Nucleation of Superconductivity in Decreasing Fields II

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Abstract

The bifurcation from a normally conducting to a superconducting state as an external magnetic field is lowered is examined using the Ginzburg-Landau theory. Linear and weakly-nonlinear stability analyses are performed near the bifurcation point, and the implications of the results for each of three examples in considered.

1 Introduction

In a previous paper [3], hereafter referred to as I, we examined the response of a superconductor to an externally applied magnetic field, and found that there are a series of bifurcations from the normally conducting (normal) state to partially superconducting states as the magnetic field is lowered (see I, fig. 2). Here we extend this analysis by performing linear and weakly-nonlinear stability analyses near the bifurcation points. We also consider the implications of the results of these analyses for three specific examples, namely one-dimensional solutions on an infinite domain, one-dimensional solutions on a halfspace, and periodic two-dimensional solutions.

The response of a superconducting material in a magnetic field is described by the time-dependent Ginzburg-Landau equations, which, together with boundary conditions and other conditions are [9, 5]

$$\frac{\alpha}{\kappa^2}\frac{\partial \Psi}{\partial t} + \frac{\alpha i}{\kappa}\Psi\Phi + \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2\Psi = \Psi(1- |\Psi|^2) \quad \text{in } \Omega, \tag{1}$$

$$-(\text{curl})^2\boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi = \frac{i}{2\kappa}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + |\Psi|^2\,\boldsymbol{A} \quad \text{in } \Omega, \tag{2}$$

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$$
\begin{aligned}
-(\mathrm{curl})^2 \boldsymbol{A} &= \varsigma_e \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \Phi \right) \quad \text{outside } \Omega, & (3)\\
\nabla^2 \Phi &= 0 \text{ outside } \Omega, & (4)\\
\boldsymbol{n} \cdot ((i/\kappa)\nabla + \boldsymbol{A})\Psi + (i/d)\Psi &= 0 \quad \text{on } \partial\Omega, & (5)\\
[\boldsymbol{n} \wedge \boldsymbol{A}] &= \mathbf{0}, & (6)\\
[\boldsymbol{n} \wedge (1/\mu)\mathrm{curl}\ \boldsymbol{A}] &= \mathbf{0}, & (7)\\
[\Phi] &= 0, & (8)\\
\left[\varepsilon \frac{\partial \Phi}{\partial n}\right] &= 0, & (9)\\
\mathrm{curl}\ \boldsymbol{A} &\to h\hat{\boldsymbol{z}} \quad \text{as } r \to \infty, & (10)\\
\Phi &\to 0 \quad \text{as } r \to \infty, & (11)\\
\mathrm{div}\ \boldsymbol{A} &= 0. & (12)
\end{aligned}
$$

Here Ψ is the (complex) superconducting order parameter, $\boldsymbol{A}$ is the (real) magnetic vector potential, Φ is the (real) electric scalar potential, ε is the relative permittivity, ς_e is the conductivity of the region external to the superconductor, scaled with the normal conductivity of the superconducting region, $\hat{\boldsymbol{z}}$ is a unit vector in the z-direction, r is the distance from the origin, $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$, $[\,]$ denotes the jump in the enclosed quantity across $\partial\Omega$, and $*$ denotes complex conjugation. Equation (3) simply states that the current in the external region is given by Ohms law, while equation (4) states that there is no charge in this region. Equations (6), (7) and (8), (9) are the usual boundary conditions on the vector potential and scalar potential respectively at an interface between two media.

We now begin our analysis of the linear stability of the solution branches found in I. We adopt the convention that underscored equation and figure numbers refer to equations and figures in I, while plain numbers refer to equations and figures in the present paper.

2 Linear Stability of the Solution Branches

2.1 Linear Stability of the Solution Branches for a Body of Arbitrary Shape

Linear Stability of the Normal State We examine first the stability of the normal state. We make a small perturbation about the normal solution ($\underline{10}$), by setting

$$
\begin{aligned}
\Psi &= \delta e^{\sigma t} \Psi_1, & (13)\\
\boldsymbol{A} &= h\boldsymbol{A}_N + \delta e^{\sigma t} \boldsymbol{A}_1, & (14)\\
\Phi &= \delta e^{\sigma t} \Phi_1, \quad 0 < \delta \ll 1. & (15)
\end{aligned}
$$

Substituting (13)-(15) into (1)-(12) and linearising in δ (to give the leading order behaviour of an asymptotic expansion in powers of δ) yields

$$
\begin{aligned}
\frac{\alpha}{\kappa^2}\sigma\Psi_1 + \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2 \Psi_1 &= \Psi_1 \quad \text{in } \Omega, && (16)\\
-(\mathrm{curl})^2 \boldsymbol{A}_1 &= \sigma \boldsymbol{A}_1 + \nabla\Phi_1 \quad \text{in } \Omega, && (17)\\
-(\mathrm{curl})^2 \boldsymbol{A}_1 &= \varsigma_e \left(\sigma \boldsymbol{A}_1 + \nabla\Phi_1\right) \quad \text{outside } \Omega, && (18)\\
\nabla^2\Phi_1 &= 0 \text{ outside } \Omega, && (19)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h\boldsymbol{A}_N)\Psi_1 + (i/d)\Psi_1 &= 0 \quad \text{on } \partial\Omega, && (20)
\end{aligned}
$$

with $\boldsymbol{A}_1$ and Φ_1 satisfying (6)-(12). For each given h, (16) and (20) determine a discrete set of real eigenvalues for σ. For $h > h_{c_2}$ all these eigenvalues will be negative and the normal state is linearly stable. For $h < h_{c_2}$ we expect at least one of these eigenvalues to be positive, and the normal state to be linearly unstable. To see this we multiply (16) by Ψ_1^* and integrate over Ω to give

$$
\begin{aligned}
&\frac{\alpha\sigma}{\kappa^2}\int_\Omega |\,\Psi_1|^2 \; dV = \int_\Omega |\,\Psi_1|^2 \; dV - \int_\Omega \Psi_1^* \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)^2 \Psi_1 \, dV,\\
&= \int_\Omega |\,\Psi_1|^2 \; dV - \int_\Omega \left|\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1\right|^2 dV - \frac{i}{\kappa}\int_{\partial\Omega} \Psi_1^* \left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1 \, dS,\\
&= \int_\Omega |\,\Psi_1|^2 \; dV - \int_\Omega \left|\left(\frac{i}{\kappa}\nabla + h\boldsymbol{A}_N\right)\Psi_1\right|^2 dV - \frac{1}{\kappa d}\int_{\partial\Omega} |\,\Psi_1|^2 \; dS.
\end{aligned}
$$

As $h \to \infty$ we see that the second term on the right-hand side dominates the first, and hence all the eigenvalues σ are negative for large h. We expect the eigenvalues σ each to depend continuously on h, and one of the eigenvalues σ will pass through zero when and only when h passes through an eigenvalue of (21), (24). The largest of these eigenvalues is h_{c_2}. Thus for $h > h_{c_2}$ all the eigenvalues σ are negative. As h passes through h_{c_2} the largest eigenvalue σ will pass through zero, and hence we expect that for $h < h_{c_2}$ there will be at least one positive eigenvalue σ. This is the case in each of the examples that we consider.

Linear Stability of the Superconducting Branch We now consider a small perturbation of the previously found superconducting solution. We set

$$
\begin{aligned}
\Psi &= \Psi_0 + \delta e^{\sigma t}\Psi_1, && (21)\\
\boldsymbol{A} &= \boldsymbol{A}_0 + \delta e^{\sigma t}\boldsymbol{A}_1, && (22)\\
\Phi &= \delta\Phi_1 e^{\sigma t}, \quad 0 < \delta \ll 1, && (23)
\end{aligned}
$$

where $(\Psi_0, \boldsymbol{A}_0)$ is the steady superconducting solution given by (40)-(42). Substituting (21)-(23) into (1)-(12) and linearising in δ yields

$$\frac{\alpha}{\kappa^2}\sigma\Psi_1 + \frac{\alpha}{\kappa}i\Psi_0\Phi_1 + \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}_0\right)^2 \Psi_1 = -\frac{2i}{\kappa}\boldsymbol{A}_1 \cdot \nabla\Psi_0 - 2\boldsymbol{A}_0 \cdot \boldsymbol{A}_1\Psi_0$$
$$+ \Psi_1 - 2\,|\Psi_0|^2\,\Psi_1 - \Psi_0^2\Psi_1^* \quad \text{in } \Omega, \tag{24}$$

