On the approximation of the eigenvalues of an annulus using complex rays

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Abstract

The use of both higher-order amplitude corrections and complex rays to increase the accuracy of the approximation of the eigenvalues of Helmholtz equation on an annulus is demonstrated.

1 Introduction

Certain families of eigenmodes and eigenvalues of interior domains have been successfully approximated using ray theory for problems in acoustics and electromagnetics and WKB theory for problems in quantum mechanics; a review of these methods is given in [2]. We consider here two extensions to the method for a particular example of a scalar Helmholtz equation in an annular domain. Firstly we consider the effects of including more than the first term in the expansion of the amplitude, giving a more accurate determination of the eigenvalues.

Secondly we show how to include the contribution due to complex rays which are reflected off the inner boundary of the annulus when it lies within the caustic of the eigenfunction.

The first modification gives corrections up to $O(k^{-n})$ where n is the extra number of terms taken in the amplitude expansion, and k is the wavenumber. The second gives corrections which are exponentially small in k. Nonetheless, we will see that these exponentially small corrections can be significant.

2 Approximation of the eigenvalues of a circle using the leading-order amplitude

We consider the scalar Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0, \tag{1}$$

with dirichlet boundary conditions on a circle of radius a, where we aim to determine the eigenvalues of the wavenumber k and the corresponding eigenfunctions. In the large k limit we apply the ray ansatz

$$\psi = e^{iku}\sum_{n=0}^{\infty}\frac{i^n A_n}{k^n} \tag{2}$$

where the phase u satisfies the eikonal equation

$$(\nabla u)^2 = 1, \tag{3}$$

while the amplitude coefficients satisfy the transport equations

$$2\nabla A^{(n)}\cdot\nabla u + A^{(n)}\nabla^2 u = \nabla^2 A^{(n-1)}, \quad n \geq 0, \tag{4}$$

with $A^{(-1)} \equiv 0$. As described in [2, 3], the rays and phase corresponding to an eigensolution are given by

$$x = b\cos s - t\sin s, \tag{5}$$

$$y = b\sin s + t\cos s, \tag{6}$$

$$u = bs + t, \tag{7}$$

with $b < a$. With u given by (7) the transport equations become

$$2\frac{\partial A^{(n)}}{\partial t} + \frac{A^{(n)}}{t} = \nabla^2 A^{(n-1)}, \quad n \geq 0, \tag{8}$$

giving the leading-order amplitude as

$$A^{(0)} = \frac{1}{\sqrt{t}}. \tag{9}$$

The circle $t = 0$, $x = b\cos s$, $y = b\sin s$, is a caustic, which forms a branch point for the phase u. When (5)-(7) are inverted the two branches of u correspond to the incoming and outgoing rayfields respectively. We find

$$t = \pm\sqrt{r^2 - b^2}, \tag{10}$$

$$s = \theta \mp \cos^{-1}(b/r), \tag{11}$$

giving the incoming field as

$$
\begin{aligned}
u_i &= b\theta + b\cos^{-1}(b/r) - \sqrt{r^2 - b^2}, \qquad r > b, && (12)\\
A_i^{(0)} &= \frac{1}{(r^2 - b^2)^{1/4}}, \qquad r > b, && (13)
\end{aligned}
$$

and the outgoing field as

$$
\begin{aligned}
u_o &= b\theta - b\cos^{-1}(b/r) + \sqrt{r^2 - b^2}, \qquad r > b, && (14)\\
A_o^{(0)} &= \frac{-i}{(r^2 - b^2)^{1/4}}, \qquad r > b. && (15)
\end{aligned}
$$

Thus, for $r > b$,

$$\phi \sim A_i e^{iku_i} + A_o e^{iku_o}. \tag{16}$$

The two Keller-Maslov conditions are

(i) ϕ is single valued as a function of θ.

(ii) $\phi = 0$ on $r = a$.

The first condition implies

$$kb = m \in \mathbb{Z}. \tag{17}$$

The second condition may be written

$$ik\left[u_o(a) - u_i(a)\right] + \left[\log A_o(a) - \log A_i(a)\right] = \pi i(2n - 1), \qquad n \in \mathbb{Z}. \tag{18}$$

Using the expressions (12)-(15) gives

$$k\left(\sqrt{a^2 - b^2} - b\cos^{-1}(b/a)\right) = (n - 1/4)\pi. \tag{19}$$

This is exactly the condition (7.9) of [2]. Equations (17) and (19) provide two equations for the two unknowns k and b. They have been derived using only the leading-order amplitude of the ray expansion. In principal, higher accuracy can be obtained by including more terms of the expansion, although, since the expansion is divergent there is an optimal number of terms to take which depends on the value of k, which in turn depends on the solutions of (17), (19). We will examine the effect of including more terms in the next section.

