shortly. From the discussion following (2.11), we know that if ϕ_0 has a singularity of strength ν (which we assume to be real) at $s = \sigma$, then ϕ_n has one of strength $\nu + 2n$. It therefore follows from (2.18) that $\gamma_1 = -2$ and $\gamma_2 = 2\nu + 1$; in view of the arguments of the gamma functions, if $\nu > -3/2$ (so that the leading-order local behaviour of $\phi_0'(s)$ as $s \to \sigma$ is a singularity of strength $\nu + 1$) then the behaviour of ϕ_n as $n \to \infty$ is dominated by the Λ_2 term of (2.18) with $M = 1$ (provided that $\Lambda_2(1) \neq 0$) and with the right-hand side of (2.18) summed over the singularities nearest to, and equidistant from, the real axis. Provided $\nu > -3/2$, then $f_2(s;\sigma)$ gives the solution to (2.16) which is least singular at $s = \sigma$. If $\nu < -3/2$, then the f_1 contribution dominates the term that is turned on across the Stokes line (corresponding to an exponentially small shift in the origin of s), but since the optimally truncated remainder problem is linear the more important f_2 contribution is still readily incorporated by superposition. For brevity, we shall assume in what follows that $\nu > -3/2$.

3. Optimal truncation. Having determined the behaviour of the expansion (2.9) for large n, we are now in a position to analyse the Stokes phenomenon for (2.6). Following Dingle [8] we expect there to be a Stokes line wherever ϕ_n and ϕ_{n+1} have the same phase as $n \to \infty$, i.e. when $-u^2$ is real and positive; in the current case the relevant Stokes lines are given by $\arg(s-\sigma) = -\pi/2$, $\arg(s-\bar{\sigma}) = \pi/2$.

Our next step is to truncate the series (2.9) optimally. Let R_N be the remainder after $N \gg 1$ terms, i.e.

$$\phi = \sum_{n=0}^{N-1} \epsilon^{2n}\phi_n + R_N. \tag{3.1}$$

Then R_N satisfies

$$R_N(s+\epsilon) - 2R_N(s) + R_N(s-\epsilon) - \epsilon^2 F'(\phi_0)R_N(s) = -2\sum_{p=N}^{\infty} \epsilon^{2p+2} \sum_{m=p-N+2}^{p+1} \frac{\phi_{p-m+1}^{(2m)}}{(2m)!} + \cdots, \tag{3.2}$$

(for the same reasons as for (2.10), the omitted terms on the right-hand side are negligible as $N \to \infty$) the homogeneous version of (3.2) having solutions with the asymptotic behaviour

$$R_N \sim e^{-2\pi i q s/\epsilon} f_1, \qquad R_N \sim e^{-2\pi i q s/\epsilon} f_2 \qquad \text{as } \epsilon \to 0 \tag{3.3}$$

for any integer q. We aim to show how multiples of the latter type are switched on across the Stokes line $\arg(s-\sigma) = -\pi/2$, with the value of q in (3.3) being given by that of M in (2.18).

Using (2.14) and Stirling's formula (since $p \gg 1$), because the range $m = O(1)$ is dominant the right-hand side of (3.2)

$$\sim \frac{2^{\gamma+4}\sqrt{\pi}}{u^{\gamma+3}} \sum_{p=N}^{\infty} \sum_{m=p-N+2}^{\infty} \frac{\epsilon^{2p+2}(-1)^p 2^{2p} p^{2p+\gamma+5/2} e^{-2p}}{u^{2p}} \frac{(-1)^m (u')^{2m}}{(2m)!} f \qquad \text{as } N \to \infty, \tag{3.4}$$

with $f = \Lambda_2 f_2$, $\gamma = \gamma_2$, since, as we have seen, this dominates the behaviour of ϕ_n as $n \to \infty$. Writing $p = N + q$, the range $q = O(1)$ dominates and reversing the order of summation yields

$$\sim \frac{2^{2N+\gamma+4}\sqrt{\pi}(-1)^N N^{2N+\gamma+5/2} e^{-2N} \epsilon^{2N+2}}{u^{2N+\gamma+3}} \sum_{m=2}^{\infty} \sum_{q=0}^{m-2} \frac{\epsilon^{2q}(-1)^q 2^{2q} N^{2q}}{u^{2q}} \frac{(-1)^m (u')^{2m}}{(2m)!} f; \tag{3.5}$$

the two summations are readily calculated exactly, giving

$$\sim \frac{2^{2N+\gamma+5}\sqrt{\pi}(-1)^N N^{2N+\gamma+5/2} e^{-2N} \epsilon^{2N+2}}{u^{2N+\gamma-1}} \frac{\sinh^2(\epsilon N u'/u)}{(2\epsilon N)^2 (u^2 + (2\epsilon N)^2)} f. \tag{3.6}$$

Setting

$$u = 2M\pi(s-\sigma) = re^{i\theta},$$

we wish to truncate optimally by choosing N to minimise (3.6); we find that this requires

$$N = \frac{r}{2\epsilon} + \alpha,$$

where $\alpha \in [0,1)$ needs to be included because N is an integer, which implies that

$$(3.7)\quad R_N(s+\epsilon) - 2R_N(s) + R_N(s-\epsilon) - \epsilon^2 F'(\phi_0)R_N(s) \sim \frac{2^{5/2}\sqrt{\pi}(-1)^N}{\epsilon^{\gamma+1/2}r^{1/2}e^{2iN\theta}e^{i(\gamma-1)\theta}}\frac{\sinh^2(M\pi e^{-i\theta})}{1+e^{2i\theta}}e^{-r/\epsilon}f.$$

Thus the remainder is roughly of order $e^{-r/\epsilon}$. Defining $A(s)$ by

$$R_N = e^{-2iM\pi(s-\sigma)/\epsilon} f A, \tag{3.8}$$

we find the left-hand side of (3.7)

$$\sim \epsilon^2 \left(\frac{d^2}{ds^2}(fA) - F'(\phi_0)fA\right) e^{-2iM\pi(s-\sigma)/\epsilon}. \tag{3.9}$$

Since, treating r as fixed, we have

$$\frac{d}{ds} = -\frac{2M\pi i e^{-i\theta}}{r}\frac{d}{d\theta}, \tag{3.10}$$

equations (3.7) and (3.9) imply

$$(3.11)\quad -\frac{(2M\pi)^2 e^{-i\theta}}{r^2}\frac{d}{d\theta}\left(e^{-i\theta}\frac{d}{d\theta}(fA)\right) - F'(\phi_0)fA \sim \frac{2^{5/2}\sqrt{\pi}(-1)^N}{\epsilon^{\gamma+5/2}r^{1/2}e^{2iN\theta}e^{i(\gamma-1)\theta}}\frac{\sinh^2(M\pi e^{-i\theta})}{1+e^{2i\theta}}e^{ire^{i\theta}/\epsilon}e^{-r/\epsilon}f.$$

We see that the right-hand side is exponentially small in ϵ except near $\theta = -\pi/2$, where it is algebraic in ϵ. Thus the main change in A occurs near this line, which is a Stokes line. To examine the Stokes switching we introduce local coordinates by setting $\theta = -\pi/2 + \hat\epsilon\hat\theta$, with $\hat\epsilon(\epsilon) \ll 1$; the precise order of $\hat\epsilon$ will be determined shortly. Because f varies only slowly with $\hat\theta$, (3.11) gives

$$\frac{(2M\pi)^2}{r^2\hat\epsilon^2}\frac{d^2A}{d\hat\theta^2} \sim 2\sqrt{2\pi}\,\frac{\hat\epsilon\hat\theta(2M\pi)^2 e^{i\pi\gamma/2}}{4r^{1/2}\epsilon^{\gamma+5/2}}e^{-r\hat\theta^2\hat\epsilon^2/(2\epsilon)}. \tag{3.12}$$

Thus we see that the correct scalings are $\hat\epsilon = \epsilon^{1/2}$, $A = \hat A/\epsilon^{\gamma+1}$, giving

$$\frac{d^2\hat A_0}{d\hat\theta^2} = \frac{\sqrt{\pi}\,r^{3/2}e^{i\pi\gamma/2}}{\sqrt{2}}\hat\theta e^{-r\hat\theta^2/2} \tag{3.13}$$

Hence, requiring that $\hat A_0 \to 0$ as $\hat\theta \to -\infty$, we have

$$\hat A_0 = -\frac{\pi e^{i\pi\gamma/2}}{2}\,\mathrm{erfc}(-\hat\theta\sqrt{r/2}). \tag{3.14}$$

This implies that for the outer solution (i.e. for $|\theta+\pi/2| = O(1)$) there is a jump in the coefficient A as we cross the Stokes line given by

$$A(\theta = (-\pi/2)^+) - A(\theta = (-\pi/2)^-) \sim -\frac{\pi e^{i\pi\gamma/2}}{\epsilon^{\gamma+1}}. \tag{3.15}$$

It is worth emphasising that the Stokes line behaviour is described by a second-order equation (3.13), unlike the case with most (but not all) ordinary differential equations [3, 6], for which a first-order equation arises. Nevertheless, the Stokes smoothing is (as usual, cf. [3]) of error function form. In view of (3.8) and (3.15) and the dominance of $M = 1$, the exponentially small, but growing, term

$$R_N(s) \sim -\Lambda_2(1)\frac{\pi e^{i\pi\gamma_2/2}}{\epsilon^{\gamma_2+1}}f_2(s;\sigma)e^{-2i\pi(s-\sigma)/\epsilon} \tag{3.16}$$

is present for $s > 0$.

4. An example. We now consider the implications of this jump for the particular form of F considered in [4], namely $F(\phi) = \sin\phi$. We also use this example to illustrate the procedure for calculating the crucial Stokes multiplier Λ_2. The leading-order problem is

$$\phi_0'' = \sin\phi_0, \tag{4.1}$$

subject to (2.7)-(2.8) with $\phi_- = 0$, $\phi_+ = 2\pi$, with solution, up to a translation in s, given by

$$\phi_0 = 4\tan^{-1} e^s. \tag{4.2}$$

The freedom to translate s will be crucial—see Section 5. The leading-order solution (4.2) has logarithmic singularities at $s = \pm i\pi/2$, so that $\nu = 0$, and there are thus Stokes lines along the imaginary axis emanating from $s = i\pi/2$ and $s = -i\pi/2$.

We now illustrate the procedure for calculating Λ_2; all that is required is the determination of the most singular term in ϕ_n at each of the nearest singularities. We have

$$\phi_0 \sim 2i\log\left(-\frac{2}{s - i\pi/2}\right) \text{ as } s \to i\pi/2, \tag{4.3}$$

and, as $n \to \infty$ with $s \to i\pi/2$, we have the sum of contributions of the form

$$\phi_n \sim -2i(2M\pi)\Lambda_1 \frac{\Gamma(2n-1)(-1)^n}{(2M\pi(s - i\pi/2))^{2n}} + \frac{i\Lambda_2}{6(2M\pi)^2}\frac{\Gamma(2n+2)(-1)^n}{(2M\pi(s - i\pi/2))^{2n}} \tag{4.4}$$

with the dominant behaviour as $n \to \infty$ coming from the second term, with $M = 1$.