$$-(\text{curl})\boldsymbol{A}_1 - \sigma\boldsymbol{A}_1 - \nabla\Phi_1 = (i/2\kappa)(\Psi_0^*\nabla\Psi_1 + \Psi_1^*\nabla\Psi_0)$$
$$- (i/2\kappa)(\Psi_0\nabla\Psi_1^* + \Psi_1\nabla\Psi_0^*) + (\Psi_0\Psi_1^* + \Psi_0^*\Psi_1)\boldsymbol{A}_0 + |\Psi_0|^2\,\boldsymbol{A}_1 \quad \text{in } \Omega, \tag{25}$$

$$-(\text{curl})^2\boldsymbol{A_1} \;=\; \varsigma_e\,(\sigma\boldsymbol{A}_1 + \nabla\Phi_1) \quad \text{outside } \Omega, \tag{26}$$

$$\nabla^2\Phi_1 \;=\; 0 \;\; \text{outside } \Omega, \tag{27}$$

$$\boldsymbol{n} \cdot ((i/\kappa)\nabla + \boldsymbol{A}_0)\Psi_1 + \boldsymbol{n} \cdot \boldsymbol{A}_1\Psi_0 \;=\; -(i/d)\Psi_1 \quad \text{on } \partial\Omega, \tag{28}$$

with $\boldsymbol{A}_1$ and Φ_1 again satisfying (6)-(12). We examine the stability close to the bifurcation point by introducing ϵ as before:

$$\Psi_0 \;=\; \epsilon^{1/2}\psi_0, \tag{29}$$

$$\boldsymbol{A}_0 \;=\; h\boldsymbol{A}_N + \epsilon\boldsymbol{a}_0, \tag{30}$$

$$\Psi_1 \;=\; \epsilon^{1/2}\psi_1, \tag{31}$$

$$\boldsymbol{A}_1 \;=\; \epsilon\boldsymbol{a}_1, \tag{32}$$

$$\Phi_1 \;=\; \epsilon\phi_1. \tag{33}$$

Note that since we are expanding in powers of ϵ after linearising in δ we are assuming that $\delta \ll \epsilon$. In particular we will equate coefficients of $\epsilon\delta$ in the equations while neglecting terms of order δ^2. Hence we require that at least $\delta = o(\epsilon)$. If we wish to be definite about the relative sizes of ϵ and δ we may take, for example, $\delta = \epsilon^2$. In the next section we will perform a weakly nonlinear analysis near the bifurcation point which will not require such a restriction on the initial data. Substituting (29)-(33) into (24)-(28) yields

$$(\alpha\sigma/\kappa^2)\psi_1 - \psi_1 + ((i/\kappa) + h\boldsymbol{A}_N)^2\psi_1 = -(\epsilon\alpha i/\kappa)\psi_0\phi_1 - 2\epsilon\,|\psi_0|^2\,\psi_1 - \epsilon\psi_1^*\psi_0^2$$
$$- (2\epsilon i/\kappa)\boldsymbol{a}_0 \cdot \nabla\psi_1 - 2\epsilon h\boldsymbol{a}_0 \cdot \boldsymbol{A}_N\psi_1 - (2\epsilon i/\kappa)\boldsymbol{a}_1 \cdot \nabla\psi_0$$
$$- 2\epsilon h\boldsymbol{A}_N \cdot \boldsymbol{a}_1\psi_0 - \epsilon^2\,|\boldsymbol{a}_0|^2\,\psi_1 - 2\epsilon^2\boldsymbol{A}_0 \cdot \boldsymbol{a}_1\psi_0 \quad \text{in } \Omega, \tag{34}$$

$$-(\text{curl})^2\boldsymbol{a}_1 - \sigma\boldsymbol{a}_1 - \nabla\phi_1 = (i/2\kappa)(\psi_0^*\nabla\psi_1 + \psi_1^*\nabla\psi_0) - (i/2\kappa)(\psi_0\nabla\psi_1^*$$
$$+ \psi_1\nabla\psi_0^*) + (\psi_0\psi_1^* + \psi_0^*\psi_1)h\boldsymbol{A}_N + \epsilon(\psi_0\psi_1^* + \psi_0^*\psi_1)\boldsymbol{a}_0 + \epsilon\,|\psi_0|^2\,\boldsymbol{a}_1 \quad \text{in } \Omega, \tag{35}$$

$$-(\text{curl})^2\boldsymbol{a}_1 \;=\; \varsigma_e(\sigma\boldsymbol{a}_1 + \nabla\phi_1) \quad \text{outside } \Omega, \tag{36}$$

$$\nabla^2\phi_1 \;=\; 0 \quad \text{outside } \Omega, \tag{37}$$

$$\boldsymbol{n} \cdot ((i/\kappa)\nabla + h\boldsymbol{A}_N)\psi_1 + (i/d)\psi_1 \;=\; -\,\epsilon\boldsymbol{n} \cdot \boldsymbol{a}_1\psi_0 - \epsilon\boldsymbol{n} \cdot \boldsymbol{a}_0\psi_1 \quad \text{on } \partial\Omega, \tag{38}$$

with $\boldsymbol{a}_1$ and ϕ_1 satisfying (6)-(12). In I we obtained expansions in powers of ϵ for $\boldsymbol{a}_0$, ψ_0 and h near $h = h^{(0)}$. We expand also $\boldsymbol{a}_1$, ψ_1, ϕ_1 and σ in powers of ϵ:

$$\psi_1 = \psi_1^{(0)} + \epsilon\psi_1^{(1)} + \cdots, \tag{39}$$

$$a_1 = a_1^{(0)} + \epsilon a_1^{(1)} + \cdots, \tag{40}$$

$$\phi_1 = \phi_1^{(0)} + \epsilon\phi_1^{(1)} + \cdots, \tag{41}$$

$$\sigma = \sigma^{(0)} + \epsilon\sigma^{(1)} + \cdots. \tag{42}$$

Substituting these expansions into equations (34)-(38) and equating powers of ϵ we find at leading order

$$(\alpha\sigma^{(0)}/\kappa^2)\psi_1^{(0)} - \psi_1^{(0)} = -((i/\kappa) + h^{(0)}\boldsymbol{A}_N)^2\psi_1^{(0)} \quad \text{in } \Omega, \tag{43}$$

$$\begin{aligned} -(\mathrm{curl})^2\boldsymbol{a}_1^{(0)} - \sigma^{(0)}\boldsymbol{a}_1 - \nabla\phi_1^{(0)} &= (i/2\kappa)(\psi_0^{(0)*}\nabla\psi_1^{(0)} + \psi_1^{(0)*}\nabla\psi_0^{(0)}) \\ &- (i/2\kappa)(\psi_0^{(0)}\nabla\psi_1^{(0)*} + \psi_1^{(0)}\nabla\psi_0^{(0)*}) + (\psi_0^{(0)}\psi_1^{(0)*} + \psi_0^{(0)*}\psi_1^{(0)})h^{(0)}\boldsymbol{A}_N \text{ in } \Omega, \end{aligned} \tag{44}$$

$$-(\mathrm{curl})^2\boldsymbol{a}_1^{(0)} = \varsigma_e(\sigma^{(0)}\boldsymbol{a}_1^{(0)} + \nabla\phi_1^{(0)}) \quad \text{outside } \Omega, \tag{45}$$

$$\nabla^2\phi_1^{(0)} = 0 \quad \text{outside } \Omega, \tag{46}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h^{(0)}\boldsymbol{A}_N)\psi_1^{(0)} = -(i/d)\psi_1^{(0)} \text{ on } \partial\Omega, \tag{47}$$

with $\boldsymbol{a}_1^{(0)}$ and $\phi_1^{(0)}$ satisfying (6)-(12). Equations (43) and (47) are exactly equations (16) and (20). As before, if $h^{(0)} < h_{c_2}$ then there exists an unstable mode. Hence the solution branches bifurcating from eigenvalues $h^{(0)} < h_{c_2}$ are linearly unstable. It remains to determine the stability of the solution branch bifurcating from $h^{(0)} = h_{c_2}$. When $h^{(0)} = h_{c_2}$ all the eigenvalues $\sigma^{(0)}$ are negative except for the eigenvalue $\sigma^{(0)} = 0$. We must proceed to higher orders in our expansions to determine the stability of this mode. We note that for $\sigma^{(0)} = 0$, $\psi_1^{(0)}$ satisfies the same equation and boundary conditions as $\psi_0^{(0)}$, and hence $\psi_1^{(0)} \propto \psi_0^{(0)}$. Since the equations are linear in ψ_1 by construction the constant of proportionality is irrelevant and we take it to be unity (in effect this defines δ). Substituting into equations (44) we find