3 Approximation of the eigenvalues of a circle using higher-order amplitude terms

Let us now calculate some of the higher-order terms in the amplitude expansion (2). By the symmetry in the problem, $A^{(n)}$ is independent of s. Then we find that

$$\nabla^2 A^{(n)} = \frac{1}{t}\frac{\partial A^{(n)}}{\partial t} + \frac{\partial^2 A^{(n)}}{\partial t^2} + \frac{b^2}{t}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial A^{(n)}}{\partial t}\right). \tag{20}$$

Hence (8) becomes

$$2\frac{\partial A^{(n)}}{\partial t} + \frac{A^{(n)}}{t} = \frac{1}{t}\frac{\partial A^{(n-1)}}{\partial t} + \frac{\partial^2 A^{(n-1)}}{\partial t^2} + \frac{b^2}{t}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial A^{(n-1)}}{\partial t}\right), \quad n \geq 1, \tag{21}$$

the solution of which is given by

$$A^{(n)} = \sum_{j=n}^{j=3n} \frac{g_{n,j}\, b^j}{t^{j+1/2}\, b^n}, \tag{22}$$

where

$$\begin{aligned} g_{n,j} &= -\frac{1}{2j}\left[(j-1/2)^2 g_{n-1,j-1} + (j-5/2)(j-1/2) g_{n-1,j-3}\right], &(23)\\ g_{0,0} &= 1. &(24)\end{aligned}$$

Solving for the first few terms we find

$$A^{(1)} = -\frac{A^{(0)}}{24}\left(\frac{3}{t} + \frac{5\,b^2}{t^3}\right), \tag{25}$$

$$A^{(2)} = \frac{A^{(0)}}{1152}\left(\frac{81}{t^2} + \frac{462\,b^2}{t^4} + \frac{385\,b^4}{t^6}\right), \tag{26}$$

$$A^{(3)} = -\frac{A^{(0)}}{414720}\left(\frac{30375}{t^3} + \frac{369603\,b^2}{t^5} + \frac{765765\,b^4}{t^7} + \frac{425425\,b^6}{t^9}\right), \tag{27}$$

$$\begin{aligned} A^{(4)} = \frac{A^{(0)}}{39813120}\Bigg(&\frac{4465125}{t^4} + \frac{94121676\,b^2}{t^6} + \frac{349922430\,b^4}{t^8} \\ &+ \frac{446185740\,b^6}{t^{10}} + \frac{185910725\,b^8}{t^{12}}\Bigg), \end{aligned} \tag{28}$$

$$\begin{aligned} A^{(5)} = -\frac{A^{(0)}}{6688604160}\Bigg(&\frac{1519035525}{t^5} + \frac{49286948607\,b^2}{t^7} + \frac{284499769554\,b^4}{t^9} \\ &+ \frac{614135872350\,b^6}{t^{11}} + \frac{566098157625\,b^8}{t^{13}} + \frac{188699385875\,b^{10}}{t^{15}}\Bigg). \end{aligned} \tag{29}$$

In applying the Keller-Maslov conditions we require $\log A$. We have

$$
\begin{aligned}
\log A \quad \sim \quad & \log A^{(0)} + \frac{iA^{(1)}}{A^{(0)}k} + \frac{{A^{(1)}}^2 - 2\,A^{(0)}\,A^{(2)}}{2\,{A^{(0)}}^2\,k^2} \\
& - \frac{i\,\left({A^{(1)}}^3 - 3\,A^{(0)}\,A^{(1)}\,A^{(2)} + 3\,{A^{(0)}}^2\,A^{(3)}\right)}{3{A^{(0)}}^3\,k^3} \\
& + \frac{-{A^{(1)}}^4 + 4\,A^{(0)}\,{A^{(1)}}^2\,A^{(2)} - 2\,{A^{(0)}}^2\,{A^{(2)}}^2 - 4\,{A^{(0)}}^2\,A^{(1)}\,A^{(3)} + 4\,{A^{(0)}}^3\,A^{(4)}}{4\,{A^{(0)}}^4\,k^4} \\
& + \frac{i}{5{A^{(0)}}^5\,k^5}\left({A^{(1)}}^5 - 5\,A^{(0)}\,{A^{(1)}}^3\,A^{(2)} + 5\,{A^{(0)}}^2\,A^{(1)}\,{A^{(2)}}^2 + 5\,{A^{(0)}}^2\,{A^{(1)}}^2\,A^{(3)}\right. \\
& \qquad \left. -5\,{A^{(0)}}^3\,A^{(2)}\,A^{(3)} - 5\,{A^{(0)}}^3\,A^{(1)}\,A^{(4)} + 5\,{A^{(0)}}^4\,A^{(5)}\right) + \cdots .
\end{aligned}
$$

Using (25)-(29) this becomes

$$
\begin{aligned}
& \log(A^{(0)}) - \frac{i\,(5b^2 + 3\,t^2)}{24t^3\,k} - \frac{(5\,b^4 + 6\,b^2\,t^2 + \,t^4)}{16\,t^6\,k^2} \\
& + \frac{i\,(5525\,b^6 + 9945\,b^4\,t^2 + 4779\,b^2\,t^4 + 375\,t^6)}{5760t^9\,k^3} \\
& + \frac{565\,b^8 + 1356\,b^6\,t^2 + 1062\,b^4\,t^4 + 284\,b^2\,t^6 + 13\,t^8}{128\,t^{12}\,k^4} \\
& - \frac{i\,(8696625\,b^{10} + 26089875\,b^8\,t^2 + 28286370\,b^6\,t^4 + 13077470\,b^4\,t^6 + 2252205\,b^2\,t^8 + 67599\,t^{10})}{322560t^{15}\,k^5} \\
& + \cdots .
\end{aligned}
$$