In view of (4.3), we now let $s = i\pi/2 + \epsilon\xi$, $\phi = 2i\log(2/\epsilon) + i\psi$, to give in a putative inner region in the complex plane

$$\psi_0(\xi+1) - 2\psi_0(\xi) + \psi_0(\xi-1) = 2e^{\psi_0(\xi)} \tag{4.5}$$

with $\psi_0 \sim -2\log(-\xi)$ as $|\xi| \to \infty$ down the negative imaginary axis. The far-field expansion of this inner solution may be written

$$\psi_0 \sim -2\log(-\xi) + \sum_{n=1}^{\infty} \frac{A_n}{\xi^{2n}},$$

and it follows by matching with (4.4) that

$$\Lambda_2(1) = 6\lim_{n\to\infty} \frac{A_n(-1)^n(2\pi)^{2n+2}}{\Gamma(2n+2)}. \tag{4.6}$$

The recurrence relation for A_n can readily be deduced from (4.5) (this being the only information we require from (4.5)) and by iterating this to large values (≈ 160) of n numerically we find that

$$\Lambda_2 \approx 356.1, \tag{4.7}$$

with Λ_2 real, and (3.16) becomes

$$R_N(s) \sim -\frac{\Lambda_2\pi i}{4\epsilon^2}\left(\sinh s + \left(s - \frac{i\pi}{2}\right)\mathrm{sech} s\right) e^{-2i\pi s/\epsilon} e^{-\pi^2/\epsilon} \tag{4.8}$$

for $s > 0$. The value of Λ_2 above agrees with the established value of Lazutkin's number $c_0 = 2237.6554...$ [21, 20], since his c_0 is $2\pi\Lambda_2$, and with $\omega_0 = 1118.827706...$ given by Gelfreich et al. [11, 12], since their ω_0 is $\pi\Lambda_2$. Gelfreich et al. include higher-order corrections to Λ_2.

For the purposes of subsequent comparison with numerical results, it is very helpful to obtain the $O(\epsilon)$ correction term in Λ_2. As is apparent from (2.9), the expansion proceeds in powers of ϵ^2; the source of the $O(\epsilon)$ correction is an $O(\epsilon^2)$ perturbation in the singularity location σ in (3.16). We note two ways to calculate this. The first involves obtaining the second term in (2.9), giving in the current case

$$\phi \sim 4\tan^{-1} e^s + \epsilon^2 \left(\frac{\sinh s}{4\cosh^2 s} - \frac{s}{12\cosh s} \right), \tag{4.9}$$

where we require $\phi_1(s)$ to be odd. In terms of the inner variables, (4.9) provides the matching condition

$$\psi \sim -2\log(-\xi) - \frac{1}{4\xi^2} + \frac{i\epsilon\pi}{24\xi} \quad \text{as } i\xi \to +\infty,\ \epsilon \to 0; \tag{4.10}$$

since

$$\log\left(-\xi + \frac{i\epsilon\pi}{48}\right) \sim \log(-\xi) - \frac{i\epsilon\pi}{48\xi} \quad \text{as } \epsilon \to 0,$$

the third term in (4.10) can be suppressed by replacing $\sigma = i\pi/2$ by $\sigma = i\pi(1 + \epsilon^2/24)/2$ and writing $s = \sigma + \epsilon\xi$. This is equivalent to replacing Λ_2 in (4.8) by

$$\Lambda_2 e^{-\epsilon\pi^2/24} \sim \Lambda_2 \left(1 - \frac{\epsilon\pi^2}{24}\right) \tag{4.11}$$

and, for reasons already indicated, it is possible to show that this is the only $O(\epsilon)$ correction term present.

The second approach is shorter but perhaps less transparent. If we write $S = 2\sinh^{-1}(\epsilon/2)x$ in place of $s = \epsilon x$, it follows for $F(\phi) = \sin\phi$ that

$$\phi(S) \sim D(\epsilon)e^S \quad \text{as } S \to -\infty \tag{4.12}$$

for some constant D, which implies that

$$\phi_n(S + 2\pi i) = \phi_n(S), \quad \phi_n(S + \pi i) = -\phi_n(S) \tag{4.13}$$

for each n, whereas the (imaginary) periodicity of $\phi(s)$ depends (albeit weakly) on ϵ. The second of (4.13) shows that the relevant singularity is located at $S = i\pi/2$ giving (to all orders) $\sigma = i\epsilon\pi/4\sinh^{-1}(\epsilon/2)$, implying that Λ_2 in (4.8) should be replaced by

$$\Lambda_2 \exp\left(\pi^2 \left(\frac{1}{\epsilon} - \frac{1}{2\sinh^{-1}(\epsilon/2)}\right)\right), \tag{4.14}$$

again leading at $O(\epsilon)$ to (4.11). The approaches just outlined generalise readily to other forms of $F(\phi)$.

5. Solutions of the difference equation. The complementary function (3.16) which is turned on across the Stokes line is exponentially growing as $s \to +\infty$. However, remembering that we are actually trying to solve a difference equation at lattice points a discrete distance ϵ apart (in terms of s), we note we only need to "sample" ϕ at the values $s = (x+a)\epsilon$ for $x \in \mathbb{Z}$, for some real constant a which allows for a change of origin, permitted by the translational invariance of (2.1). Thus, for a solution of the discrete problem (2.1)-(2.3) to exist, we require $R_N((x+a)\epsilon)$ to be bounded for integer $x \to +\infty$ for some a. Using (3.16) we find that, provided there are precisely two singularities of ϕ_0 nearest to the real axis (at $s = \sigma$ and $s = \bar{\sigma}$), then

$$R_N((x+a)\epsilon) \sim -\mathrm{Re}\left[\frac{2\Lambda_2 \pi e^{i\pi\gamma_2/2}}{\epsilon^{\gamma_2+1}} f_2((x+a)\epsilon;\sigma) e^{2i\pi\sigma/\epsilon} e^{-2i\pi a}\right], \tag{5.1}$$

for $x \in \mathbb{Z}$ with $s > \mathrm{Re}(\sigma)$. Hence the condition on a for the solution to satisfy (2.8) is

$$\cos\left(\Phi_2 + \frac{\pi\gamma_2}{2} + \frac{2\pi \mathrm{Re}(\sigma)}{\epsilon} - 2\pi a\right) = 0, \tag{5.2}$$

where $\Phi_2 = \arg(\Lambda_2)$; because $\phi_0' \to 0$ as $s \to +\infty$ it follows from (2.17) that the dominant contribution to f_2 is real for large real s. There are exactly two solutions to this equation with a lying in the range $[0, 1)$. Hence, up to translation of x by integers, there are exactly two solutions to (2.1)-(2.3). We note that, because $f_2((x+a)\epsilon;\sigma) - f_2((x+a)\epsilon;\mathrm{Re}(\sigma))$ is a multiple of $\phi_0'(s)$, when (5.2) is imposed, the quantity turned on is equal for $x \in \mathbb{Z}$, $s > \mathrm{Re}(\sigma)$ to an exponentially small multiple of $\phi_0'(s)$.

When $F(\phi)$ is odd about $(\phi_+ + \phi_-)/2$, we may choose $\phi_0'(s)$ to be even about $s = 0$, in which case $\mathrm{Re}(\sigma) = 0$ and we may take $\phi_n(-s) = -\phi_n(s)$ for $n \geq 1$. It then follows from (2.18) that $\Phi_2 = -\pi(\gamma_2 \pm 1)/2$ (if there are precisely two nearest singularities) and (5.2) has solutions $a = 0$ and $a = 1/2$. This is to be expected since, for this class of F, if $\phi(x)$ satisfies (2.1)-(2.3) then so does $\phi_+ + \phi_- - \phi(-x)$, implying that if a is a possible root then so are $-a$ and $1-a$; the solution with $a = 0$ satisfies

$$\phi(x) = \phi_+ + \phi_- - \phi(-x) \tag{5.3}$$

and that with $a = 1/2$

$$\phi(x) = \phi_+ + \phi_- - \phi(-(1+x)); \tag{5.4}$$

the two steady state solutions in fact possess these symmetry properties for all values of ϵ. To recover these symmetry properties in our asymptotic solution, an exponentially small multiple of $\phi_0'(s)$ needs to be included in the complementary function for R_N in (3.7), corresponding to an exponentially small shift in the origin of s, to give $R_N = 0$ at the centre of the Stokes line (i.e. at $s = 0$, implying at $x = 0$ for $a = 0$ and at $x = -1/2$ for $a = 1/2$). We stress that the asymptotic procedure also applies when $F(\phi)$ is not odd, but the value of Φ_2 is then unknown a priori.

6. Steady states with forcing. Let us consider now the effect of adding a constant forcing term onto the right-hand side of (2.1), so that the problem becomes

$$\phi(x+1) - 2\phi(x) + \phi(x-1) = \epsilon^2(F(\phi(x)) + \lambda), \tag{6.1}$$

$$\phi(x) \to \phi_- \quad \text{as } x \to -\infty, \tag{6.2}$$

$$\phi(x) \to \phi_+ \quad \text{as } x \to +\infty, \tag{6.3}$$

where $F(\phi_-) + \lambda = F(\phi_+) + \lambda = 0$, $F'(\phi_\pm) > 0$, ϕ_- and ϕ_+ (which depend on λ) being neighbouring values of ϕ with these properties. We will find that λ is required to be exponentially small in ϵ for a solution to (6.1)-(6.3) to exist as $\epsilon \to 0$, and in consequence the leading-order quantities $\phi_\pm$ in the next four sections are in fact solutions of $F(\phi) = 0$. For brevity we again consider the case where there are precisely two singularities of ϕ_0 nearest to the real line, at σ and $\bar{\sigma}$ with $\mathrm{Im}(\sigma) > 0$.

Now, away from the Stokes line R_N is given, to leading order, by the sum of solutions to

$$R_N'' - F'(\phi_0)R_N = \lambda, \tag{6.4}$$

and terms of the form (3.3). As we shall see, the regime which needs to be analysed to identify a critical λ for propagation failure requires λ to be of the same order as the Stokes multiplier; see (6.9) below. Hence, for real s, we have

$$R_N \sim \lambda f_2(s;\mathrm{Re}(\sigma)) \int_{-\infty}^{s} f_1(s')\,ds' - \lambda f_1(s) \int_{\mathrm{Re}(\sigma)}^{s} f_2(s';\mathrm{Re}(\sigma))\,ds' + \mathrm{Re}\left(\beta f_2(s;\sigma) e^{-2i\pi(s-\sigma)/\epsilon}\right), \tag{6.5}$$

where, by matching with the solution in the vicinity of the Stokes line (cf. (3.16)), we have

$$\beta = 0 \text{ for } s < 0, \tag{6.6}$$

$$\beta = -\frac{2\Lambda_2 \pi e^{i\pi\gamma_2/2}}{\epsilon^{\gamma_2+1}} \text{ for } s > 0. \tag{6.7}$$

Changing the lower limit for the second integral in (6.5) simply corresponds to changing the origin of s; the lower limit of the first integral is chosen so that R_N is bounded as $s \to -\infty$. Now, in view of (2.17), $f_1(s)\int_{\mathrm{Re}(\sigma)}^{s} f_2(s';\mathrm{Re}(\sigma))\,ds'$ is bounded as $s \to +\infty$ and $f_2(s;\mathrm{Re}(\sigma))\int_{-\infty}^{s} f_1(s')\,ds' = (\phi_0(s) - \phi_-)f_2(s;\mathrm{Re}(\sigma))$. Hence the growing component of R_N as $s \to +\infty$ is