$$\begin{aligned} -(\mathrm{curl})^2\boldsymbol{a}_1^{(0)} - \nabla\phi_1^{(0)} &= (i/\kappa)(\psi_0^{(0)*}\nabla\psi_0^{(0)} - \psi_0^{(0)}\nabla\psi_0^{(0)*}) + 2\,|\,\psi_0^{(0)}|^2\, h_{c_2}\boldsymbol{A}_N \\ &= -2(\mathrm{curl})^2\boldsymbol{a}_0^{(0)} \text{ in } \Omega, \end{aligned} \tag{48}$$

by equation (22). Taking the divergence of this equation we find

$$\nabla^2\phi_1^{(0)} = 0, \text{ in } \Omega.$$

This, together with equation (46) and the boundary conditions on $\phi_1^{(0)}$ implies

$$\phi_1^{(0)} \equiv 0.$$

Now by comparing equations (48) and (45) with equations (22) and (23) we see that

$$\boldsymbol{a}_1^{(0)} = 2\boldsymbol{a}_0^{(0)}. \tag{49}$$

Equating powers of ϵ in equations (34) and (38) we find

$$\begin{aligned} -\psi_1^{(1)} + ((i/\kappa) + h_{c_2}\boldsymbol{A}_N)^2\psi_1^{(1)} &= -(\alpha\sigma^{(1)}/\kappa^2)\psi_0^{(0)} - 3\,|\,\psi_0^{(0)}\,|^2\,\psi_0^{(0)} - (6i/\kappa)\boldsymbol{a}_0^{(0)}\cdot\nabla\psi_0^{(0)} \\ &\quad - 6h_{c_2}\boldsymbol{a}_0^{(0)}\cdot\boldsymbol{A}_N\psi_0^{(0)} - (2i/\kappa)h^{(1)}\boldsymbol{A}_N\cdot\nabla\psi_0^{(0)} - 2h_{c_2}h^{(1)}\,|\,\boldsymbol{A}_N\,|^2\,\psi_0^{(0)} \qquad \text{in } \Omega, \end{aligned} \tag{50}$$

$$\boldsymbol{n}\cdot((i/\kappa)\nabla + h_{c_2}\boldsymbol{A}_N)\psi_1^{(1)} + (i/d)\psi_1^{(1)} \;=\; -3\boldsymbol{n}\cdot\boldsymbol{a}_0^{(0)}\psi_0^{(0)} - h^{(1)}\boldsymbol{n}\cdot\boldsymbol{A}_N\psi_0^{(0)} \text{ on } \partial\Omega. \tag{51}$$

Now, $\psi_0^{(0)}$ is a solution of the inhomogeneous version of equation (50) and boundary condition (51), namely (21) and (24). Hence there is a solution for $\psi_1^{(1)}$ if and only if an appropriate solvability condition is satisfied. To derive this condition we multiply (50) by $\psi_0^{(0)*}$ and integrate over Ω. After integration by parts a calculation similar to to that which led to (38) yields

$$\begin{aligned} \sigma^{(1)} &= -\frac{2\kappa^2}{\alpha\,|\,C\,|^2}\int_\Omega \left[|\,\psi^{(0)}\,|^4 - 2\boldsymbol{a}^{(0)}\cdot(\mathrm{curl})^2\boldsymbol{a}^{(0)}\right] dV \\ &= -(4\kappa^2 h^{(1)}/\alpha)\int_\Omega \boldsymbol{A}_N\cdot\mathrm{curl}^2\hat{\boldsymbol{a}}_0^{(0)}\,dV, \end{aligned} \tag{52}$$

by (38). We see that the sign of $\sigma^{(1)}$ depends on both the sign of $h^{(1)}$ and the sign of $\boldsymbol{A}_N\cdot\mathrm{curl}^2\hat{\boldsymbol{a}}_0^{(0)}$.

Let us now try and relate the above result about classical stability to the problem of free energy minimisation. If D represents the region outside Ω then the total dimensionless Gibbs free energy of the superconducting state is given by [8, 2]

$$\begin{aligned} \mathcal{G}_{sH} &= \int_\Omega \left|\frac{1}{\kappa}\nabla\Psi - i\boldsymbol{A}\Psi\right|^2 - |\,\Psi\,|^2 + \frac{|\,\Psi\,|^4}{2} + |\,\mathrm{curl}\ \boldsymbol{A}\,|^2 - 2h\,\mathrm{curl}\ \boldsymbol{A}\cdot\mathrm{curl}\ \boldsymbol{A}_N\,dV \\ &\quad + \frac{1}{\mu_e}\int_D |\,\mathrm{curl}\ \boldsymbol{A}\,|^2 - 2h\,\mathrm{curl}\ \boldsymbol{A}\cdot\mathrm{curl}\ \boldsymbol{A}_N\,dV + \int_{\partial\Omega}\frac{|\,\Psi\,|^2}{\kappa d}\,dS, \end{aligned}$$

where μ_e is the permeability of the external region normalised with that of the superconducting region. In the normal state $\Psi \equiv 0$, $\boldsymbol{A} = h\boldsymbol{A}_N$ and the total dimensionless Gibbs free energy is then

$$\mathcal{G}_{nH} = -h^2\int_\Omega |\,\mathrm{curl}\ \boldsymbol{A}_N\,|^2\ dV - \frac{h^2}{\mu_e}\int_D |\,\mathrm{curl}\ \boldsymbol{A}_N\,|^2\ dV.$$

Hence

$$
\begin{aligned}
\mathcal{G}_{sH}-\mathcal{G}_{nH} &= \int_\Omega \left|\frac{1}{\kappa}\nabla\Psi - i\boldsymbol{A}\Psi\right|^2 - |\,\Psi\,|^2 + \frac{|\,\Psi\,|^4}{2} + |\,\mathrm{curl}(\boldsymbol{A}-h\boldsymbol{A}_N)\,|^2 \; dV \\
&\quad + \frac{1}{\mu_e}\int_D |\,\mathrm{curl}(\boldsymbol{A}-h\boldsymbol{A}_N)\,|^2 \; dV + \int_{\partial\Omega}\frac{|\,\Psi\,|^2}{\kappa d}\,dS \\
&= \int_\Omega -\frac{|\,\Psi\,|^4}{2} + (\boldsymbol{A}-h\boldsymbol{A}_N)\cdot(\mathrm{curl})^2(\boldsymbol{A}-h\boldsymbol{A}_N)\,dV,
\end{aligned}
$$

on integration by parts, using (1), (3), (5)-(8) and (10)-(12). Substituting the equations (16), (17) and the expansions (19), (20) into this expression yields the leading order approximation to $\mathcal{G}_{sH}-\mathcal{G}_{nH}$ as

$$\epsilon^2\int_\Omega -\frac{|\,\psi^{(0)}\,|^4}{2} + \boldsymbol{a}^{(0)}\cdot(\mathrm{curl})^2\boldsymbol{a}^{(0)}\,dV.$$

Thus we see that $\sigma^{(0)} < 0$ if and only if $\mathcal{G}_{sH} < \mathcal{G}_{nH}$, i.e. the superconducting solution is stable if and only if it has a lower Gibbs free energy than that of the normal state.

2.2 A one-dimensional example

In the infinite one-dimensional case $\Phi \equiv 0$, and (16)-(20) become

$$
\begin{aligned}
-(\alpha\sigma/\kappa^2)\psi + (1/\kappa^2)\psi'' &= -\psi + h^2x^2\psi, && (53)\\
a'' &= \sigma a, && (54)\\
\psi' &\to 0 \qquad \text{as } |\,x\,|\to\infty, && (55)\\
a' &\to 0 \qquad \text{as } |\,x\,|\to\infty. && (56)
\end{aligned}
$$

where $' \equiv d/dx$. Equation (53) with the boundary condition (55) has solutions

$$e^{-h\kappa x^2/2}H_n(\sqrt{2h\kappa}\,x),$$

when

$$\sigma = \sigma_n = \frac{\kappa h}{\alpha}\left(\frac{\kappa}{h} - 2n - 1\right).$$

When $h > \kappa = h_{c_2}$ all the eigenvalues σ are negative and the normal state solution is linearly stable. When $h < \kappa$ at least one eigenvalue is positive, and the normal state is linearly unstable. Thus it ought to be possible to 'supercool' a type I superconductor down to a field $h = \kappa$, at which point the normal state becomes linearly unstable, and the conversion to a superconducting state will occur.