The outer boundary for the incoming rayfield corresponds to $t = -\sqrt{a^2 - b^2}$, while for the outgoing rayfield it corresponds to $t = +\sqrt{a^2 - b^2}$. Hence the jump in $\log A$ between the two rays is given by

$$
\begin{aligned}
& [\log A]_{t=-\sqrt{a^2-b^2}}^{t=+\sqrt{a^2-b^2}} = \\
& \quad -\frac{i\pi}{2} - \frac{i(5b^2 + 3(a^2 - b^2))}{12(a^2 - b^2)^{3/2}k} \\
& \quad + \frac{i\,(5525\,b^6 + 9945\,b^4\,(a^2 - b^2) + 4779\,b^2\,(a^2 - b^2)^2 + 375\,(a^2 - b^2)^6)}{2880(a^2 - b^2)^{9/2}\,k^3} \\
& \quad - \frac{i}{161280(a^2 - b^2)^{15/2}\,k^5}\left(8696625\,b^{10} + 26089875\,b^8\,(a^2 - b^2) + 28286370\,b^6\,(a^2 - b^2)^2\right. \\
& \qquad \left. + 13077470\,b^4\,(a^2 - b^2)^3 + 2252205\,b^2\,(a^2 - b^2)^4 + 67599\,(a^2 - b^2)^5\right) \\
& \quad + \cdots .
\end{aligned}
$$

m	n	b	Exact	$j=0$	$j=1$	$j=3$	$j=5$	$j=7$	$j=9$
1	1	0.26098	3.83171	3.79444	3.83370	3.83096	3.83240	3.83047	3.83521
1	2	0.14254	7.01559	6.99700	7.01582	7.01557	7.01559	7.01559	7.01559
2	1	0.389437	5.13562	5.10039	5.13762	5.13484	5.13636	5.13431	5.13936
2	2	0.237607	8.41724	8.40014	8.41745	8.41723	8.41725	8.41724	8.41725

Table 1: Comparision of eigenvalues of a circle of unit radius approximated by an amplitude series up to $A^{(j)}$ with the exact value. The approximate values are obtained by solving (17) together with (30) truncated at order k^{-j} for the unknowns b and k.

Thus the condition (19) becomes

$$\begin{aligned}
k\left(\sqrt{a^2-b^2}-b\cos^{-1}(b/a)\right) &= (n-1/4)\pi \\
&+\frac{(5b^2+3(a^2-b^2))}{24(a^2-b^2)^{3/2}k} \\
&-\frac{(5525\,b^6+9945\,b^4\,(a^2-b^2)+4779\,b^2\,(a^2-b^2)^2+375\,(a^2-b^2)^6)}{5760(a^2-b^2)^{9/2}\,k^3} \\
&+\frac{1}{322560(a^2-b^2)^{15/2}\,k^5}\Big(8696625\,b^{10}+26089875\,b^8\,(a^2-b^2)+28286370\,b^6\,(a^2-b^2)^2 \\
&\qquad +13077470\,b^4\,(a^2-b^2)^3+2252205\,b^2\,(a^2-b^2)^4+67599\,(a^2-b^2)^5\Big) \\
&\qquad +O(k^{-7}). \qquad (30)
\end{aligned}$$

The two unknowns k and b are now determined by solving (17) and (30). Of course, including more terms of the amplitude expansion is equivalent to including more terms in the Debye expansion of $e^{im\theta}J_m(kr)$, which is the exact solution to (1).

In Tables 1 and 2 we show the results of including higher-order amplitude corrections in the Keller-Maslov conditions. Note that since the amplitude expansion is divergent there is an optimal number of terms to take in this expansion to achieve the greatest accuracy in the approximation.

4 Approximation of the eigenvalues of an annulus using complex rays

We have seen how including higher-order terms in the amplitude expansion can increase the accuracy of the approximation to the eigenvalues of a circle. Now suppose that the domain is an annulus $c \leq r \leq a$. There will be two families of eigenfunctions: those for which the rays

m	1	1	2	2	10
n	1	2	1	2	10
k	3.83171	7.01559	5.13562	8.41724	45.23157
$j = 0$	9.726×10^{-3}	2.649×10^{-3}	6.861×10^{-3}	2.032×10^{-3}	6.971×10^{-5}
$j = 1$	-5.199×10^{-4}	-3.288×10^{-5}	-3.895×10^{-4}	-2.486×10^{-5}	-2.960×10^{-8}
$j = 3$	1.942×10^{-4}	2.739×10^{-6}	1.517×10^{-4}	2.172×10^{-6}	9.261×10^{-11}
$j = 5$	-1.814×10^{-4}	-5.755×10^{-7}	-1.435×10^{-4}	-4.736×10^{-7}	-7.440×10^{-13}
$j = 7$	3.217×10^{-4}	2.254×10^{-7}	2.557×10^{-4}	1.893×10^{-7}	1.114×10^{-14}
$j = 9$	-9.138×10^{-4}	-1.415×10^{-7}	-7.279×10^{-4}	-1.201×10^{-7}	-2.667×10^{-16}

Table 2: Relative error in the eigenvalues of a circle of unit radius approximated by an amplitude series up to $A^{(j)}$. The approximate values are obtained by solving (17) together with (30) truncated at order k^{-j} for the unknowns b and k.

bounce off both the inside and outside walls, shown in Figure 1a, and those for which the inner wall is within the caustic of the rays, shown in Figure 1b. The eigenvalues of the first family can be calculated in the usual way. We will concentrate here on the second family, for which complex rays play a role.