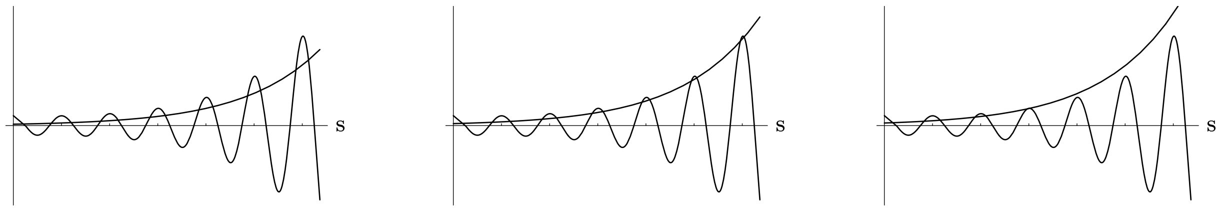
$$\left(\lambda(\phi_+ - \phi_-) - \mathrm{Re}\left[\frac{2\Lambda_2\pi e^{i\pi\gamma_2/2}}{\epsilon^{\gamma_2+1}} e^{-2i\pi(s-\sigma)/\epsilon}\right]\right) f_2(s;\mathrm{Re}(\sigma)). \tag{6.8}$$

As before, we sample only for $s = (x+a)\epsilon$, $x \in \mathbb{Z}$. Hence, for there to be a solution, we require that a satisfy

$$\lambda = \frac{2|\Lambda_2|\pi}{(\phi_+ - \phi_-)\epsilon^{\gamma_2+1}} e^{-2\pi\mathrm{Im}(\sigma)/\epsilon} \cos\left(\Phi_2 + \frac{\pi\gamma_2}{2} + \frac{2\pi\mathrm{Re}(\sigma)}{\epsilon} - 2\pi a\right). \tag{6.9}$$

There are exactly two solutions of this equation for a in the range $[0, 1)$ when $|\lambda|$ is less than the exponentially small quantity

$$\lambda_c^0 \equiv \frac{2|\Lambda_2|\pi}{(\phi_+ - \phi_-)\epsilon^{\gamma_2+1}} e^{-2\pi\mathrm{Im}(\sigma)/\epsilon}; \tag{6.10}$$



(a) $|\lambda| < \lambda_c^0$ (b) $|\lambda| = \lambda_c^0$ (c) $|\lambda| > \lambda_c^0$

FIG. 6.1. *Schematic diagram showing the sum of the first two terms (monotonic) and the third term (oscillatory) in (6.5). For $|\lambda| < \lambda_c^0$ the curves intersect twice per period as $s \to +\infty$ corresponding to the two solutions of (2.6)-(2.8), for $|\lambda| = \lambda_c^0$ the curves intersect only once per period as $s \to +\infty$ corresponding to the single solution of (2.6)-(2.8), while for $|\lambda| > \lambda_c^0$ the curves do not intersect for sufficiently large s and there are no solutions to (2.6)-(2.8).*

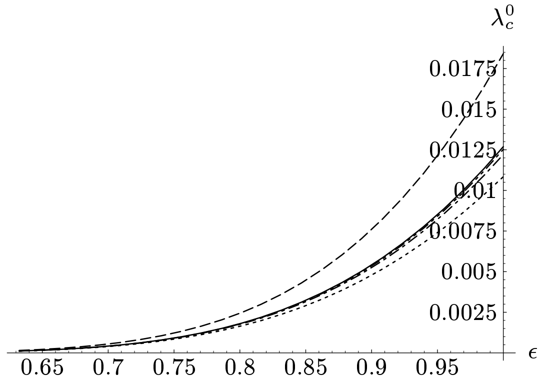
this is perhaps the first time a Peierls stress (given by (6.10)) has been systematically calculated by the methods of asymptotics beyond all orders. When $\lambda = \lambda_c^{(\pm)}(\epsilon)$, where

$$\lambda_c^{(+)}(\epsilon) \sim \lambda_c^0, \qquad \lambda_c^{(-)}(\epsilon) \sim -\lambda_c^0 \qquad \text{as } \epsilon \to 0, \tag{6.11}$$

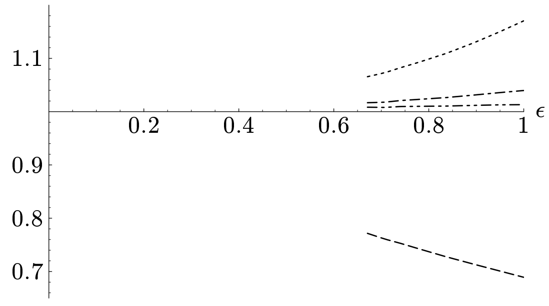
these solutions come together and there is exactly one solution of (6.1)-(6.3). For $\lambda > \lambda_c^{(+)}$ there are no solutions. A schematic diagram of the three possibilities is given in Fig. 6.1; a comparison between asymptotic and numerical results is given in Fig. 6.2. We should note that similar conclusions are likely to apply for any sufficiently small ϵ, whereby in general two solutions to (6.1)-(6.3) exist for $\lambda_c^{(-)}(\epsilon) < \lambda < \lambda_c^{(+)}(\epsilon)$, one for $\lambda = \lambda_c^{(\pm)}(\epsilon)$, and none for $\lambda > \lambda_c^{(+)}(\epsilon)$ or $\lambda < \lambda_c^{(-)}(\epsilon)$. Expressions as $\epsilon \to \infty$ which complement (6.11) are given in the Appendix (see (A.18) and (A.28)).

As already noted, the preceeding analysis assumes that there are precisely two singularities of ϕ_0 nearest the real axis. When there are four or more, the behaviour is essentially of the form above, but the Λ_2 term in (6.8) is replaced by a sum over the contributions from each of the Stokes lines connecting the pairs of singularities with equal real part. However, for special choices of $\epsilon \ll 1$ (for given $F(\phi)$) the contributions from the various Stokes lines may be made to cancel exactly (cf. [6]), implying that no term of the latter form in (3.3) with $q = 1$ is present as $s \to +\infty$ for those particular ϵ. A non-zero term with $q = 2$ will in general still be turned on across the Stokes lines, leading in the absence of the $q = 1$ term to the appearance of $4\pi a$ rather than $2\pi a$ in the analogue of (6.9) and hence to four distinct steady states when $|\lambda|$ is less than the corresponding $\lambda_c^{(\pm)}$; these quantities are then roughly of order $\exp(-4\pi \mathrm{Im}(\sigma)/\epsilon)$, so the Peierls stress will be much reduced for these special values of ϵ. Even less generic scenarios are possible with $6, 8, \ldots$ etc. steady states, according to how many values of q can be exactly suppressed through the combination of Stokes switchings.

Finally we should mention the existence of pathological examples in which the naive expansion (2.9) truncates, and therefore does not exhibit Stokes phenomenon at all, leading to an infinity of solutions. For



(a)



(b)

FIG. 6.2. *(a) The leading-order asymptotic value (6.10) of λ_c^0 (solid) compared with the value $\lambda_c^{(+)}$ obtained by direct numerical simulation (dashed). Also shown is the value of λ_c calculated using the first-order correction to Λ_2 (right-hand side of (4.11) ;dotted), the value calculated by replacing Λ_2 by $\Lambda_2 e^{-\epsilon\pi^2/24}$ as in the left-hand side of (4.11) (dot-dashed), and the value calculated by replacing Λ_2 by (4.14) (dot-dot-dashed). These last three are asymptotic (and equivalent) only up to order ϵ, but including the full exponential is shown to be numerically better for larger ϵ. Since it is difficult to see the agreement due to the decay of $\lambda_c^{(+)}$ with ϵ, in (b) the ratio of the asymptotic value to the value determined numerically is shown. We see that the solution using (4.14) is accurate to about 1% even at $\epsilon = 1$. Numerical calculation becomes difficult for values of ϵ smaller than* 0.6*, due in part to the beyond-all-orders character of the problem.*

example, if

$$F(\phi;\epsilon) = \frac{4}{\epsilon^2}\tan^{-1}\left(\frac{\tanh^2(\epsilon/2)\sin\phi}{1-\tanh^2(\epsilon/2)\cos\phi}\right) \tag{6.12}$$

then $\phi = 4\tan^{-1} e^s$ is the exact solution to (2.6)-(2.8), which can be "sampled" at $s = \epsilon(x+a)$, $x \in \mathbb{Z}$ for arbitrary a, giving a one-parameter family of solutions. However, (6.12) is not really of the form (2.6), since F depends on ϵ. A slightly less artificial example is

$$F = \phi(x) - \frac{1}{2}\phi(x)^2\left(\phi(x+1)+\phi(x-1)\right), \tag{6.13}$$

which also gives an infinity of solutions and in which F is independent of ϵ, although F now depends on $\phi(x+1)$ and $\phi(x-1)$ as well as on $\phi(x)$.

7. Time dependence. Let us consider now the time-dependent behaviour of the solution with an applied force, so that the problem becomes

$$\phi(x+1,t) - 2\phi(x,t) + \phi(x-1,t) = \epsilon^2(\dot{\phi}(x,t) + F(\phi(x,t)) + \lambda), \tag{7.1}$$

$$\phi(x,t) \to \phi_- \quad \text{as } x \to -\infty, \tag{7.2}$$

$$\phi(x,t) \to \phi_+ \quad \text{as } x \to +\infty, \tag{7.3}$$

with suitable initial data, where the dot denotes $\partial/\partial t$ and, as before, $F(\phi_-)+\lambda = F(\phi_+)+\lambda = 0$, $F'(\phi_\pm) > 0$. Then proceeding as before, with $\lambda = O(\lambda_c^0)$, the leading-order solution is (after an initial transient $t = O(1)$

over which the leading-order behaviour is described by the continuum limit (1.2) with $\lambda = 0$)

$$\phi_0 = \phi_0(\eta), \tag{7.4}$$

where $\eta = s - c(t;\epsilon)$ and $c(t;\epsilon)$ is to be determined. As we might expect, dc/dt will prove to be exponentially small (i.e. the evolution occurs on an exponentially slow timescale), and the analysis proceeds as for the steady case until we reach the equation for R_N. The Stokes switching is as before, but equation (6.4) is replaced by

$$R_N'' - F'(\phi_0)R_N = \lambda - \dot{c}\phi_0', \tag{7.5}$$

(here λ and $\dot{c}$ must both be of the same order as the Stokes multiplier to give the necessary non-trivial balance) where $'$ denotes $\partial/\partial\eta$, and the solution for real η (and for exactly two nearest singularities, σ and $\bar{\sigma}$) is of the form

$$\begin{aligned} R_N = {}& f_2(\eta;\mathrm{Re}(\sigma))\int_{-\infty}^{\eta} f_1(\eta')(\lambda - \dot{c}\phi_0'(\eta'))\,d\eta' - f_1(\eta)\int_{\mathrm{Re}(\sigma)}^{\eta} f_2(\eta';\mathrm{Re}(\sigma))(\lambda - \dot{c}\phi_0'(\eta'))\,d\eta' \\ & + \mathrm{Re}\left[\beta f_2(\eta;\sigma)e^{-2i\pi(\eta-\sigma)/\epsilon}\right], \end{aligned} \tag{7.6}$$

where

$$\beta = 0 \text{ for } \eta < 0, \tag{7.7}$$

$$\beta = -\frac{2\Lambda_2\pi e^{i\pi\gamma_2/2}}{\epsilon^{\gamma_2+1}} \text{ for } \eta > 0, \tag{7.8}$$

as before. Hence the coefficient of f_2 as $\eta \to +\infty$ in this case is

$$\lambda(\phi_+ - \phi_-) - \dot{c}\int_{-\infty}^{\infty}(\phi_0')^2\,d\eta - \mathrm{Re}\left[\frac{2\Lambda_2\pi e^{i\pi\gamma_2/2}}{\epsilon^{\gamma_2+1}}e^{-2i\pi(\eta-\sigma)/\epsilon}\right]. \tag{7.9}$$