Examining the stability of the superconducting solution branch, (43)-(47) become

$$-(\alpha\sigma^{(0)}/\kappa^2)\psi_1^{(0)} + (1/\kappa^2)\psi_1^{(0)\prime\prime} = -\psi_1^{(0)} + (h^{(0)})^2 x^2 \psi_1^{(0)}, \tag{57}$$

$$a_1^{(0)\prime\prime} = \sigma^{(0)} a_1^{(0)} + 2h^{(0)} x \psi_0^{(0)} \psi_1^{(0)}, \tag{58}$$

$$\psi_1^{(0)\prime} \to 0, \qquad \text{as } |x| \to \infty, \tag{59}$$

$$a_1^{(0)\prime} \to 0, \qquad \text{as } |x| \to \infty. \tag{60}$$

Equation (57) with the boundary conditions (60) is exactly equation (53) with corresponding boundary conditions (56). Hence there are non-zero solutions for $\psi_1^{(0)}$ when

$$\sigma^{(0)} = \frac{\kappa h^{(0)}}{\alpha}\left(\frac{\kappa}{h^{(0)}} - 2n - 1\right).$$

For the solution branches bifurcating at eigenvalues $h^{(0)} < \kappa$, we see that there is at least one positive eigenvalue for σ, namely the case $n = 0$. Thus there is at least one unstable mode, and the superconducting solution branch will be linearly unstable. For the solution branch bifurcating at $h^{(0)} = \kappa$ all the eigenvalues $\sigma^{(0)}$ are negative except for the eigenvalue $\sigma^{(0)} = 0$. To determine the stability of this mode we need to proceed to higher orders in our expansions.

When $\sigma^{(0)} = 0$ we have $\psi_1^{(0)} \propto e^{-\kappa^2 x^2/2}$. Since all the equations are linear in $\psi_1, \boldsymbol{a}_1$ and ϕ_1 by construction, the constant of proportionality is irrelevant and we take it to be $C\kappa^{1/2}/\pi^{1/4}$, so that $\psi_1^{(0)} = \psi_0^{(0)}$ (in effect this defines δ). The solution for $a_1^{(0)}$ is then

$$a_1^{(0)} = -\frac{C^2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-\kappa^2 \tilde{x}^2}\, d\tilde{x} \tag{61}$$

Hence, using (52) we have

$$\begin{aligned}\sigma^{(1)} &= (4\kappa^2 h^{(1)}/\alpha\sqrt{\pi}) \int_{-\infty}^{\infty} \kappa^2 x^2 e^{-\kappa^2 x^2}\, dx \\ &= 2\kappa h^{(1)}/\alpha, \\ &= \frac{\kappa(1-2\kappa^2)C^2}{\alpha\sqrt{2\pi}}.\end{aligned}$$

Note that $\sigma^{(1)} < 0$ if and only if $\kappa > 1/\sqrt{2}$, i.e. if the superconductor is of type II. Thus for type I superconductors the bifurcation at $h = \kappa$ is subcritical, for type II superconductors it is supercritical. Fig. 1 shows the stability of the solution branches in the response diagrams of type I and type II superconductors.

When $\kappa = 1/\sqrt{2}$ we find $\sigma^{(1)} = 0$. This was expected, since we know that in this case there exist solutions of arbitrary amplitude at $h = \kappa$. Infact, we expect $\sigma^{(n)} = 0 \quad \forall n$. The stability of vertical bifurcations is an interesting, and as yet unapproached, problem. We will return to this point in the next section when we consider the weakly nonlinear stability of the normal state solution.

2.3 Surface superconductivity

We have, for a perturbation of the superconducting solution branch $f_1^{(0)} = f_0^{(0)}$, $a_1^{(0)} = 2a_0^{(0)}$. Substituting into equation (52) we find

$$\sigma^{(1)} = \frac{4\kappa^2 h^{(1)}}{\alpha C^2 h_{c_3}} \int_0^\infty (h_{c_3}x + c^{(0)})^2 (f_0^{(0)})^2 \, dx.$$

Hence $\sigma^{(1)} < 0$ if and only if $h^{(1)} < 0$.

Figure 7 shows a plot of the value of κ at which the solution becomes stable for different values of γ $(= \sinh^{-1}(1/d))$. We also show the value of κ at which the critical field h_{c_3} becomes greater than the critical field H_c, which is usually taken to be the criterion for the observation of surface superconductivity. As we can see the two values differ. Figure 2 shows the value of h_{c_3} at which the solution becomes stable with the critical field H_c as a reference. We see that in fact the solution is stable for values of h_{c_3} less than H_c.

2.4 The mixed state

Before we begin our analysis of the linear stability of the superconducting solution branches, we note that each solution contained an arbitrary parameter k, such that the period in the y direction was $2\pi/k$. In I we varied k so as to minimize the free energy. However, the steady state solution existed for all k, and hence a classical linear stability analysis of the effects of perturbations of k would simply reveal neutral stability. Nonetheless, a linear stability analysis is useful to determine the stability of the various solutions to perturbations of the C_n, for fixed k and N, since a solution may be locally stable, even though it is globally unstable. The problem can be thought of as an initial value problem, in which we perturb the C_n slightly from their equilibrium values and observe the evolution of the solution, requiring that the period in both the x and y-direction be fixed.

For a perturbation of the superconducting solution branch we find that in this case equations (43)-(47) become

$$\frac{\alpha\sigma^{(0)}}{\kappa^2}\psi_1^{(0)} - \frac{1}{\kappa^2}\left(\frac{\partial^2\psi_1^{(0)}}{\partial x^2} + \frac{\partial^2\psi_1^{(0)}}{\partial y^2}\right) + \frac{2ih^{(0)}x}{\kappa}\frac{\partial\psi_1^{(0)}}{\partial y} + (h^{(0)})^2x^2\psi_1^{(0)} = \psi_1^{(0)}, \tag{62}$$

$$-\frac{\partial^2 a_1^{(0)}}{\partial x\partial y} - \frac{\partial\phi_1^{(0)}}{\partial x} = \frac{i}{2\kappa}\left(\psi_0^{(0)*}\frac{\partial\psi_1^{(0)}}{\partial x} + \psi_1^{(0)*}\frac{\partial\psi_0^{(0)}}{\partial x} - \psi_0^{(0)}\frac{\partial\psi_1^{(0)*}}{\partial x} - \psi_1^{(0)}\frac{\partial\psi_0^{(0)*}}{\partial x}\right), \tag{63}$$

$$\begin{aligned}\frac{\partial^2 a_1^{(0)}}{\partial x^2} - \sigma^{(0)}a_1^{(0)} - \frac{\partial\phi_1^{(0)}}{\partial y} &= \frac{i}{2\kappa}\left(\psi_0^{(0)*}\frac{\partial\psi_1^{(0)}}{\partial y} + \psi_1^{(0)*}\frac{\partial\psi_0^{(0)}}{\partial y} - \psi_0^{(0)}\frac{\partial\psi_1^{(0)*}}{\partial y} - \psi_1^{(0)}\frac{\partial\psi_0^{(0)*}}{\partial y}\right) \\ &\quad + h^{(0)}x(\psi_0^{(0)}\psi_1^{(0)*} + \psi_0^{(0)*}\psi_1^{(0)}),\end{aligned} \tag{64}$$

with $\partial\psi_1^{(0)}/\partial x$, $(i/\kappa)\partial\psi_1^{(0)}/\partial y + h^{(0)}x\psi_1^{(0)}$, $\partial a_1^{(0)}/\partial x$, and $\phi_1^{(0)}$ periodic. As in one dimension, equation (62) has solutions when

$$\sigma^{(0)} = \frac{\kappa h^{(0)}}{\alpha}\left(\frac{\kappa}{h^{(0)}} - 2n - 1\right).$$

For $h^{(0)} < \kappa$ there is always an unstable mode. Hence the superconducting branches bifurcating from eigenvalues $h^{(0)} < \kappa$ are unstable. For $h^{(0)} = \kappa$ all modes are stable except the $n = 0$ mode which has $\sigma^{(0)} = 0$. To check the stability of this mode we must proceed to higher powers of ϵ in our expansions.

When $h^{(0)} = \kappa$ the solution of (62) with periodic boundary conditions is

$$\psi_1^{(0)} = \sum_{n=-\infty}^{\infty} B_n e^{inky}\psi_n(x), \tag{65}$$

where ψ_n is as before, and

$$B_{n+N} = B_n, \quad \forall n. \tag{66}$$

Substituting this expression back into equations (63), (64) we find

$$\begin{aligned}
\phi_1^{(0)} &\equiv 0, && (67)\\
a_1^{(0)} &= -\frac{1}{2\kappa}\sum_{m,n=-\infty}^{\infty}(C_n^* B_m + B_n^* C_m)e^{i(m-n)ky}\int^x \psi_m\psi_n\,dx. && (68)
\end{aligned}$$

Notice that in this case because there is not a unique superconducting solution we do not necessarily have $\boldsymbol{a}_1^{(0)} = 2\boldsymbol{a}_0^{(0)}$, $\psi_1^{(0)} = \psi_0^{(0)}$.