Since the real rays do not meet the inner wall, the calculation involving real rays only gives the same approximation for the eigenvalues as for those of a circle radius a, no matter how many terms of the amplitude expansion are included.

The wavefield inside the caustic is exponentially small in k, and therefore the correction to the eigenvalues due to the inner boundary will also be exponentially small in k. This wavefield may be described in terms of complex rays [1]; let us first see how to calculate it.

If we analytically continue the ray solution (12), (13) for complex r we find that the Riemann surface for the phase function u is as shown in Figure 2, where the cuts in the Riemann surface are not branch cuts but Stokes lines across which the ray contribution corresponding to one of the branches of u is switched on or off. We see that both branches contribute in r real and greater than b, corresponding to the incoming and outgoing rays (the incoming ray is the upper of the two branches as shown), while only one branch contributes in r real and less than b, corresponding to the exponentially small ray contribution in the interior.

The local structure near $r = b$ is that of the Airy function $\mathrm{Ai}(-z)$, shown in Figure 3, as we would expect, and the Stokes lines are given by the lines of equal phase of the two branches of u. Note that the incoming rayfield is the upper of the two sheets of the Riemann

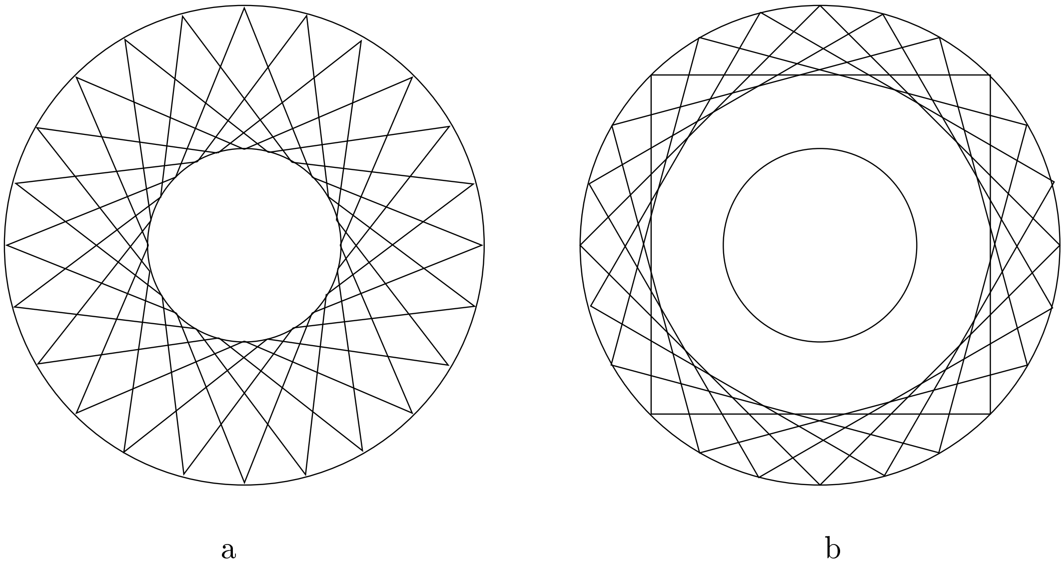

Figure 1: The two families of eigenfunctions in an annulus.

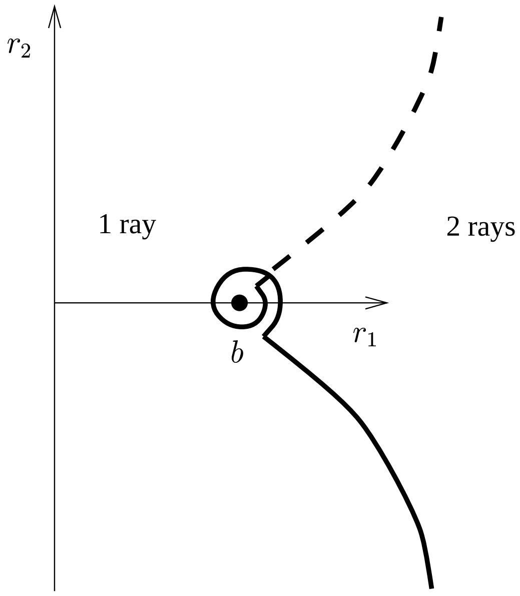

Figure 2: Diagram showing the Riemann surface of the phase u for the circular caustic in the plane $r = r_1 + ir_2$, with r_1, $r_2 \in \mathbb{R}$. The Riemann surface ends at the Stokes lines, which are indicated by a dotted line on the lower sheet and a solid line on the upper sheet.

surface in Figure 2, so that the diagram indicates that we must circle the branchpoint $r = b$ anticlockwise to get from the incoming rayfield to the outgoing rayfield (via an exponentially small rayfield in $r < b$). This in turn tells us which branch of $(r^2 - b^2)^{1/4}$ we pass onto for the outgoing ray, and therefore tells us the phase change in A_0 in the outgoing ray, giving the $-i$ in equation (15).