This time we are sampling at $\eta = \epsilon x - c(t)$, giving the leading-order ordinary differential equation for c

$$\dot{c}\int_{-\infty}^{\infty}(\phi_0')^2\,d\eta = \lambda(\phi_+ - \phi_-) - \mathrm{Re}\left[\frac{2\Lambda_2\pi e^{i\pi\gamma_2/2}}{\epsilon^{\gamma_2+1}}e^{2i\pi\sigma/\epsilon}e^{2i\pi c/\epsilon}\right], \tag{7.10}$$

so that

$$\dot{c}\int_{-\infty}^{\infty}(\phi_0')^2\,d\eta = \lambda(\phi_+ - \phi_-) - \frac{2|\Lambda_2|\pi}{\epsilon^{\gamma_2+1}}e^{-2\pi\mathrm{Im}(\sigma)/\epsilon}\cos\left(\Phi_2 + \frac{\pi\gamma_2}{2} + \frac{2\pi(\mathrm{Re}(\sigma)+c)}{\epsilon}\right). \tag{7.11}$$

The required scalings are thus

$$\lambda = \lambda_c^0\hat{\lambda} = \frac{2|\Lambda_2|\pi}{(\phi_+ - \phi_-)\epsilon^{\gamma_2+1}}e^{-2\pi\mathrm{Im}(\sigma)/\epsilon}\,\hat{\lambda}, \tag{7.12}$$

$$t = \frac{\epsilon}{2\pi\lambda_c^0(\phi_+ - \phi_-)}\int_{-\infty}^{\infty}(\phi_0')^2\,d\eta\,\hat{t}, \tag{7.13}$$

$$c = \frac{\epsilon}{2\pi}\left(\hat{c} - \Phi_2 - \frac{\pi\gamma_2}{2}\right) - \mathrm{Re}(\sigma), \tag{7.14}$$

with $\hat{\lambda}$, $\hat{t}$ and $\hat{c} = O(1)$, giving a "universal" law of motion, whose form is independent of $F(\phi)$, namely

$$\frac{d\hat{c}}{d\hat{t}} = \hat{\lambda} - \cos\hat{c}, \tag{7.15}$$

(the 2π periodicity in (7.15) corresponding to the lattice spacing) with solution for $|\hat{\lambda}| < 1$

$$\hat{c} = -2\tan^{-1}\left(\sqrt{\frac{1-\hat{\lambda}}{1+\hat{\lambda}}}\tanh\left(\frac{(\hat{t}-\mu)\sqrt{1-\hat{\lambda}^2}}{2}\right)\right) \tag{7.16}$$

or

$$\hat{c} = -2\tan^{-1}\left(\sqrt{\frac{1-\hat{\lambda}}{1+\hat{\lambda}}}\coth\left(\frac{(\hat{t}-\mu)\sqrt{1-\hat{\lambda}^2}}{2}\right)\right) \tag{7.17}$$

depending on the initial conditions, which also determine the branch of $\tan^{-1}$ and the value of the constant μ; for $\hat{\lambda} = 1$ we have

$$\hat{c} = -2\cot^{-1}(\hat{t}-\mu), \tag{7.18}$$

and for $\hat{\lambda} > 1$

$$\hat{c} = 2\tan^{-1}\left(\sqrt{\frac{\hat{\lambda}-1}{\hat{\lambda}+1}}\tan\left(\frac{(\hat{t}-\mu)\sqrt{\hat{\lambda}^2-1}}{2}\right)\right). \tag{7.19}$$

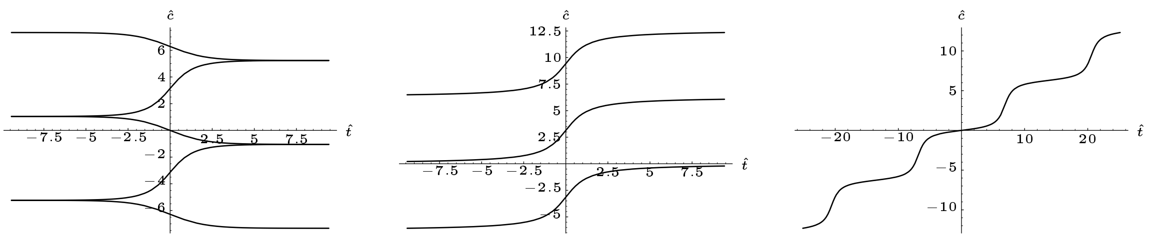
Hence, with $|\hat{\lambda}| \leq 1$, $\hat{c}$ tends to a constant as $\hat{t} \to \infty$ (exponentially quickly for $|\hat{\lambda}| < 1$ but only algebraically so for $\hat{\lambda} = 1$), so that the solution tends to one of the steady states (for $\hat{\lambda} = 1$ the steady state is stable from below but unstable from above). Equation (7.15) also tells us which of the two steady states is stable for $0 \leq \hat{\lambda} < 1$, with $\hat{c} = \cos^{-1}\hat{\lambda} \in (0, \pi/2]$ being unstable and $\hat{c} = -\cos^{-1}\hat{\lambda} \in [-\pi/2, 0)$ being stable, the unstable solution representing a "saddle point" configuration between two energy minima. The quantity a in the steady-state analysis (see (6.9)) is then given (up to integer translations) by

$$a = \frac{1}{2\pi}\left(\Phi_2 + \frac{\pi\gamma_2}{2} + \frac{2\pi\mathrm{Re}(\sigma)}{\epsilon} - \hat{c}\right). \tag{7.20}$$

We are free to translate s to set $\mathrm{Re}(\sigma) = 0$ and, in the case of odd $F(\phi)$, the result $\Phi_2 = -\pi(\gamma_2 \pm 1)/2$ noted above then gives

$$a = \mp\frac{1}{4} - \frac{\hat{c}}{2\pi}; \tag{7.21}$$

This implies in the case of the lower sign that $a \in [0, 1/4)$ is unstable and $a \in (1/4, 1/2]$ is stable, while for the upper sign we have $a \in [1/2, 3/4)$ being unstable and $a \in (3/4, 1]$ stable. The former is consistent with the results of the Appendix for the opposite limit and it seems plausible that the lower sign is the one that actually arises when there are only two nearest singularities (Λ_2 being positive in (4.7) implies that this is the case for $F(\phi) = \sin\phi$), with the branch of steady states starting from (5.3) at $\lambda = 0$ being unstable for all ϵ and that starting from (5.4) being stable. However, with more nearest singularities the Stokes multiplier



(a) $\lambda = 0.5\lambda_c^0$ (b) $\lambda = \lambda_c^0$ (c) $\lambda = 1.1\lambda_c^0$

FIG. 7.1. *Plots of (7.16)-(7.19) illustrating the behaviour of $\hat{c}$ as a function of $\hat{t}$.*

corresponding to $\Lambda_2(1)$ can go through zero (as discussed at the end of Section 6) as ϵ decreases, presumably with an exchange of stabilities; this will happen repeatedly as $\epsilon \to 0+$.

With $\hat{\lambda} > 1$, the speed $\dot{c}$ does not tend to a constant as $t \to +\infty$ since with our current formulation the dislocation "clicks" through the lattice points. To work out the average speed we calculate the time t^* it takes for c to increase by ϵ, which corresponds to the time it takes $\hat{c}$ to increase by 2π, namely $2\pi/\sqrt{\hat{\lambda}^2 - 1}$. In the original scaling this time is given by

$$t^* \sim \frac{\epsilon}{(\phi_+ - \phi_-)\sqrt{\lambda^2 - (\lambda_c^0)^2}} \int_{-\infty}^{\infty} (\phi_0')^2 \, d\eta. \tag{7.22}$$

Hence, in terms of s, the speed of the dislocation

$$\sim \frac{\sqrt{\lambda^2 - (\lambda_c^0)^2}(\phi_+ - \phi_-)}{\int_{-\infty}^{\infty} (\phi_0')^2 \, d\eta} \qquad \text{as } \epsilon \to 0 \tag{7.23}$$

for $\lambda > \lambda_c^0$.

8. Travelling waves. Writing

$$\phi = \phi(\zeta), \qquad \zeta = s - qt \tag{8.1}$$

yields the ordinary differential-difference equation

$$-\epsilon^2 q \frac{d\phi}{d\zeta}(\zeta) = \phi(\zeta + \epsilon) - 2\phi(\zeta) + \phi(\zeta - \epsilon) - \epsilon^2(F(\phi(\zeta)) + \lambda). \tag{8.2}$$

We expect that for $\lambda > \lambda_c^{(+)}$ the large time behaviour can be written in the form (8.1). The analysis of Section 7 did not lead to such a solution because the continuation from integer n to real (and indeed complex) ζ implicit in (8.2) differs from the continuation resulting from replacing differences by derivatives in using a Taylor expansion in ϵ. We return to such matters below. The required soution of (8.2) is of a multiple scales character, with fast scale $z = \zeta/\epsilon$ that of the lattice spacing. Applying the multiple scales approach, with λ and q exponentially small, we have

$$\phi_0 = \phi_0(\hat{\zeta}), \tag{8.3}$$

where ϕ_0 is a solution to (2.5) and

$$\hat{\zeta} = \zeta + \epsilon\Omega(z;\epsilon), \tag{8.4}$$

where Ω satisfies $\Omega(z+1;\epsilon) = \Omega(z;\epsilon)$ with $\Omega(0;\epsilon) = 0$ but otherwise remains to be determined (cf. Kuzmak's method; see for example [18]). The terms $\phi_n(\hat{\zeta})$ in the expansion to all algebraic orders then coincide with those of previous sections, but since

$$\frac{d\phi}{d\zeta} = \frac{1}{\epsilon}\frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial\zeta}, \tag{8.5}$$

the appropriate remainder equation away from the Stokes line now reads

$$R_N'' - F'(\phi_0)R_N = \lambda - q\left(1 + \frac{d\Omega_0}{dz}\right)\phi_0', \tag{8.6}$$

with $'$ denoting $\partial/\partial\hat{\zeta}$. This is identical in form to (7.5) and the solvability conditions which determine q and Ω_0 can essentially be read off from (7.11); since (8.2) holds for all (not just integer) ζ, the exponentially growing term must now be suppressed for all real $\zeta \to \infty$ and, since $\hat{\zeta}/\epsilon = z + \Omega$, this implies that, to leading order,

$$q\left(1 + \frac{d\Omega_0}{dz}\right)\int_{-\infty}^{\infty}(\phi_0')^2\, d\hat{\zeta} = \lambda(\phi_+ - \phi_-) - \frac{2|\Lambda_2|\pi}{\epsilon^{\gamma_2+1}}e^{-2\pi \mathrm{Im}(\sigma)/\epsilon}\cos\left(\Phi_2 + \frac{\pi\gamma_2}{2} + \frac{2\pi\mathrm{Re}(\sigma)}{\epsilon} - 2\pi(z+\Omega_0)\right). \tag{8.7}$$

Hence $z + \Omega_0$ satisfies essentially the same ordinary differential equation as $c(t)$, so that

$$\Omega_0 = -z - \frac{1}{\epsilon}c\left(-\frac{\epsilon z}{q}\right), \tag{8.8}$$

q being determined as having the leading-order value (7.23) by the constraint that $\Omega_0(z+1) = \Omega_0(z)$, and μ chosen to satisfy $\Omega_0(0) = 0$; a plot of Ω_0 is shown in Fig. 8.1. It is noteworthy that in this problem the solvability condition which determines Ω_0 in the leading-order solution (8.3)-(8.4) occurs at exponentially-small orders. The asymptotic approximation to the travelling wave profile is shown in Figure 8.2, along with a direct numerical simulation using the long-time behaviour of a simple explicit forward time-stepping of the difference equation.