The solvability condition for the first order terms (52) is now

$$\begin{aligned}
\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_n}{(1-2\kappa^2)} &= \sum_{m,p} 2C_pC_m^*B_{n-p+m}\exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2+(p-m)^2\right]\right\}\\
&\quad + \sum_{m,p} C_pC_{n-p+m}B_m^*\exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2+(p-m)^2\right]\right\}\\
&\quad - \sum_{m,p}\frac{C_pC_m^*C_{n-p+m}B_n}{C_n}\exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2+(p-m)^2\right]\right\},
\end{aligned} \tag{69}$$

The problem now is to examine the growth rates of the various modes. Let us consider first the mode where $B_n = C_n$. Then

$$\frac{\sqrt{2}\,\alpha\sigma^{(1)}C_n}{(1-2\kappa^2)} = \sum_{m,p} C_pC_m^*C_{n-p+m}\exp\left\{-\frac{k^2}{2\kappa^2}\left[(p-n)^2+(p-m)^2\right]\right\}.$$

If we multiply by C_n^* and sum over n we find

$$\sigma^{(1)} = \left(\frac{1-2\kappa^2}{\alpha}\right)\frac{\overline{|\,\psi_0^{(0)}\,|^4}}{\overline{|\,\psi_0^{(0)}\,|^2}} = \frac{2\kappa h^{(1)}}{\alpha}. \tag{70}$$

We see that

$$\sigma^{(1)} < 0 \text{ if and only if } \kappa > 1/\sqrt{2}.$$

Hence for a type I superconductor, where $\kappa < 1/\sqrt{2}$, the mixed state is always unstable to this mode, whatever the value of N and the coefficients C_n. For a type II superconductor the mode $B_n = C_n$ is always stable.

In order to examine the growth rates of other modes we need to specify which value of N we are considering. For the case $N = 1$ the only possible mode is that examined above. We examine the case when $N = 2$. Let us consider first the rectangular lattice of Abrikosov, so that $C_n = C\ \forall n$. Then (69) becomes

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_0}{(1-2\kappa^2)} &= (|\,C\,|^2\,B_0 + C^2B_0^*)S_{2,0}^2 + (2\,|\,C\,|^2\,B_1 + C^2B_0^* - |\,C\,|^2\,B_0)S_{2,1}^2\\ &\quad + 2(|\,C\,|^2\,B_1 + C^2B_1^*)S_{2,0}S_{2,1}, \end{aligned}\tag{71}$$

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_1}{(1-2\kappa^2)} &= (|\,C\,|^2\,B_1 + C^2B_1^*)S_{2,0}^2 + (2\,|\,C\,|^2\,B_0 + C^2B_1^* - |\,C\,|^2\,B_1)S_{2,1}^2\\ &\quad + 2(|\,C\,|^2\,B_0 + C^2B_0^*)S_{2,0}S_{2,1}. \end{aligned}\tag{72}$$

We subtract (71) from (72) to obtain

$$\frac{2\sqrt{2}\,\alpha\sigma^{(1)}}{|\,C\,|^2\,(1-2\kappa^2)} = (1+E)(S_{2,0}-S_{2,1})^2 - 4S_{2,1}^2,$$

where $E = C(B_0 - B_1)^*/C^*(B_0 - B_1)$, $|\,E\,| = 1$. We are interested in the case $\kappa > 1/\sqrt{2}$. Then, when $E = -1$, we see

$$\sigma^{(1)} = -\frac{\sqrt{2}\,S_{2,1}^2\,|\,C\,|^2\,(1-2\kappa^2)}{\alpha} > 0.$$

Thus the mode given by $E = -1$ is unstable. Hence, when $N = 2$ the solution $C_n = C \quad \forall n$ is linearly unstable.

Let us now consider the solution $C_1 = iC_0$, $C_{n+2} = C_n\ \forall n$. Then (69) becomes

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_0}{(1-2\kappa^2)} &= (|\,C_0\,|^2\,B_0 + C_0^2B_0^*)S_{2,0}^2 + (2i\,|\,C_0\,|^2\,B_1 - C_0^2B_0^* + |\,C_0\,|^2\,B_0)S_{2,1}^2\\ &\quad + 2i(-\,|\,C_0\,|^2\,B_1 + C_0^2B_1^*)S_{2,0}S_{2,1}, \end{aligned}\tag{73}$$

$$\begin{aligned}\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_1}{(1-2\kappa^2)} &= (|\,C_0\,|^2\,B_1 - C_0^2B_1^*)S_{2,0}^2 + (-2i\,|\,C_0\,|^2\,B_0 + C_0^2B_1^* - |\,C_0\,|^2\,B_1)S_{2,1}^2\\ &\quad + 2i(|\,C_0\,|^2\,B_0 + C_0^2B_0^*)S_{2,0}S_{2,1}. \end{aligned}\tag{74}$$

We multiply (74) by i and add to (74) to obtain

$$\frac{2\sqrt{2}\,\alpha\sigma^{(1)}}{|\,C_0\,|^2\,(1-2\kappa^2)} \;=\; (1+E)(S_{2,0}-S_{2,1})^2+2(1-E)S_{2,1}^2,$$

where $E = C_0(B_0{+}iB_1)^*/C_0^*(B_0{+}iB_1)$. Since $|\,E\,|= 1$ we see that for $\kappa > 1/\sqrt{2}$, $\Re\{\sigma^{(1)}\} < 0$. Hence, when $N = 2$ the solution $C_1 = iC_0$, $C_{n+2} = C_n$ is linearly stable, for all k. Thus we are in agreement with the free energy argument of Abrikosov, namely that the solution $C_1 = iC_0$, $C_{n+2} = C_n$ is preferred to the solution $C_n = C$. However, we do not know whether this solution is stable for other values of N, e.g. $N = 4$.

We note that the one-dimensional solution found previously can be represented by (82), but with $C_0 = C$, $C_n = 0$, $n \neq 0$, so that in this case $N = \infty$. We now check the stability of this solution with respect to a two-dimensional perturbation. Equation (69) becomes

$$\frac{2\sqrt{2}\,\alpha\sigma^{(1)}B_n}{(1-2\kappa^2)} =|\,C\,|^2\,B_n(2e^{-\frac{k^2n^2}{2\kappa^2}}-1)+C^2B_{-n}^*e^{-\frac{k^2n^2}{\kappa^2}}. \tag{75}$$

Taking note of the equation obtained by substituting $-n$ for n in (75) we see that this system of equations has solutions $|\,B_N\,|=|\,B_{-N}\,|\neq 0$, $B_n = 0$, $n \neq \pm N$, when

$$\sigma^{(1)} = \sigma_N^{(1)} = \frac{|\,C\,|^2\,(1-2\kappa^2)}{2\sqrt{2}\,\alpha}\left(2e^{-\frac{k^2N^2}{2\kappa^2}}-1+Ee^{-\frac{k^2N^2}{\kappa^2}}\right),$$

where $E = CB_N^*/C^*B_{-N}$, $|\,E\,|= 1$. We see that for modes $N \neq 0$ ($N = 0$ corresponds to a purely one-dimensional pertubation),

$$\sigma_N^{(1)} = -\frac{|\,C\,|^2\,(1-2\kappa^2)}{2\sqrt{2}\,\alpha}(1-e^{-\frac{k^2N^2}{2\kappa^2}})^2,$$

when $E = 1$. Therefore $\sigma^{(1)} > 0$ when $\kappa > 1/\sqrt{2}$ for this mode. Hence the one-dimensional solution is linearly unstable with respect to a two-dimensional perturbation.

Thus we see that the true form of the mixed state in a type II superconductor will be two-dimensional, even when the geometry allows a one-dimensional solution. This is in agreement with the following common heuristic argument. It can be shown that the surface energy of a normal/superconducting transition region in a type II superconductor is negative [4]. Thus it is energetically advantageous for the superconductor to form many small blips of superconducting material rather than sheets of superconducting material, leading to a two-dimensional mixed state as described above and in I. This form of the mixed state has also been observed experimentally [7].

3 Weakly-nonlinear Stability of the Normal State Solution

We complete our analysis with a weakly-nonlinear analysis near the bifurcation point.