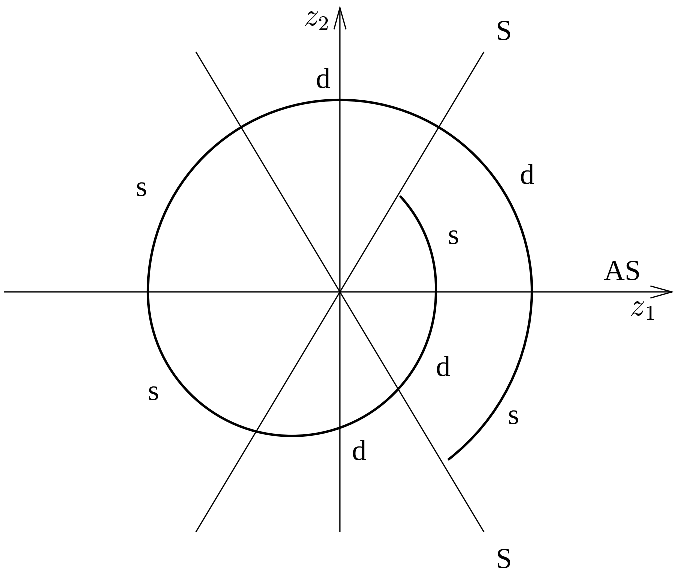

Figure 3: Diagram indicating the ray contributions in the WKB expansion of $\mathrm{Ai}(-z)$; d and s indicate the dominant and subdominant rays respectively, while S and AS indicate the Stokes and anti-Stokes lines.

Now, the complex ray in the interior of the caustic will be reflected from the inner boundary of the annulus at $r = c$. We may calculate the value of u and $A^{(0)}$ on the boundary from (12), (13). We find that

$$u = b\theta + b\cos^{-1}(b/c) - \sqrt{c^2 - b^2}, \tag{31}$$

$$A^{(0)} = \frac{1}{(c^2 - b^2)^{1/4}}. \tag{32}$$

The reflected complex ray may then be calculated, and is given by

$$x = b\cos s - t\sin s, \tag{33}$$

$$y = b\sin s + t\cos s, \tag{34}$$

$$u = bs + t + 2i\left(b\log\left(\frac{b + \sqrt{b^2 - c^2}}{c}\right) - \sqrt{b^2 - c^2}\right), \tag{35}$$

$$A^{(0)} = -\frac{1}{\sqrt{t}}, \tag{36}$$

where the branch of u is the opposite to that of the incoming complex ray, that is, the ray is exponenially decaying away from the inner boundary. This reflected rayfield again forms a caustic at $r = b$, and in (33)-(36) we have shifted the origin of t so that it measures distance from this caustic, rather than from the inner boundary. On the outside of the caustic the complex ray forms real rays again, but with an exponentially small amplitude prefactor. When we include these rays in our eigenvalue conditions we will see the effect of the inner boundary on the eigenvalues.

However, the Stokes line structure for this reflected wavefield is different to that shown in Figure 2. In the local problem near $r = b$ we now must have a solution which is exponentially growing in the interior of the caustic (since our reflected field is decaying away from the inner boundary, it must be growing away from the caustic). Hence the local behaviour of the solution is not now described by $\mathrm{Ai}(-z)$, but by either $\mathrm{Ai}(-e^{2i\pi/3}z)$ or $\mathrm{Ai}(-e^{4i\pi/3}z)$, both of which contain an exponentially growing component as $z \to -\infty$, and whose local Stokes line structure is illustrated in Figure 4.