It follows from (8.2) that

$$q\int_{-\infty}^{\infty}\left(\frac{d\phi}{d\zeta}\right)^2 d\zeta = \lambda\left(\phi_+(\lambda) - \phi_-(\lambda)\right) + \int_{\phi_-(\lambda)}^{\phi_+(\lambda)} F(\phi)\, d\phi; \tag{8.9}$$

since the right-hand side is positive for $\lambda > 0$, (8.9) implies in particular that

$$q\int_{-\infty}^{\infty}\left(\frac{d\phi}{d\zeta}\right)^2 d\zeta \to \infty \quad \text{as } \lambda \downarrow \lambda_c^{(+)}, \tag{8.10}$$

so the slope becomes infinite for some value of ζ. The constraint (2.4) involves $\phi_-(0)$ and $\phi_+(0)$; taking the limit $\lambda \to 0$ in (8.9), or multiplying (8.7) by $(1 + d\Omega_0/dz)$ and integrating with respect to z over the interval $[0,1]$, yields

$$q\left(1 + \left\langle\left(\frac{d\Omega_0}{dz}\right)^2\right\rangle\right)\int_{-\infty}^{\infty}(\phi_0')^2\, d\hat{\zeta} = \lambda(\phi_+ - \phi_-) \tag{8.11}$$

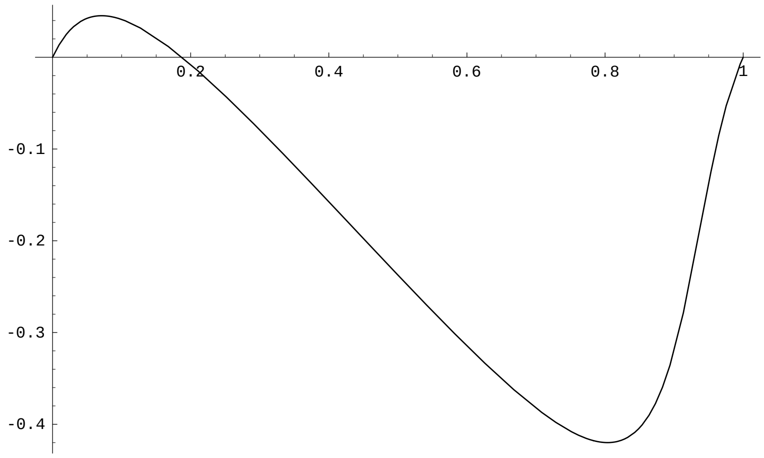


FIG. 8.1. *The profile* Ω_0 *for* $F(\phi) = \sin\phi$, $\epsilon = 0.632$ *and* $\lambda = 5 \times 10^{-5}$.

at leading order, where

$$< \cdot >= \int_0^1 \cdot \, dz.$$

Hence as $\lambda \downarrow \lambda_c^{(+)}$ we have

$$\left\langle \left(\frac{d\Omega_0}{dz} \right)^2 \right\rangle \to \infty,$$

as is apparent from (8.8). The formula (8.9) is well-known in the case of the continuum ("reaction-diffusion") limit of (8.2), namely

$$-q\frac{d\phi}{d\zeta} = \frac{d^2\phi}{d\zeta^2} - (F(\phi) + \lambda), \tag{8.12}$$

$$\phi \to \phi_\pm \quad \text{as } \zeta \to \pm\infty. \tag{8.13}$$

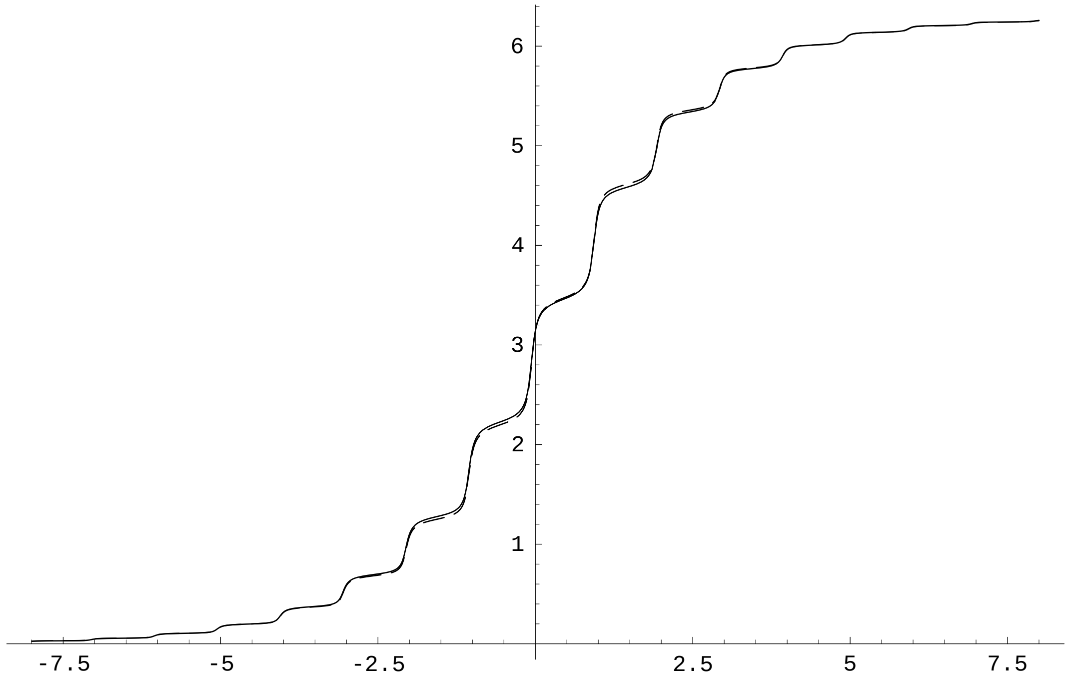


FIG. 8.2. *The profile of the travelling wave obtained from the asymptotic analysis (solid) and by direct numerical simulation (dashed) for* $F(\phi) = \sin\phi$, $\epsilon = 0.632$ *and* $\lambda = 5 \times 10^{-5}$*. In the asymptotic formula we have used the value of* λ_c *calculated to first order using the right-hand side of (4.11). The direct numerical simulation uses the long-time behaviour of a simple explicit forward time-stepping of the difference equation.*

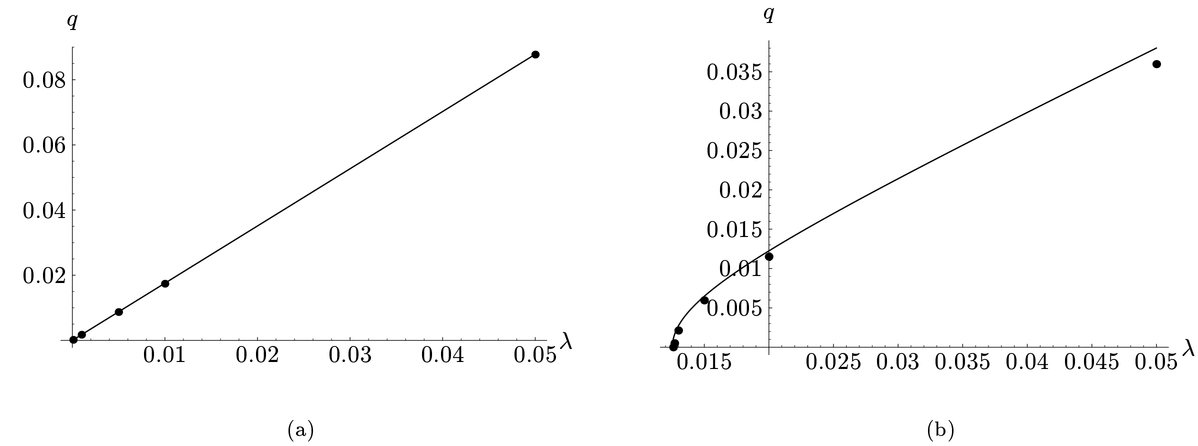


FIG. 8.3. *(a) The speed of the travelling wave for $F(\phi) = \sin\phi$ as a function of λ for (a) $\epsilon = 0.447214$ and (b) $\epsilon = 1$. The dots are numerically calculated values, while the solid curve is the asymptotic value, using (4.14) in case (b).*

When $\lambda = O(1)$ expression (7.23) no longer applies, the wavespeed q being determined in the usual way from the requirement that (8.12)-(8.13) have a solution (so that discreteness plays no role at leading order). A comparison of the numerical and asymptotic values of the speed of the travelling wave is shown in Fig. 8.3; the agreement is remarkably good even for $\epsilon = 1$.

9. Interacting dislocations. Here we discuss the interaction between two neighbouring dislocations, which may take either of the following forms.

(i) Dislocations of opposite type. In this case, on the appropriate exponentially slow timescales we take

$$
\begin{aligned}
\phi &\sim \phi_0(c_+(t) - s) \quad \text{for } s - c_+ = O(1),\\
\phi &\sim \phi_0(s - c_-(t)) \quad \text{for } s - c_- = O(1),
\end{aligned}
\tag{9.1}
$$

with (here and in (ii) below) $c_- < c_+$ and $c_+ - c_- \gg 1$; in (9.1) we have, as before,

$$\phi_0(\eta) \to \phi_\pm \text{ as } \eta \to \pm\infty.$$

(ii) Dislocations of like type. In this case we require there to be an additional stable equilibrium, $\phi = \phi_-^\dagger$ say, and we take

$$
\begin{aligned}
\phi &\sim \phi_0(c_+(t) - s) \quad \text{for } s - c_+ = O(1),\\
\phi &\sim \phi_0^\dagger(s - c_-(t)) \quad \text{for } s - c_- = O(1),
\end{aligned}
\tag{9.2}
$$

where $\phi_0^\dagger$ as a second heteroclinic connection for $\phi'' = F(\phi)$ with

$$\phi_0^\dagger \to \phi_\pm^\dagger \text{ as } \eta \to \pm\infty,$$

with $\phi_-^\dagger < \phi_+^\dagger = \phi_-$.

In each case we locate the origin of η such that

$$
\begin{aligned}
\phi_0(\eta) &\sim \phi_\pm \mp B e^{\mp\mu_\pm \eta} \quad \text{as } \eta \to \pm\infty,\\
\phi_0^\dagger(\eta) &\sim \phi_\pm^\dagger \mp B^\dagger e^{\mp\mu_\pm^\dagger \eta} \quad \text{as } \eta \to \pm\infty,
\end{aligned}
\tag{9.3}
$$

for some constants B, $B^\dagger$, $\mu_\pm$ and $\mu_\pm^\dagger$ with $\mu_+^\dagger = \mu_-$. From (2.17) we thus have (with the notation in the latter case being an obvious one)

$$f_2(\eta;\sigma) \sim \frac{e^{\mu_+\eta}}{2\mu_+ B}, \quad f_2^\dagger(\eta;\sigma^\dagger) \sim \frac{e^{\mu_+^\dagger \eta}}{2\mu_+^\dagger B^\dagger}, \qquad \text{as } \eta \to +\infty. \tag{9.4}$$