3.1 Weakly-nonlinear Stability of the Normal State in a Body of Arbitrary Shape

We seek a solution to the time-dependent Ginzburg-Landau equations (1)-(12) near the bifurcation point $h = h_{c_2}$. To this end we set

$$h = h_{c_2} + \epsilon h^{(1)}. \tag{76}$$

We again introduce $\psi, \boldsymbol{a}$, and ϕ by setting

$$\Psi = \epsilon^{1/2}\psi, \tag{77}$$

$$\boldsymbol{A} = h\boldsymbol{A}_N + \epsilon\boldsymbol{a}, \tag{78}$$

$$\Phi = \epsilon\phi. \tag{79}$$

When we examined the linear stability of the normal state and superconducting state solutions near the bifurcation point we found that one mode had growth/decay timescale of $O(\epsilon^{-1})$ while all other modes had a decay timescale of $O(1)$. Thus we expect when we examine the nonlinear behaviour of the solution that there will be two timescales: an $O(1)$ timescale and an $O(\epsilon^{-1})$ timescale. We note that in the case of weakly nonlinear hydrodynamic stability [6] the problem often requires a multiple scales analysis, since the linearised short-time solution is often a wave, and the long-time solution a modulated wave. In the present situation the short-time solution tends to a single real exponential as time increases, and we may proceed via matched asymptotic expansions, treating each timescale separately.

A. Short timescale : $t = O(1)$. We denote the short-time solution by $\psi_s(\boldsymbol{r}, t)$, $\boldsymbol{a}_s(\boldsymbol{r}, t)$, $\phi_s(\boldsymbol{r}, t)$, and expand all quantities in powers of ϵ as before:

$$\psi_s = \psi_s^{(0)} + \epsilon\psi_s^{(1)} + \cdots, \tag{80}$$

$$\boldsymbol{a}_s = \boldsymbol{a}_s^{(0)} + \epsilon\boldsymbol{a}_s^{(1)} + \cdots, \tag{81}$$

$$\phi_s = \phi_s^{(0)} + \epsilon\phi_s^{(1)} + \cdots. \tag{82}$$

Substituting (76)-(82) into equations (1)-(12) and equating powers of ϵ yields at leading order

$$\begin{aligned}
\frac{\alpha}{\kappa^2}\frac{\partial\psi_s^{(0)}}{\partial t} + \left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2\psi_s^{(0)} &= \psi_s^{(0)} \text{ in } \Omega, && (83)\\
-(\mathrm{curl})^2\boldsymbol{a}_s^{(0)} - \frac{\partial\boldsymbol{a}_s^{(0)}}{\partial t} - \nabla\phi_s^{(0)} &= \frac{i}{2\kappa}(\psi_s^{(0)*}\nabla\psi_s^{(0)} - \psi_s^{(0)}\nabla\psi_s^{(0)*}) \\
&\qquad + h_{c_2}\,|\,\psi_s^{(0)}|^2\,\boldsymbol{A}_N \text{ in } \Omega, && (84)\\
-(\mathrm{curl})^2\boldsymbol{a}_s^{(0)} &= \varsigma_e\left(\frac{\partial\boldsymbol{a}_s^{(0)}}{\partial t} + \nabla\phi_s^{(0)}\right) \text{ outside } \Omega, && (85)\\
\nabla^2\phi_s^{(0)} &= 0 \text{ outside } \Omega, && (86)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h_{c_2}\boldsymbol{A}_N)\psi_s^{(0)} &= -(i/d)\psi_s^{(0)} \text{ on } \partial\Omega, && (87)
\end{aligned}$$

with $\boldsymbol{a}_s^{(0)}$ and $\phi_s^{(0)}$ satisfying (6)-(12). Equation (83) with the boundary condition (87) has solution

$$\psi_s^{(0)}(x,t) = \sum_{n=-\infty}^{\infty} C_n e^{\sigma_n t}\theta_n(\boldsymbol{r}), \tag{88}$$

where σ_n are the eigenvalues of

$$\begin{aligned}
\left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2\theta - \theta &= -\frac{\alpha}{\kappa^2}\sigma\theta \quad \text{ in } \Omega, && (89)\\
\boldsymbol{n}\cdot((i/\kappa)\nabla + h_{c_2}\boldsymbol{A}_N)\theta &= -(i/d)\theta \quad \text{ on } \partial\Omega, && (90)
\end{aligned}$$

with corresponding eigenfunctions θ_n, and C_n are constants. Note that equations (89), (90) are exactly equations (16), (20) with $h = h_{c_2}$, and hence the eigenvalues are real and the eigenfunctions corresponding to distinct eigenvalues are orthogonal. We know the largest eigenvalue is zero, so we specify $\sigma_0 = 0$. The C_n must be chosen such that

$$\sum_{n=-\infty}^{\infty} C_n\theta_n(\boldsymbol{r}) = \psi_s^{(0)}(\boldsymbol{r},0). \tag{91}$$

Multiplying (91) by $\theta_m^*(\boldsymbol{r})$ and integrating over Ω yields

$$C_m = \int_\Omega \psi_s^{(0)}(\boldsymbol{r},0)\theta_m^*(\boldsymbol{r})\,dV. \tag{92}$$

Thus

$$\psi_s^{(0)}(\boldsymbol{r},t) = \int_\Omega\left(\sum_{n=-\infty}^{\infty}\theta_n^*(\tilde{\boldsymbol{r}})e^{\sigma_n t}\theta_n(\boldsymbol{r})\right)\psi_s^{(0)}(\tilde{\boldsymbol{r}},0)\,d\tilde{V}. \tag{93}$$

We can then solve for $\boldsymbol{a}_s^{(0)}$ and $\phi_s^{(0)}$.

This leading-order solution ignores the growth of the unstable mode since the growth happens on a timescale of $O(\epsilon^{-1})$. We expect that if we proceed to determine the first order terms that we will find secular terms appearing, and that the solution will cease to be valid when $t = O(\epsilon^{-1})$.

B. Long timescale : $t = O(\epsilon^{-1})$. We now consider the long-time behaviour of the solution. We define

$$\tau = \epsilon t$$

and consider τ to be $O(1)$. We denote the long-time solution by $\psi_l(\boldsymbol{r},\tau)$, $\boldsymbol{a}_l(\boldsymbol{r},\tau)$, $\phi_l(\boldsymbol{r},\tau)$. Equations (1)-(12) become

$$\begin{aligned}\epsilon\frac{\alpha}{\kappa^2}\frac{\partial\psi_l}{\partial\tau} + \left(\frac{i}{\kappa}\nabla + (h_{c_2} + \epsilon h^{(1)})\boldsymbol{A}_N\right)^2 \psi_l - \psi_l = -\epsilon\frac{\alpha i}{\kappa}\psi_l\phi_l + \epsilon\psi_l \mid\psi_l\mid^2 \\ + 2\epsilon(h_{c_2} + \epsilon h^{(1)})\psi_l(\boldsymbol{A}_N\cdot\boldsymbol{a}_l) + \frac{2\epsilon i}{\kappa}(\boldsymbol{a}_l\cdot\nabla\psi_l) - \epsilon^2 \mid\boldsymbol{a}_l\mid^2 \psi_l \text{ in } \Omega,\end{aligned} \tag{94}$$

$$-(\text{curl})^2\boldsymbol{a}_l - \epsilon\frac{\partial\boldsymbol{a}_l}{\partial\tau} - \nabla\phi_l = \frac{i}{2\kappa}(\psi_l^*\nabla\psi_l - \psi_l\nabla\psi_l^*) + \mid\psi_l\mid^2 (h_{c_2} + \epsilon h^{(1)})\boldsymbol{A}_N + \epsilon\mid\psi_l\mid^2 \boldsymbol{a}_l \text{ in } \Omega, \tag{95}$$

$$\begin{aligned}-(\text{curl})^2\boldsymbol{a}_l &= \varsigma_e\left(\epsilon\frac{\partial\boldsymbol{a}_l}{\partial\tau} + \nabla\phi_l\right) \text{ outside } \Omega, &(96)\\ \nabla^2\phi_l &= 0 \text{ outside } \Omega, &(97)\\ \boldsymbol{n}\cdot((i/\kappa)\nabla + (h_{c_2} + \epsilon h^{(1)})\boldsymbol{A}_N)\psi_l + (i/d)\psi_l &= -\epsilon(\boldsymbol{n}\cdot\boldsymbol{a}_l)\psi_l \text{ on } \partial\Omega, &(98)\end{aligned}$$

with $\boldsymbol{a}_l$ and ϕ_l satisfying (6)-(12). We expand all quantities in powers of ϵ as before:

$$\begin{aligned}\psi_l &= \psi_l^{(0)} + \epsilon\psi_l^{(1)} + \cdots, &(99)\\ \boldsymbol{a}_l &= \boldsymbol{a}_l^{(0)} + \epsilon\boldsymbol{a}_l^{(1)} + \cdots, &(100)\\ \phi_l &= \phi_l^{(0)} + \epsilon\phi_l^{(1)} + \cdots. &(101)\end{aligned}$$