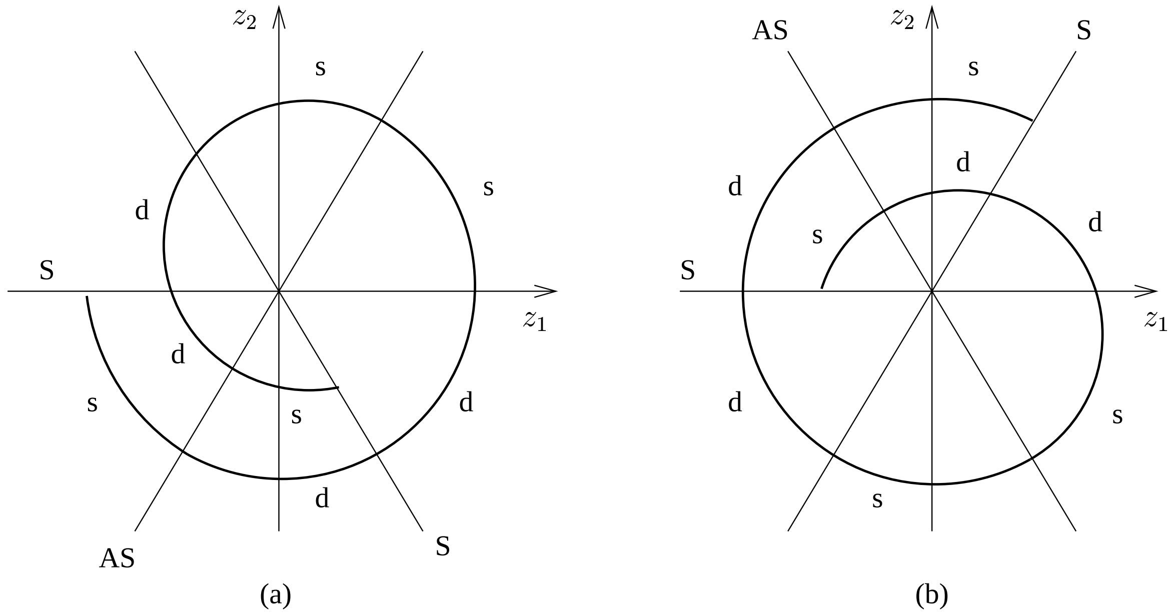

Figure 4: Stokes lines and ray structure for $\mathrm{Ai}(-e^{2i\pi/3}z)$ or $\mathrm{Ai}(-e^{4i\pi/3}z)$; d and s indicate the dominant and subdominant rays respectively, while S and AS indicate the Stokes lines and anti-Stokes lines.

Thus the outer Stokes line structure must be one of those illustrated in Figure 5. However, the difference between these two solutions is simply a multiple of our original eigensolution, so that by choosing different combinations of them we are simply multiplying our whole

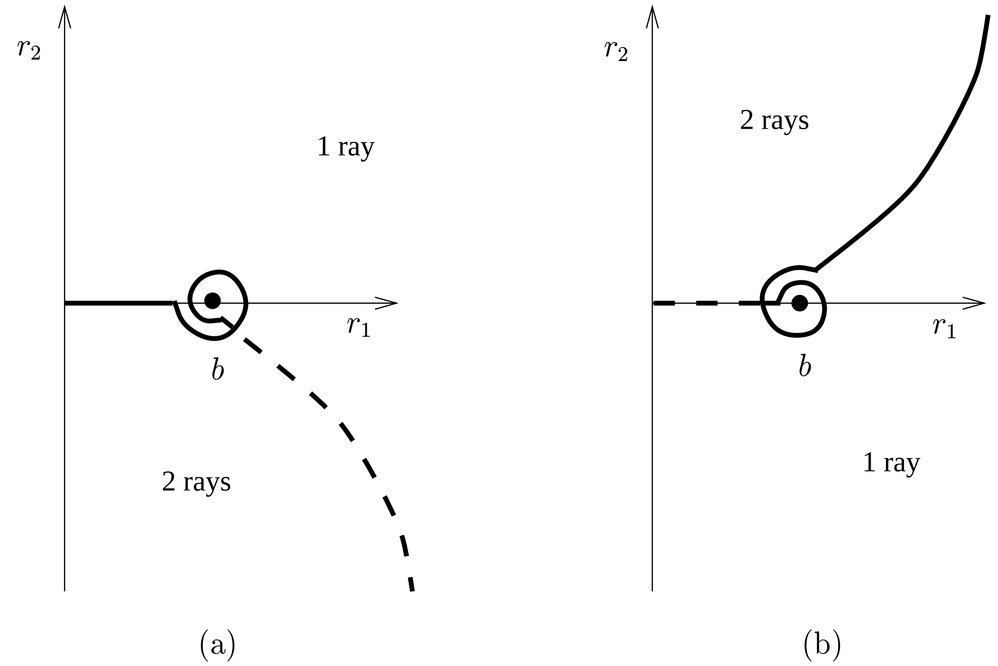

Figure 5: Possible Riemann surfaces for the phase u of the reflected field; $r = r_1 + ir_2$, r_1, $r_2 \in \mathbb{R}$. As in figure 2, the Riemann surface ends at the Stokes lines.

eigensolution by a prefactor. The quantisation conditions are therefore independent of which structure we choose, and therefore without loss of generality we choose the reflected field to be of the form illustrated in Figure 5(a). In this case in the exterior of the caustic the reflected field becomes the outgoing real rayfield, and we therefore need to include an extra exponentially small multiple of this rayfield when calculating the quantisation conditions.

From (33)-(36) we find

$$u = b\theta - b\cos^{-1}(b/r) + \sqrt{r^2 - b^2} + 2i\left(b\log\left(\frac{b+\sqrt{b^2-c^2}}{c}\right) - \sqrt{b^2-c^2}\right). \tag{37}$$

Therefore the extra multiple of the outgoing wavefield is

$$-A_o^{(0)} i \exp\left(-2k\left(b\log\left(\frac{b+\sqrt{b^2-c^2}}{c}\right) - \sqrt{b^2-c^2}\right)\right).$$

The factor of i here arises because to get from the original outgoing wavefield to the exponentially small interior we must circle the point $r = b$ clockwise. The exponentially decaying ray is reflected into the exponentially growing ray (with an exponentially small prefactor). Then to get back to the outgoing ray outside the caustic we must continue to circle the point $r = b$ clockwise, completing a full circle. Thus there is a phase change between the original outgoing ray and the exponentially small addition due to the extra factor of i we pickup from the amplitude (15) when we circle the branchpoint $r = b$ clockwise.