The exponentially-growing term turned on crossing the Stokes line in the left-hand dislocation (with $s - c_-$ increasing) should in this case no longer be completely suppressed as $s - c_- \to +\infty$; rather, it must match with the relevant exponential in the right-hand dislocation, given in (9.3). Accordingly, in case (i) we have, retaining all the terms which may be relevant (though it may prove subsequently that they are not all of the same order),

$$(\phi_+ - \phi_-)\lambda_c^0 \left(\hat\lambda - \frac{d\hat c_-}{d\hat t} - \cos\hat c_-\right) \frac{e^{\frac{-\epsilon\mu_+\hat c_-}{2\pi}}}{2\mu_+ B} = -Be^{\frac{-\epsilon\mu_+\hat c_+}{2\pi}}, \tag{9.5}$$

where we have matched what is turned on across the left-hand Stokes line (cf. (7.9), (7.12)-(7.14)) with the relevant exponential in the first of (9.3) with $\eta = \sigma_+ - s \to +\infty$. Defining

$$K = \frac{\mu_+ B^2}{|\Lambda_2|\pi} > 0,$$

and performing the corresponding matching of the quantity turned on as the right-hand Stokes line is crossed with s decreasing, yields the system

$$\begin{aligned} \frac{d\hat c_-}{d\hat t} &= \hat\lambda - \cos\hat c_- + K\epsilon^{\gamma_2+1} e^{2\pi \mathrm{Im}(\sigma)/\epsilon} e^{-\frac{\epsilon\mu_+}{2\pi}(\hat c_+ - \hat c_-)}, \\ \frac{d\hat c_+}{d\hat t} &= -\hat\lambda + \cos\hat c_+ - K\epsilon^{\gamma_2+1} e^{2\pi \mathrm{Im}(\sigma)/\epsilon} e^{-\frac{\epsilon\mu_+}{2\pi}(\hat c_+ - \hat c_-)}. \end{aligned} \tag{9.6}$$

We take $\hat\lambda = O(1)$. It follows from (9.6) that the pertinent scaling for the distance between the dislocations is $c_+ - c_- = O(1/\epsilon)$, so that $\hat c_+ - \hat c_- = O(1/\epsilon^2)$. If we have

$$\epsilon^2 \mu_+(\hat c_+ - \hat c_-) < 4\pi^2 \mathrm{Im}(\sigma)$$

initially, then the final term dominates each of the right-hand sides of (9.6) and the dislocations approach one another at a rate that is exponentially slow in t (this balance being that which occurs for the corresponding continuum model—see, for example, [23]); (9.6) then implies that the dislocations annihilate each other in finite time, with discreteness coming in at leading order over the final stage of the process, for which $\hat c_+ - \hat c_- = O(1)$. If, however,

$$\epsilon^2 \mu_+(\hat c_+ - \hat c_-) > 4\pi^2 \mathrm{Im}(\sigma)$$

holds at $\hat t = 0$ then the two dislocations act independently, being pinned if $|\hat\lambda| < 1$ and evolving to travelling waves of the same speed but opposite direction if $|\hat\lambda| > 1$. There is thus a critical "shielding" distance

$$\epsilon(c_+ - c_-) \sim \frac{2\pi \mathrm{Im}(\sigma)}{\mu_+} \tag{9.7}$$

beyond which the discreteness of the lattice serves to eliminate the attraction between the two dislocations if $\hat{\lambda} < 1$, the effect being absent in the continuum version. If $\hat{\lambda} > 1$ the two dislocations are driven together by the imposed force, so their spacing ultimately drops below (9.7) and they annihilate each other in finite time; for $\hat{\lambda} < -1$ they move apart.

To complete the analysis of case (i) we need to discuss the borderline regime, for which we write

$$
\begin{aligned}
\hat{c}_+ &= \frac{2\pi^2 \mathrm{Im}(\sigma)}{\epsilon^2 \mu_+} + \frac{(\gamma_2+1)\pi\log\epsilon}{\epsilon\mu_+} + \frac{\pi\log K}{\epsilon\mu_+} + \frac{L}{\epsilon} + \kappa_+ + C_+, \\
\hat{c}_- &= -\frac{2\pi^2 \mathrm{Im}(\sigma)}{\epsilon^2 \mu_+} - \frac{(\gamma_2+1)\pi\log\epsilon}{\epsilon\mu_+} - \frac{\pi\log K}{\epsilon\mu_+} - \frac{L}{\epsilon} + \kappa_- + C_-,
\end{aligned}
\tag{9.8}
$$

where L, κ_+ and κ_- are $O(1)$ constants, with κ_+ and κ_- included to make $\hat{c}_+ - C_+$ and $\hat{c}_- - C_-$ multiples of 2π; the origin has been chosen such that $\hat{c}_+ = -\hat{c}_- + O(1)$ and the choice of the value of L will play a key role. Moreover, we shall subsequently need to allow L to vary on the long timescale $\hat{t} = O(1/\epsilon)$, pursuing (without going into details) a multiple timescale approach. To leading order, C_+ and C_- satisfy

$$
\begin{aligned}
\frac{d\hat{C}_-}{d\hat{t}} &= \hat{\lambda} + \hat{\lambda}_L - \cos\hat{C}_-, \\
\frac{d\hat{C}_+}{d\hat{t}} &= -\hat{\lambda} - \hat{\lambda}_L + \cos\hat{C}_+,
\end{aligned}
\tag{9.9}
$$

which thus decouple, being equivalent to (7.15) with $\hat{\lambda}$ shifted by $\hat{\lambda}_L \equiv \exp(-\mu_+ L/\pi) > 0$; the force on each dislocation is made up of $\hat{\lambda}$ (the externally imposed force) and $\hat{\lambda}_L$ (the force due to the other dislocation). If the initial conditions have $\hat{\lambda} + \hat{\lambda}_L > 1$ then the two dislocations move together, the evolution over the longer timescale $\hat{t} = O(1/\epsilon)$ proceeding through successively larger values of $\hat{\lambda}_L$, whereby

$$
\frac{d}{d\hat{t}}\left(\frac{L}{\epsilon}\right) = \left(\left(e^{-\mu_+ L/\pi} + \hat{\lambda}\right)^2 - 1\right)^{1/2}
\tag{9.10}
$$

(because, in terms of $\hat{c}(\hat{t})$, the speed (7.23) is $(\hat{\lambda}^2 - 1)^{1/2}$), until the $\hat{\lambda}_L$ terms in (9.9) become dominant, leading to the annihilation regime outlined above. If $|\hat{\lambda} + \hat{\lambda}_L| < 1$ the two dislocations are pinned, while if $\hat{\lambda} < -1 - \hat{\lambda}_L$ they move apart, with $\hat{\lambda}_L$ decreasing on the timescale $\hat{t} = O(1/\epsilon)$, in a manner described by (9.10) with the negative square root on the right-hand side; in this case the $\hat{\lambda}_L$ terms in (9.9) ultimately become negligible. Hence if $\hat{\lambda} > 1$ the dislocations always annihilate, if $|\hat{\lambda}| < 1$ they are pinned if the initial conditions are such that $|\hat{\lambda} + \hat{\lambda}_L| < 1$ but annihilate if $\hat{\lambda} + \hat{\lambda}_L > 1$, and if $\hat{\lambda} < -1$ they move apart if $\hat{\lambda} + \hat{\lambda}_L < -1$, are pinned if $|\hat{\lambda} + \hat{\lambda}_L| < 1$ and annihilate if $\hat{\lambda} + \hat{\lambda}_L > 1$; the various possibilities described in Section 7 can thus all arise here witin the same problem, simply by varying the initial spacing.

In the continuum version of this problem an unstable homoclinic connection exists which can be regarded as providing the borderline between those solutions in which annihilation occurs and those in which two travelling waves move apart. On introducing discreteness, this single connection is replaced in the range $|\hat{\lambda} + \hat{\lambda}_L| < 1$ by a large family of stable "pinned" configurations (for two dislocations, only a quarter of the steady states will be stable, since stable and unstable states alternate within each half of the connection). The effects of discreteness are thus particularly pronounced in this context, generating stable homoclinic connections, with the exponential sensitivity of the interactions between widely spaced heteroclinic connections (even in the continuum case) implying that the consequences of discreteness, though exponentially small, are long-range and qualitatively significant.

The analysis for case (ii) proceeds in a very similar way, and we restrict ourselves for brevity to the case in which $F(\phi)$ is odd about ϕ_- (so that $\phi_-^\dagger = 2\phi_- - \phi_+$ etc.). In this case (9.6) is replaced by

$$\begin{aligned} \frac{d\hat{c}_-}{d\hat{t}} &= \hat{\lambda} - \cos\hat{c}_- - K\epsilon^{\gamma_2+1}e^{2\pi\mathrm{Im}(\sigma)/\epsilon}e^{-\frac{\epsilon\mu_+}{2\pi}(\hat{c}_+-\hat{c}_-)}, \\ \frac{d\hat{c}_+}{d\hat{t}} &= \hat{\lambda} - \cos\hat{c}_+ + K\epsilon^{\gamma_2+1}e^{2\pi\mathrm{Im}(\sigma)/\epsilon}e^{-\frac{\epsilon\mu_+}{2\pi}(\hat{c}_+-\hat{c}_-)}. \end{aligned} \tag{9.11}$$

Now if

$$\epsilon^2\mu_+(\hat{c}_+ - \hat{c}_-) > 4\pi^2\mathrm{Im}(\sigma)$$

the dislocations act independently and, since they are of the same type, they move with the same velocity for $|\hat{\lambda}| > 1$ and hence to leading order maintain their original spacing whether they move ($|\hat{\lambda}| > 1$) or are pinned ($|\hat{\lambda}| < 1$). If

$$\epsilon^2\mu_+(\hat{c}_+ - \hat{c}_-) < 4\pi^2\mathrm{Im}(\sigma)$$

initially then the dislocations repel (in a manner described by the continuum model) until the inequality ceases to be satisfied and effects due to discreteness come into play. This is best described in terms of the transition regime scaling (9.8) which now yields

$$\begin{aligned} \frac{d\hat{C}_-}{d\hat{t}} &= \hat{\lambda} - \hat{\lambda}_L - \cos\hat{C}_-, \\ \frac{d\hat{C}_+}{d\hat{t}} &= \hat{\lambda} + \hat{\lambda}_L - \cos\hat{C}_+. \end{aligned} \tag{9.12}$$

Restricting attention to $\hat{\lambda} \geq 0$ without loss of generality, if the initial conditions satisfy $|\hat{\lambda} \pm \hat{\lambda}_L| < 1$ then both dislocations are pinned. If $-1 < \hat{\lambda} - \hat{\lambda}_L < 1 < \hat{\lambda} + \hat{\lambda}_L$ initially then the left-hand dislocation is pinned but the right-hand one moves to the right; in terms of L, the spacing then increases on the timescale $\hat{t} = O(1/\epsilon)$, corresponding to a decrease in $\hat{\lambda}_L$. Hence if $\hat{\lambda} < 1$ the right-hand dislocation ceases moving when $\hat{\lambda}_L$ drops to $1 - \hat{\lambda}$, whereas if $\hat{\lambda} > 1$ the left-hand dislocation starts moving to the right when $\hat{\lambda}_L$ drops below $\hat{\lambda} - 1$, with $\hat{\lambda}_L$ ultimately becoming negligible. Finally, if $\hat{\lambda} - \hat{\lambda}_L < -1$ holds at $\hat{t} = 0$ then the left-hand dislocation initially moves to the left with $\hat{\lambda}_L$ again decreasing on the timescale $\hat{\tau} = \epsilon\hat{t} = O(1)$, so if $\hat{\lambda} > 1$ it is pinned while $\hat{\lambda} - 1 < \hat{\lambda}_L(\hat{\tau}) < \hat{\lambda} + 1$ and then moves to the right for $\hat{\lambda}_L(\hat{\tau}) < \hat{\lambda} - 1$; the other details should now be apparent. The key points to make about the role of discreteness in this case are that if the critical spacing (9.7) is initially exceeded then the dislocations act independently, whereas if they start closer than (9.7) they repel until the critical spacing is attained and then act as if they were a bound pair, maintaining their distance. The expression (9.7) can therefore be regarded as giving an equilibrium spacing between like dislocations.