Substituting the expansions (99)-(101) into equations (94)-(98) and equating powers of ϵ yields at leading order

$$\begin{aligned}\left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2 \psi_l^{(0)} - \psi_l^{(0)} &= 0 \text{ in } \Omega, &(102)\\ -(\text{curl})^2\boldsymbol{a}_l^{(0)} - \nabla\phi_l^{(0)} &= \frac{i}{2\kappa}(\psi_l^{(0)*}\nabla\psi_l^{(0)} - \psi_l^{(0)}\nabla\psi_l^{(0)*}) + h_{c_2}\mid\psi_l^{(0)}\mid^2 \boldsymbol{A}_N \text{ in } \Omega, &(103)\\ -(\text{curl})^2\boldsymbol{a}_l^{(0)} &= \varsigma_e\nabla\phi_l^{(0)} \text{ outside } \Omega, &(104)\\ \nabla^2\phi_l^{(0)} &= 0 \text{ outside } \Omega, &(105)\\ \boldsymbol{n}\cdot((i/\kappa)\nabla + h_{c_2}\boldsymbol{A}_N)\psi_l^{(0)} &= -(i/d)\psi_l^{(0)} \text{ on } \partial\Omega, &(106)\end{aligned}$$

with $\boldsymbol{a}_l^{(0)}$ and $\phi_l^{(0)}$ satisfying (6)-(12). Equations (102) and (106) are exactly equations (21) and (24) with $h^{(0)} = h_{c_2}$, and as such have solution

$$\psi_l^{(0)} = C(\tau)\theta_0, \tag{107}$$

where $C(\tau)$ is an unknown function of τ and θ_0 is as before. Now by comparing (103) with (29) we see

$$-(\mathrm{curl})^2\boldsymbol{a}_l^{(0)} - \nabla\phi_l^{(0)} = -\mid C(\tau)\mid^2 (\mathrm{curl})^2\hat{\boldsymbol{a}}_0^{(0)} \text{ in } \Omega, \tag{108}$$

where $\hat{\boldsymbol{a}}_0$ is the previously found steady-state superconducting solution, which is independent of τ. Hence, in the same way as before, $\phi_l^{(0)} \equiv 0$, $\boldsymbol{a}_l^{(0)} = \mid C(\tau)\mid^2 \hat{\boldsymbol{a}}_0^{(0)}$. To determine $C(\tau)$ we must proceed to higher orders in our expansions in ϵ. Equating powers of ϵ in (94), (98) yields

$$\begin{aligned}\left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2\psi_l^{(1)} - \psi_l^{(1)} &= -\frac{\alpha}{\kappa^2}\frac{\partial\psi_l^{(0)}}{\partial\tau} - \mid\psi_l^{(0)}\mid^2\psi_l^{(0)} - 2h_{c_2}h^{(1)}\mid\boldsymbol{A}_N\mid^2\psi_l^{(0)}\\ &- \frac{2ih^{(1)}}{\kappa}(\boldsymbol{A}_N\cdot\nabla\psi_l^{(0)}) + 2h_{c_2}(\boldsymbol{A}_N\cdot\boldsymbol{a}_l^{(0)})\psi_l^{(0)} + \frac{2i}{\kappa}(\boldsymbol{a}_l^{(0)}\cdot\nabla\psi_l^{(0)}) \text{ in } \Omega,\end{aligned} \tag{109}$$

$$\boldsymbol{n}\cdot\left(\frac{i}{\kappa} + h_{c_2}\boldsymbol{A}_N\right)\psi_l^{(1)} - \frac{i}{d}\psi_l^{(1)} = -\boldsymbol{n}\cdot(\boldsymbol{a}_l^{(0)} + h^{(1)}\boldsymbol{A}_N)\psi_l^{(0)} \text{ on } \partial\Omega. \tag{110}$$

Substituting in our expressions for $\psi_l^{(0)}$ and $\boldsymbol{a}_l^{(0)}$ we find

$$\begin{aligned}\left(\frac{i}{\kappa}\nabla + h_{c_2}\boldsymbol{A}_N\right)^2\psi_l^{(1)} - \psi_l^{(1)} &= -\frac{\alpha}{\kappa^2}\frac{dC}{d\tau}\theta_0 - \mid C\mid^2 C\mid\theta_0\mid^2\theta_0 - 2Ch_{c_2}h^{(1)}\mid\boldsymbol{A}_N\mid^2\theta_0\\ &- \frac{2iCh^{(1)}}{\kappa}(\boldsymbol{A}_N\cdot\nabla\theta_0) + 2\mid C\mid^2 Ch_{c_2}(\boldsymbol{A}_N\cdot\hat{\boldsymbol{a}}_0^{(0)})\theta_0 + \frac{2i\mid C\mid^2 C}{\kappa}(\hat{\boldsymbol{a}}_0^{(0)}\cdot\nabla\theta_0) \text{ in } \Omega,\end{aligned} \tag{111}$$

$$\boldsymbol{n}\cdot\left(\frac{i}{\kappa} + h_{c_2}\boldsymbol{A}_N\right)\psi_l^{(1)} - \frac{i}{b}\psi_l^{(1)} = -C\boldsymbol{n}\cdot(\mid C\mid^2\hat{\boldsymbol{a}}_0^{(0)} + h^{(1)}\boldsymbol{A}_N)\theta_0 \text{ on } \partial\Omega. \tag{112}$$

As before, θ_0 is a solution of the homogeneous versions of equations (111), (112) and therefore there is a solution for $\psi_l^{(1)}$ if and only if an appropriate solvability condition is satisfied. This condition is derived by multiplying by θ_0^* and integrating over Ω. A calculation very similar to that preceding (39) yields

$$\frac{\alpha}{\kappa^2}\frac{dC}{d\tau} = \mid C\mid^2 C\left[2\int_\Omega \hat{\boldsymbol{a}}_0^{(0)}\cdot(\mathrm{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV - \int_\Omega\mid\theta_0\mid^4\,dV\right] + 2h^{(1)}C\int_\Omega \boldsymbol{A}_N\cdot(\mathrm{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV. \tag{113}$$

Equation (113) is the Landau equation for our weakly nonlinear stability analysis. The boundary condition for this equation at $\tau = 0$ is given by matching with the short-time solution. We find

$$C(0)\theta_0 = \lim_{t\to\infty}\psi_s^{(0)} = C_0\theta_0,$$

since all the other eigenvalues σ_n in the expression (88) are negative. Hence

$$C(0) = C_0 = \int_\Omega \psi_s^{(0)}(\boldsymbol{r},0)\theta_0^*(\boldsymbol{r})\,dV. \tag{114}$$

The coefficients in equation (113), although given by integrals of the steady state solution, are simply real numbers. We simplify our expressions by writing

$$p = 2\int_\Omega \hat{\boldsymbol{a}}_0^{(0)}\cdot(\mathrm{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV - \int_\Omega |\theta_0|^4\ dV, \tag{115}$$

$$q = 2h^{(1)}\int_\Omega \boldsymbol{A}_N\cdot(\mathrm{curl})^2\hat{\boldsymbol{a}}_0^{(0)}\,dV. \tag{116}$$

Note that these are also the quantities that determine the sign of $h^{(1)}$ and the linear stability of the superconducting solution branch. We have

$$\frac{\alpha}{\kappa^2}\frac{dC}{d\tau} = p\,|C|^2\,C + qC. \tag{117}$$

Let $C = re^{i\vartheta}$, $C_0 = r_0 e^{i\vartheta_0}$. Then

$$\begin{aligned}\frac{\alpha}{\kappa^2}\frac{dr}{d\tau} &= pr^3 + qr,\\ \frac{d\vartheta}{d\tau} &= 0,\\ r(0) = r_0 \quad &, \quad \vartheta(0) = \vartheta_0.\end{aligned}$$

Therefore

$$\vartheta \equiv \vartheta_0,$$

$$r^2 = \begin{cases} \frac{q}{p}\left(\frac{De^{(2\kappa^2 q/\alpha)\tau}}{1-De^{(2\kappa^2 q/\alpha)\tau}}\right) & \text{if } q/p>0,\\ \begin{cases} -\frac{q}{p}\left(\frac{De^{(2\kappa^2 q/\alpha)\tau}}{De^{(2\kappa^2 q/\alpha)\tau}-1}\right) & \text{if } r_0^2 > -q/p\\ -\frac{q}{p}\left(\frac{De^{(2\kappa^2 q/\alpha)\tau}}{De^{(2\kappa^2 q/\alpha)\tau}+1}\right) & \text{if } r_0^2 < -q/p\end{cases} & \text{if } q/p<0.\end{cases} \tag{118}$$

where

$$D = \frac{r_0^2}{|r_0^2 + q/p|}.$$

3.2 One-dimensional Example

A. Short timescale : $t = O(1)$. In the infinite one-dimensional case the leading-order short-time equations (83)-(87) are

$$-\frac{\alpha}{\kappa^2}\frac{\partial\psi_s^{(0)}}{\partial t} + \frac{1}{\kappa^2}\frac{\partial^2\psi_s^{(0)}}{\partial x^2} = -\psi_s^{(0)} + \kappa^2 x^2\psi_s^{(0)}, \tag{119}$$

$$\frac{\partial^2 a_s^{(0)}}{\partial x^2} = \frac{\partial a_s^{(0)}}{\partial t} + \kappa x(\psi_s^{(0)})^2, \tag{120}$$

$$\frac{\partial\psi_s^{(0)}}{\partial x} \to 0, \text{ as } |x|\to\infty, \tag{121}$$

$$\frac{\partial a_s^{(0)}}{\partial x} \to 0, \text{ as } |x|\to\infty. \tag{122}$$