Finally, when we add the extra ray contribution to the previous ray expansion, we find that

$$
\begin{aligned}
u_o &= b\theta - b\cos^{-1}(b/r) + \sqrt{r^2 - b^2}, && (38)\\
A_o &\sim \sum_{n=0}^{\infty} \frac{i^n A_n}{k^n} - i\exp\left(-2k\left(b\log\left(\frac{b+\sqrt{b^2-c^2}}{c}\right) - \sqrt{b^2-c^2}\right)\right)\sum_{n=0}^{\infty}\frac{i^n A_n}{k^n}. && (39)
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\log A]_{t=-\sqrt{a^2-b^2}}^{t=+\sqrt{a^2-b^2}} &\sim -\frac{i\pi}{2} - \frac{i}{12k}\left(\frac{3}{(a^2-b^2)^{1/2}} + \frac{5b^2}{(a^2-b^2)^{3/2}}\right) + \cdots \\
&\quad - i\exp\left(-2k\left(b\log\left(\frac{b+\sqrt{b^2-c^2}}{c}\right) - \sqrt{b^2-c^2}\right)\right). \qquad (40)
\end{aligned}
$$

The new condition (ii) is

$$
\begin{aligned}
2k\left(\sqrt{a^2-b^2} - b\cos^{-1}(b/a)\right) &= 2(n-1/4)\pi + \frac{1}{12k}\left(\frac{3}{(a^2-b^2)^{1/2}} + \frac{5b^2}{(a^2-b^2)^{3/2}}\right) + \cdots \\
&\quad + \exp\left(-2k(b\log\left(\frac{b+\sqrt{b^2-c^2}}{c}\right) - \sqrt{b^2-c^2})\right). \qquad (41)
\end{aligned}
$$

We now have to solve (17) and (41) for the two unknowns k and b. Of course, we have only included the leading-order complex ray contribution in (41). The Stokes line structure shown in Figure 5(a) shows that an exponentially subdominant complex ray is "reflected" from the caustic back towards the inner boundary. This ray will again reflect off the inner boundary and reform a caustic at $r = b$, giving an additional contribution

$$
-\frac{1}{2}\exp\left(-4k\left(b\log\left(\frac{b+\sqrt{b^2-c^2}}{c}\right) - \sqrt{b^2-c^2}\right)\right)\sum_{n=0}^{\infty}\frac{i^n A_n}{k^n}, \qquad (42)
$$

to the amplitude of the outgoing reflected ray. Note the factor of a half, which comes from the fact that the amplitude of the exponentially small ray on the inside of the caustic in Figure 5(a) is only half of that of the exponentially large ray, due to the fact that we are evaluating exactly on the Stokes line. The caustic of these rays will again send an exponentially subdominant ray back towards the inner boundary, which is again reflected. Including all such multiple reflections we find that the amplitude of the outgoing rayfield we should be

$$
\left(\sum_{n=0}^{\infty}\frac{i^n A_n}{k^n}\right)\left[-1 + 2\sum_{j=0}^{\infty}\frac{(-i)^j}{2^j}\exp\left(-2jk\left(b\log\left(\frac{b+\sqrt{b^2-c^2}}{c}\right) - \sqrt{b^2-c^2}\right)\right)\right]. \qquad (43)
$$

This expression can be summed to give

$$\left(\frac{2-iE}{2+iE}\right)\sum_{n=0}^{\infty}\frac{i^n A_n}{k^n}, \tag{44}$$

where

$$E=\exp\left(-2k\left(b\log\left(\frac{b+\sqrt{b^2-c^2}}{c}\right)-\sqrt{b^2-c^2}\right)\right). \tag{45}$$

Condition (41) then becomes

$$\begin{aligned} 2k\left(\sqrt{a^2-b^2}-b\cos^{-1}(b/a)\right) &= 2(n-1/4)\pi+\frac{5}{12k}\left(\frac{3}{(a^2-b^2)^{1/2}}+\frac{b^2}{(a^2-b^2)^{3/2}}\right)+\cdots \\ &\quad +2\tan^{-1}\left[\frac{1}{2}\exp\left(-2k(b\log\left(\frac{b+\sqrt{b^2-c^2}}{c}\right)-\sqrt{b^2-c^2})\right)\right]. \end{aligned} \tag{46}$$

We now must solve (17) and (46) for k and b. In Table 3 we show a comparison of the approximation to the eigenvalues of an annulus using the leading-order amplitude with and without the complex rays included (i.e. we compare the results of solving (17) and (19) for k and b to those of solving (17) and (46) for k and b). We see that the contribution from these exponentially small corrections may be significant.

Finally, we should comment on the meaning of expressions such as (39), in which an exponentially small term has been added to an infinite algebraic asymptotic expansion. Formally, the exponentially small term is only significant asymptotically if the algebraic expansion is truncated optimally, since in this case the error between the exact solution and the truncated algebraic expansion is also exponentially small. However, in practice we can see from Table 3 that the exponentially small term can have a significant numerical effect, even when only the first term in the algebraic series is taken.