The analysis above readily generalises to interactions between multiple dislocations. If, for example, we begin with an array of dislocations of alternating type, with $\lambda = 0$, then those which start sufficiently close together will annihilate, so that the array undergoes "phase coarsening"; however, in contrast to the continuum version, this process will cease when the spacing between all neighbours exceeds the shielding distance (9.7). Such steady dislocation arrays evidently correspond to non-monotonic heteroclinic connections for (2.1).

10. Discussion. As we have already indicated, there have been a number of previous beyond all orders studies of equations of the form (2.1) using somewhat different methods from those applied here. The following features of the current (optimal truncation/matched asymptotic expansions) approach are accordingly worth highlighting.

(i) We have been able to address a whole class of nonlinearities $F(\phi)$ at the same time; only the calculation of Λ_2 depends on any of the details of $F(\phi)$ and even this involves only its leading-order behaviour at a singularity.

(ii) The complementary error function smoothing across the Stokes line (cf. [3]) has been made manifest for what we believe is the first time in the current, discrete, context.

(iii) Our approach does not necessitate consideration of the solution off the real line (even the calculation of Λ_2 via formulations such as (4.5) can be viewed purely as an algorithm for determining the coefficient of the late terms in the expansion (2.9)—they need not (and in some circumstances should not) be interpreted as describing the behaviour of ϕ in some inner region in the complex plane). This makes the method very well-suited to being generalised to time-dependent problems, in particular, as has been done in Sections 7-9 above.

Most of our remaining comments relate to the features that arise specifically from the discreteness of (1.1). We now write (2.6) in the form

$$(10.1) \qquad \sum_{m=1}^{\infty} \frac{2\epsilon^{2m}}{(2m)!}\frac{d^{2m}\phi}{ds^{2m}} \quad \left(= 4\sinh^2\left(\frac{\epsilon}{2}\frac{d}{ds}\right)\phi\right) = \epsilon^2 F(\phi)$$

(care obviously needs to be taken in attaching meaning to the infinite sum; here we interpret it as being asymptotic in the limit $\epsilon \to 0$). A common ad hoc way to seek to mimic some of the effects of discreteness within a continuum framework is to truncate such a sum after a finite number of terms, $m = m_T$ say. The asymptotic solution of the resulting differential equation will reproduce that of (2.6) correctly up to $n = m_T - 1$ in (2.9); nevertheless, such approaches have the very serious drawback that however large m_T is chosen they fail to capture correctly phenomena which occur beyond all orders, the importance of which has been indicated above. The fact that discreteness features in an essential way within the analysis above can be illustrated by the infinite sum (2.13) which determines u' and in the appearance of the sinh in (3.6). Because

$$\frac{d\phi}{ds}\frac{d^{2m}\phi}{ds^{2m}} = \frac{d}{ds}\left\{\sum_{k=1}^{m-1}(-1)^{k-1}\frac{d^k\phi}{ds^k}\frac{d^{2m-k}\phi}{ds^{2m-k}} + \frac{1}{2}(-1)^{m-1}\left(\frac{d^m\phi}{ds^m}\right)^2\right\},$$

equation (10.1) can, at any order, be integrated once, providing ready confirmation that ϕ_n in (2.9) does exist at any order. While this procedure can be applied term by term to the asymptotic expansion, applying it directly to (2.6) by multiplying by $d\phi/ds$ and integrating (the corresponding approach for (8.2) leading to (8.9)) requires $d\phi/ds$ to be defined and this leads to the issue of how the solution to (2.6) is to be continued away from lattice points. The representation (10.1), in which s is treated as continuous, provides one such continuation, but it is evidently not the only one. Indeed, the transformation

$$\hat{s} = s + \epsilon\Omega(s/\epsilon;\epsilon) \tag{10.2}$$

maps (2.6) into itself for any function Ω which satisfies $\Omega(x+1;\epsilon) = \Omega(x;\epsilon)$ (so that $\Omega(x;\epsilon) = \sum_{k=-\infty}^{\infty} a_k(\epsilon)e^{2ik\pi x}$ for some a_k); the difference equation can then be written in the form (10.1) with s replaced by $\hat{s}$, giving an infinity of continuations away from the lattice points, corresponding to different ways of sampling off the lattice points. On seeking travelling wave solutions (8.1), the ordinary differential-difference equation (8.2) by necessity holds for continuous ζ and represents a different continuation from (10.1); this difference is the reason the analysis of Section 7 leads to modulated (rather than a true) travelling waves. The representations of Sections 7 and 8 are equivalent on, but not away from, lattice points and it is no coincidence that a transformation of the form (10.2) naturally arises when studying travelling waves (see (8.4)); such considerations have broad implications for the application of similarity methods to discrete systems.

The example we have studied here illustrates a widespread phenomenon in which exponentially small terms select a subset of solutions (usually two in the current case) from among the infinite number that exist to all orders. Thus the leading-order solution $\phi_0(s)$ can apparently be sampled on any lattice with spacing ϵ, this freedom being crucial in the analysis of the time-dependent problem in particular; the members of this infinite set of lattices act independently until the optimal truncation point is reached and the Stokes phenomenon comes into play to couple the various samplings together for a given continuation (so that, for example, the manner in which the sampling is undertaken in (8.4) is determined via exponentially small terms in (8.7)). The Stokes switching is described by (3.13), which represents a quite different type of continuum limit from that satisfied by ϕ_0 (and by successive terms in the algebraic expansion (2.9)). The Stokes line thickness $s = O(\epsilon^{1/2})$ is much greater than the lattice spacing ϵ but much smaller than the range $s = O(1)$ over which ϕ_0 varies; both of these types of continuum limit need to be captured in order to incorporate correctly the discreteness of the problem (cf. the discussion above of ad hoc approximations), with key aspects of qualitative behaviour for which discreteness is responsible (notably the non-zero Peierls stress) becoming apparent only at optimal truncation.

A final feature of discreteness worth emphasising is that an infinite number of different quantities get turned on across a Stokes line (see (3.3)), in contrast to the situation with ordinary differential equations of finite order (cf. [6]). In consequence it is not possible in general to supress all of them by special choices of the values of a finite number of parameters (in $F(\phi)$, for example), meaning in particular that the Peierls stress cannot be eliminated altogether; nevertheless, increasing q by one leads to a decrease in size of around $\exp(-2\pi \mathrm{Im}(\sigma)/\epsilon)$ in the quantity turned on, so suppression even of just the first term (that with $q=1$) leads to a very dramatic reduction in the critical stress.

We conclude by noting that the analysis generalises in a relatively straightforward way to problems such as discrete analogs of systems of reaction-diffusion equations and to examples in which t, as well as x, is treated discretely. Moreover, our results have obvious implications for finite difference treatments of ordinary differential equations and reaction diffusion equations in particular.

Acknowledgements. The authors would like to thank Prof R. MacKay for bringing to their attention relevant dynamical systems literature.

Appendix A. Large ϵ — the weak-coupling limit.

A.1. Formulation. Here we consider the opposite limit to the rest of the paper, denoting ϵ by $1/\delta$ and taking the limit $\delta \to 0$; see [9] for related considerations. The time-dependent problem takes the form

$$\frac{\partial \phi}{\partial t}(x,t) = \delta^2 \left(\phi(x+1,t) - 2\phi(x,t) + \phi(x-1,t)\right) - \left(F(\phi(x,t)) + \lambda(\delta)\right); \tag{A.1}$$

the dependence of the source term on the forcing λ can readily be made more complicated than the additive form in (A.1), but it will suffice to consider the current case. We again impose

$$\phi(x,t) \to \phi_\pm(\lambda) \text{ as } x \to \pm\infty, \tag{A.2}$$

with $F(\phi_\pm) + \lambda = 0$, these being two neighbouring stable steady states of (A.1) (so that $F'(\phi_\pm) > 0$). Additionally, we have an unstable steady state with $F(\phi_u) + \lambda = 0$, $F'(\phi_u) < 0$, $\phi_- < \phi_u < \phi_+$. As might be expected, we shall see that for small δ a wave can propagate in the positive direction only if $-\lambda$ is close to F_{min}, the minimum value of F in the relevant range of ϕ (namely $\phi_- < \phi < \phi_+$); the constraint $F_{min} \le -\lambda \le F_{max}$ is required in order that both ϕ_- and ϕ_+ exist. Taking F_{min} to occur at $\phi = \phi_{min}$, with

$$F \sim F_{min} + \beta(\phi - \phi_{min})^2 \quad \text{as } \phi \to \phi_{min}$$

for some positive constant $\beta = F''(\phi_{min})/2$, the appropriate scaling turns out to be $\lambda + F_{min} = O(\delta^2)$ and it is convenient to define $\alpha > 0$ by

$$\lambda = -F_{min} - \frac{\delta^2 \beta \alpha^2}{4}; \tag{A.3}$$

note that in the dislocation setting $\lambda = -F_{min}$ corresponds to the theoretical shear strength of a perfect crystal. It then follows that

$$\phi_+ \sim \phi_{min} + \frac{\delta\alpha}{2}, \qquad \phi_u \sim \phi_{min} - \frac{\delta\alpha}{2} \quad \text{as } \delta \to 0 \text{ with } \alpha = O(1)$$

and that

$$F(\phi) + \lambda \sim -\beta(\phi_+ - \phi)(\phi - \phi_+ + \delta\alpha) \quad \text{for } \phi_+ - \phi = O(\delta); \tag{A.4}$$

we take F (and hence β) and α to be independent of δ. We denote

$$\phi_+^0 = \lim_{\delta \to 0} \phi_+ = \phi_{min}, \qquad \phi_-^0 = \lim_{\delta \to 0} \phi_-,$$

and write

$$F(\phi) + \lambda \sim F_0(\phi) \text{ as } \delta \to 0 \text{ with } \phi_+ - \phi = O(1), \tag{A.5}$$

so that $F_0(\phi) = F(\phi) - F_{min}$, with

$$F_0(\phi) \sim \beta(\phi_+^0 - \phi)^2 \text{ as } \phi \to \phi_+^0. \tag{A.6}$$

For simplicity of the subsequent discussion we take initial conditions which are non-decreasing in x with

$$\phi_+ - \phi(x,0) \gg \delta \text{ for } x \leq -1, \tag{A.7}$$

$$\phi_+ - \phi(x,0) \;=\; O(\delta) \text{ for } x \geq 0, \tag{A.8}$$

and $\phi(x,0) \to \phi_+$ as $x \to +\infty$.

A.2. $t = O(1)$. For $x \leq -1$ we have, to leading order as $\delta \to 0$,

$$\frac{\partial \phi_0}{\partial t} = -F_0(\phi_0), \tag{A.9}$$

so that

$$\phi_0 \to \phi_-^0 \text{ as } t \to \infty. \tag{A.10}$$

For $x \geq 0$ we write $\phi(x,t) = \phi_+ - \delta\Phi(x,t)$, and obtain from (A.4) that

$$\frac{\partial \Phi_0}{\partial t} = 0. \tag{A.11}$$

A.3. $t = O(1/\delta)$. Writing $t = \tau/\delta$ we have

$$\phi \sim \phi_- \text{ for } x \leq -1, \tag{A.12}$$

and, using (A.4),

$$\frac{d\Phi_0}{d\tau}(0,\tau) = \phi_+^0 - \phi_-^0 - \beta\Phi_0(0,\tau)(\alpha - \Phi_0(0,\tau)), \tag{A.13}$$

$$\frac{\partial \Phi_0}{\partial \tau} = -\beta\Phi_0(\alpha - \Phi_0), \quad x \geq 1. \tag{A.14}$$

We now need to distinguish between two cases.

(I) $\alpha^2 > 4(\phi_+^0 - \phi_-^0)/\beta$ (steady states). Equation (A.13) has two steady states

$$\Phi_0(0,\tau) = S_\pm \equiv \frac{1}{2}\left(\alpha \pm \left(\alpha^2 - \frac{4(\phi_+^0 - \phi_-^0)}{\beta}\right)^{1/2}\right), \tag{A.15}$$

S_+ being unstable and S_- stable. If $\Phi_0(0,0) < S_+$ then by assumption $\Phi_0(x,0) < S_+ < \alpha$ for $x \geq 1$, so as $\tau \to +\infty$ we have $\Phi_0(0,\tau) \to S_-$, $\Phi_0 \to 0$ for $x \geq 1$, and a steady-state solution is established. If $\Phi_0(0,0) = S_+$ then $\Phi_0(0,\tau) \equiv S_+$ and $\Phi_0 \to 0$ as $\tau \to \infty$ for $x \geq 1$, so a second, non-generic (unstable), steady state is attained. However, if $\Phi_0(0,0) > S_+$ then $\Phi_0(0,\tau)$ blows up at some finite time τ_c, with

$$\Phi_0(0,\tau) \sim \frac{1}{\beta(\tau_c - \tau)} \text{ as } \tau \to \tau_c^-. \tag{A.16}$$

In this case we must consider the inner temporal scaling $\tau = \tau_c + \delta T$ with $\Phi \sim \Phi_0(x,\tau_c)$ for $x \geq 1$ and

$$\frac{d\phi_0}{dT}(0,T) = -F_0(\phi_0(0,T)), \tag{A.17}$$

with (A.16) providing the matching condition $\phi_0(0,T) \sim 1/\beta T$ as $T \to -\infty$. Hence $\phi_0(0,T) \to \phi_-^0$ as $T \to +\infty$ and for $\tau - \tau_c = O(1)$ we recover the previous formulation (A.13)-(A.14), except that $\Phi_0(1,\tau)$ replaces $\Phi_0(0,\tau)$ in (A.13) and $x \geq 2$ applies in (A.14). The behaviour now follows that outlined above, according to the sign of $\Phi_0(1,\tau_c) - S_+$. This sequence can repeat itself a finite number of times, the ultimate behaviour being a translated version of one of the steady states noted above. That it can repeat only a finite number of times follows most simply from the observation that for x sufficiently large that $\Phi_0(x,0) < S_+$ we have $\Phi_0(x,\tau) < S_+$ for all τ.

At the critical value of α, namely $\alpha = \alpha_c \sim 2((\phi_+^0 - \phi_-^0)/\beta)^{1/2}$, the results above remain valid, except that $S_+ = S_- = \alpha/2$ and there is only one steady state. In the limit $\delta \to 0$, we thus have

$$\lambda_c^{(+)} \sim -F_{min} - \delta^2(\phi_+^0 - \phi_-^0). \tag{A.18}$$

(II) $\alpha^2 < 4(\phi_+^0 - \phi_-^0)/\beta$ (travelling waves). Equation (A.13) now has no real steady states and $\Phi_0(0,\tau)$ always behaves as (A.16) for some $\tau_c > 0$, with $\phi_0(0,T) \to \phi_-^0$ as $T \to +\infty$, as before. $\Phi_0(1,\tau)$ in turn blows up at some $\tau > \tau_c$, and so on for each successive $\Phi_0(x,\tau)$. The blow-up times are readily determined for specified initial conditions and the ultimate travelling wave speed (whereby $\phi(x,t) \sim \phi(x + x_\infty - qt)$ as $t \to \infty$ for some constant x_∞) is given by $q \sim \delta/\tau_c^*$ as $\delta \to 0$, where τ_c^* is the blow-up time of the following problem for $\Phi^*(\tau^*)$:

$$\begin{aligned} \frac{d\Phi^*}{d\tau^*} &= \phi_+^0 - \phi_-^0 - \beta\Phi^*(\alpha - \Phi^*), \\ \Phi^* &= 0 \text{ at } \tau^* = 0, \end{aligned} \tag{A.19}$$

and is readily calculated as

$$\tau_c^* = \frac{2}{\beta((\alpha_c^0)^2 - \alpha^2)^{1/2}}\left(\pi - \tan^{-1}\left(\frac{((\alpha_c^0)^2 - \alpha^2)^{1/2}}{\alpha}\right)\right). \tag{A.20}$$

The wave thus "clicks" through the lattice points, the interval between clicks approaching τ_c^* as $\tau \to \infty$, $\delta \to 0$ and with (A.19) describing the slow phase of each click, the fast variation being of the form (A.17).

A.4. Travelling waves. We now analyse the travelling wave ordinary differential-difference equation

$$-\delta Q \frac{d\phi}{d\zeta}(\zeta) = \delta^2 \left(\phi(\zeta+1) - 2\phi(\zeta) + \phi(\zeta+1)\right) - \left(F(\phi(\zeta)) + \lambda\right), \tag{A.21}$$

where $Q(\delta) = q(\delta)/\delta$, and show consistency with the results above. We locate the origin of ζ such that there is an inner region with $\zeta = \delta Z$ where

$$Q_0 \frac{d\phi_0}{dZ} = F_0(\phi_0) \tag{A.22}$$

with

$$\phi_0(Z) \sim \phi_-^0 + e^{F_0'(\phi_-^0)Z/Q_0} \text{ as } Z \to -\infty. \tag{A.23}$$

We then have the outer solution

$$\phi(\zeta) \to \phi_- \text{ as } \delta \to 0 \tag{A.24}$$

for $\zeta = O(1)$ with $\zeta < 0$. Equation (A.22) implies

$$\phi_0 \sim \phi_+^0 - \frac{Q_0}{\beta Z} \text{ as } Z \to +\infty. \tag{A.25}$$

In $0 < \zeta < 1$, writing $\phi = \phi_+ - \delta\Phi$ and matching with (A.25) gives

$$-Q_0 \frac{d\Phi_0}{d\zeta} = \phi_+^0 - \phi_-^0 - \beta\Phi_0(\alpha - \Phi_0), \tag{A.26}$$

$$\Phi_0 \sim \frac{Q_0}{\beta\zeta} \text{ as } \zeta \to 0^+. \tag{A.27}$$

In order to match forward we require that the solution to (A.26) reach $\Phi_0 = 0$ at $\zeta = 1$ and, by comparison with (A.19), we thus see that $Q_0 = 1/\tau_c^*$, as expected. It is worth noting that this small δ formulation is independent of the precise details of the nonlinearity $F(\phi)$.

The next inner region has $\zeta - 1 = O(\delta)$, $\Phi = O(\delta)$ and the next outer lies in $1 < \zeta < 2$ with $\Phi = O(\delta^2)$; there is a succession of such layers with the deviation of ϕ from ϕ_+ progressing through successively larger powers of δ.

The constraint (8.9) now implies

$$Q_0 \int_{-\infty}^{\infty} \left(\frac{d\phi_0}{dZ}\right)^2 dZ = \int_{\phi_-^0}^{\phi_{min}} F_0(\phi)\, d\phi,$$

the region $Z = O(1)$ dominating; this is evidently consistent with (A.22). Because ϕ_0 in (A.22) varies on the scale $Z = O(Q_0)$, the condition (8.10) is clearly satisfied as $\alpha \to \alpha_c^+$, $Q \to 0$.

A.5. Summary. The analysis above can be summarised as follows:

(1) Non-propagation regime, $\lambda_c^{(-)} < \lambda < \lambda_c^{(+)}$. The asymptotic behaviour of $\lambda_c^{(+)}$ as $\delta \to 0$ is given by (A.18), while

$$\lambda_c^{(-)} \sim -F_{max} + \delta^2(\phi_+^0 - \phi_-^0) \quad \text{as } \delta \to 0. \tag{A.28}$$

In this range the solution generically evolves to a steady state for which, as $\delta \to 0$ with $|\lambda - \lambda_c^{(\pm)}| = O(1)$, we have

$$\begin{aligned} \phi &\sim \phi_- && \text{for } x < x_\infty, \\ \phi &\sim \phi_+ && \text{for } x \geq x_\infty, \end{aligned} \tag{A.29}$$

for some integer x_∞ which depends on the initial data (the correction terms to (A.29) are $O(\delta^2)$ for $x = x_\infty - 1$ and $x = x_\infty$, decaying through successively larger powers of δ as $|x - x_\infty|$ increases; similar comments also apply in the corresponding expressions below); for $\lambda_c^{(+)} - \lambda = O(\delta^2)$ we instead have

$$\begin{aligned} \phi &\sim \phi_- && \text{for } x < x_\infty, \\ \phi &\sim \phi_+ - \delta S_- && \text{for } x = x_\infty, \\ \phi &\sim \phi_+ && \text{for } x > x_\infty, \end{aligned} \tag{A.30}$$

where, from (A.15), $S_- \sim (\phi_+^0 - \phi_-^0)/\alpha\beta$ as $\alpha \to +\infty$, $S_- \to (\alpha/2)^-$ as $\alpha \to \alpha_c^+$; similar considerations apply for $\lambda - \lambda_c^{(-)} = O(\delta^2)$. For initial conditions lying on the borderline between those which evolve to (A.29) or (A.30) with $x_\infty = y_\infty$ and with $x_\infty = y_\infty + 1$, the large time behaviour is given by the non-generic (unstable) steady state, whereby as $\delta \to 0$ we have, respectively,

$$\begin{aligned} \phi &\sim \phi_- && \text{for } x < y_\infty, \\ \phi &\sim \phi_u && \text{for } x = y_\infty, \\ \phi &\sim \phi_+ && \text{for } x > y_\infty, \end{aligned} \tag{A.31}$$

or

$$\begin{aligned} \phi &\sim \phi_- && \text{for } x < y_\infty, \\ \phi &\sim \phi_+ - \delta S_+ && \text{for } x = y_\infty, \\ \phi &\sim \phi_+ && \text{for } x > y_\infty, \end{aligned} \tag{A.32}$$

where $S_+ \sim \alpha$ as $\alpha \to +\infty$, $S_+ \to (\alpha/2)^+$ as $\alpha \to \alpha_c^+$. As $\alpha \to \alpha_c^+$ the solutions (A.30) and (A.32) merge, there being only one steady state for $\alpha = \alpha_c$. We note that even less generic steady states are possible with (A.31) replaced by

$$\begin{aligned} \phi &\sim \phi_- && \text{for } x < y_\infty, \\ \phi &\sim \phi_+ - \delta S_+ && \text{for } x = y_\infty, y_\infty + 1, \cdots, y_\infty + Y, \\ \phi &\sim \phi_+ && \text{for } x > y_\infty + Y, \end{aligned} \tag{A.33}$$

say, where the integer Y can be arbitrarily large, but these are non-monotonic for $Y \neq 0$. It seems likely that there are generically precisely two monotonic steady states.

(2) Propagation regime, $\lambda_c^{(+)} < \lambda < -F_{min}$. Here the structure of the travelling wave for $|\lambda_c^{(+)} - \lambda| = O(\delta^2)$ is as outlined in Section A.4, with analogous results applying for $-F_{max} < \lambda < \lambda_c^{(-)}$.

We conclude by noting the various parameter values for the special case $F(\phi) = \sin\phi$, when we have $F_{min} = -1$, $\beta = 1/2$, $\phi_+^0 = 3\pi/2$, $\phi_-^0 = -\pi/2$, and

$$\lambda_c^{(+)} \sim 1 - 2\pi\delta^2, \quad \lambda_c^{(-)} \sim -1 + 2\pi\delta^2, \quad \alpha_c \sim 4\sqrt{\pi}, \text{ as } \delta \to 0.$$

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