5 Conclusion

We have shown how to increase the accuracy of the approximation to the eigenvalues of an annulus by both increasing the number of terms retained in the expansion of the amplitude of the rays, and by including complex rays. The former gives corrections which are $O(k^{-n})$, where n is the number of extra terms in the amplitude expansion included, while the latter gives corrections which are exponentially small in k.

Both the extensions we have used can be applied to more general problems and domains. A key step in approximating eigenvalues in this way is identifying a finite number of ray families which join together at boundaries and caustics (see [2, 3]). This is true also when

m	n	b	c	Exact	A_0	Frac. error	A_0+ exp	Frac. error
1	1	0.26098	0	3.83171	3.79444	9.729×10^{-3}	—	—
1	1	0.25905	0.05	3.86033	3.79444	1.707×10^{-2}	3.82887	8.149×10^{-3}
1	1	0.25375	0.1	3.94094	3.79444	3.717×10^{-2}	3.93017	2.734×10^{-3}
1	1	0.23609	0.2	4.23575	3.79444	1.042×10^{-1}	4.22384	2.810×10^{-3}
1	2	0.14254	0	7.01559	6.99700	2.649×10^{-3}	—	—
1	2	0.14076	0.05	7.10455	6.99700	1.514×10^{-2}	7.10706	-3.539×10^{-4}
1	2	0.13642	0.1	7.33057	6.99700	4.550×10^{-2}	7.35736	-3.655×10^{-3}
1	3	0.09829	0	10.1735	10.1609	1.233×10^{-3}	—	—
1	3	0.09664	0.05	10.3473	10.1609	1.801×10^{-2}	10.3748	-2.661×10^{-3}
2	1	0.38944	0	5.13562	5.10039	6.861×10^{-3}	—	—
2	1	0.38940	0.05	5.13607	5.10039	6.948×10^{-3}	5.10087	6.854×10^{-3}
2	1	0.38893	0.1	5.14235	5.10039	8.160×10^{-3}	5.10777	6.724×10^{-3}
2	1	0.38712	0.15	5.16643	5.10039	1.278×10^{-2}	5.13544	5.998×10^{-3}
2	1	0.38712	0.2	5.22177	5.10039	2.325×10^{-2}	5.20251	3.687×10^{-3}
2	1	0.37595	0.25	5.31987	5.10039	4.126×10^{-2}	5.32314	-6.156×10^{-4}
2	1	0.36562	0.3	5.47024	5.10039	6.761×10^{-2}	5.48048	-1.873×10^{-3}
2	1	0.35199	0.35	5.68201	5.10039	1.024×10^{-1}	5.59550	1.523×10^{-2}
2	2	0.23761	0	8.41724	8.40014	2.032×10^{-3}	—	—
2	2	0.23752	0.05	8.42022	8.40014	2.385×10^{-3}	8.40342	1.996×10^{-3}
2	2	0.23648	0.1	8.45741	8.40014	6.771×10^{-3}	8.44667	1.270×10^{-3}
2	2	0.23319	0.15	8.57689	8.40014	2.061×10^{-2}	8.59276	-1.851×10^{-3}
2	2	0.22718	0.2	8.80394	8.40014	4.587×10^{-2}	8.81309	-1.039×10^{-3}
3	1	0.47021	0	6.38016	6.34519	5.482×10^{-3}	—	—
3	1	0.47021	0.05	6.38017	6.34519	5.483×10^{-3}	6.34519	5.482×10^{-3}
3	1	0.47019	0.1	6.38045	6.34519	5.527×10^{-3}	6.34549	5.480×10^{-3}
3	1	0.46999	0.15	6.38312	6.34519	5.942×10^{-3}	6.34833	5.450×10^{-3}
3	1	0.46915	0.2	6.39460	6.34519	7.727×10^{-3}	6.36096	5.260×10^{-3}
3	1	0.46199	0.3	6.49372	6.34519	2.287×10^{-2}	6.47980	2.143×10^{-3}
3	1	0.44120	0.4	6.79964	6.34519	6.684×10^{-2}	6.79359	8.900×10^{-4}

Table 3: Comparision of eigenvalues of an annulus $c \leq r \leq 1$ approximated using the amplitude expansion to $A^{(0)}$ with and without the exponential correction. The values neglecting complex rays are calculated by solving (17) and (19) for k and b. Those including the complex ray contribution are calculated by solving (17) and (46) (with the amplitude expansion truncated at order k^0) for k and b.

the rays are complex. It is important that the reflected complex rays should reform a caustic at the same place as the incident complex rays, so that the real rays that are formed on the outside of this caustic belong to the same families as the original real rays. Thus the inner boundary must be of the right form (in the problem considered it is crucial that the inner boundary is a circle concentric with the outer boundary). The situation is analagous to that for the first family of eigenfunctions, in which the real rays hit the inner boundary (Fig 1a). If the inner boundary is not a circle concentric with the outer then the family of rays reflected from the inner boundary will not be the same as the family reflected from the outer boundary.

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