The solution to (119) with boundary condition (122) is

$$\psi_s^{(0)}(x,t) = \sum_{n=0}^{\infty} C_n e^{\sigma_n t}\theta_n(x),$$

where $\sigma_n = -2n\kappa^2/\alpha$, with corresponding eigenfunctions

$$\theta_n(x) = \frac{\kappa^{1/2} e^{-\kappa^2 x^2/2}}{(2^n n! \sqrt{\pi})^{1/2}} H_n(\sqrt{2}\,\kappa x),$$

and

$$C_m = \int_{-\infty}^{\infty} \psi_s^{(0)}(x,0)\theta_m(x)\,dx. \tag{123}$$

Hence

$$\psi_s^{(0)}(x,t) = \int_{-\infty}^{\infty} \psi_s^{(0)}(\xi,0)\left(\sum_{n=0}^{\infty}\theta_n(\xi)\theta_n(x)e^{\sigma_n t}\right)\,d\xi. \tag{124}$$

However, in one dimension we can write the Green's function in (124) in closed form. We find

$$\psi_s^{(0)}(x,t) = \int_{-\infty}^{\infty} \psi_s^{(0)}(\xi,0)G(\xi;x,t)\,d\xi, \tag{125}$$

where

$$G(\xi;x,t) = \frac{\kappa}{\sqrt{2\pi\sinh(2t)}}\exp\left\{t - \frac{\kappa^2}{2}[(x^2+\xi^2)\coth(2t) - 2x\xi\,\mathrm{cosech}(2t)]\right\}. \tag{126}$$

When $\psi_s^{(0)}$ is given by (125) we can then solve (120), (122) for $a_s^{(0)}$.

B. Long timescale : $t = O(\epsilon^{-1})$ The leading-order long-time solution is now

$$\begin{aligned} \psi_l^{(0)} &= C(\tau)\frac{\kappa^{1/2}}{\pi^{1/4}}e^{-\kappa^2 x^2/2}, & (127)\\ a_l^{(0)} &= -C(\tau)^2\frac{1}{2\pi^{1/2}}\int_{-\infty}^{x} e^{-\kappa^2\xi^2}\,d\xi, & (128)\end{aligned}$$

where $C(\tau)$ is determined by the Landau equation (117) with

$$p = \frac{\kappa}{\sqrt{2\pi}}\left[\frac{1}{2\kappa^2} - 1\right],\ q = -\frac{h^{(1)}}{\kappa},$$

and the boundary condition

$$C(0) = C_0 = \frac{\kappa^{1/2}}{\pi^{1/4}}\int_{-\infty}^{\infty}\psi_s^{(0)}(\xi,0)e^{-\kappa^2\xi^2/2}\,d\xi. \tag{129}$$

Hence

$$C^2 = \begin{cases} \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right| \left(\frac{De^{-(2h^{(1)}\kappa/\alpha)\tau}}{1-De^{-(2h^{(1)}\kappa/\alpha)\tau}}\right) & \text{if } \frac{2\sqrt{2\pi}\,h^{(1)}}{1-2\kappa^2} < 0 \\ \begin{cases} \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right| \left(\frac{De^{-(2h^{(1)}\kappa/\alpha)\tau}}{De^{-(2h^{(1)}\kappa/\alpha)\tau}-1}\right) & \text{if } C_0^2 > \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right| \\ \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right| \left(\frac{De^{-(2h^{(1)}\kappa/\alpha)\tau}}{De^{-(2h^{(1)}\kappa/\alpha)\tau}+1}\right) & \text{if } C_0^2 < \left|\frac{2\sqrt{2\pi}}{1-2\kappa^2}\right| \end{cases} & \text{if } \frac{2\sqrt{2\pi}\,h^{(1)}}{1-2\kappa^2} > 0 \end{cases},$$

where

$$D = \frac{C_0^2}{|\,C_0^2 - h^{(1)}\frac{2\sqrt{2\pi}}{1-2\kappa^2}\,|}.$$

There are four cases to consider.

A. Type I superconductors : $\kappa < 1/\sqrt{2}$

(1) $h < \kappa$ **:** $h^{(1)} < 0$. In this case

$$C^2 = \frac{2\sqrt{2\pi}}{1-2\kappa^2}\left(\frac{De^{-(2h^{(1)}\kappa/\alpha)\tau}}{1-De^{-(2h^{(1)}\kappa/\alpha)\tau}}\right).$$

The solution blows up in finite time $\tau = \tau_\infty = -(\alpha/2\kappa h^{(1)})\log(1/D)$, as shown in Fig. 3

(2) $h > \kappa$ **:** $h^{(1)} > 0$. In this case

$$C^2 = \begin{cases} \frac{2\sqrt{2\pi}}{1-2\kappa^2}\left(\frac{De^{-(2h^{(1)}\kappa/\alpha)\tau}}{De^{-(2h^{(1)}\kappa/\alpha)\tau}-1}\right) & \text{if } C_0^2 > \frac{2\sqrt{2\pi}}{1-2\kappa^2} \\ \frac{2\sqrt{2\pi}}{1-2\kappa^2}\left(\frac{De^{-(2h^{(1)}\kappa/\alpha)\tau}}{De^{-(2h^{(1)}\kappa/\alpha)\tau}+1}\right) & \text{if } C_0^2 < \frac{2\sqrt{2\pi}}{1-2\kappa^2}. \end{cases}$$

Note that $C^2 = \frac{2\sqrt{2\pi}}{1-2\kappa^2}$ is the (in this case unstable) steady state solution found previously. We see that if C_0 is small enough the solution will decay exponentially to zero. However, if C_0 is greater than a critical value the solution will blow up in finite time $\tau = \tau_\infty = (\alpha/2\kappa h^{(1)})\log D$. The dividing line between these two types of behaviour is the unstable steady state solution (see Fig. 4). Thus although the normal state solution is linearly stable in this parameter regime we see that it is unstable to sufficiently large initial perturbations.

B. Type II superconductors : $\kappa > 1/\sqrt{2}$.

(1) $h < \kappa$ **:** $h^{(1)} < 0$. In this case

$$C^2 = \begin{cases} \frac{2\sqrt{2\pi}}{2\kappa^2-1}\left(\frac{De^{-(2h^{(1)}\kappa/\alpha)\tau}}{De^{-(2h^{(1)}\kappa/\alpha)\tau}-1}\right) & \text{if } C_0^2 > \frac{2\sqrt{2\pi}}{2\kappa^2-1} \\ \frac{2\sqrt{2\pi}}{2\kappa^2-1}\left(\frac{De^{-(2h^{(1)}\kappa/\alpha)\tau}}{De^{-(2h^{(1)}\kappa/\alpha)\tau}+1}\right) & \text{if } C_0^2 < \frac{2\sqrt{2\pi}}{2\kappa^2-1}. \end{cases}$$

In either case we see that

$$C^2 \to \frac{2\sqrt{2\pi}}{2\kappa^2-1}, \text{ as } \tau \to \infty,$$

(see Fig. 5). Thus given any initial data the solution tends to the stable superconducting state solution as $\tau \to \infty$.

(2) $h > \kappa$ **:** $h^{(1)} > 0$**.** In this case

$$C^2 = \frac{2\sqrt{2\pi}}{2\kappa^2-1}\left(\frac{De^{-(2h^{(1)}\kappa/\alpha)\tau}}{1-De^{-(2h^{(1)}\kappa/\alpha)\tau}}\right).$$

The solution decays exponentially to zero (see Fig. 6). Thus the normal state solution is both linearly and nonlinearly stable in this parameter regime.

The finite-time blow up of the solution under certain conditions is worth further comment. This does not mean that the solution of the Ginzburg-Landau equations is unbounded, rather that the expansion in powers of ϵ ceases to be valid, since Ψ is no longer $\ll 1$. A complete determination of the solution would involve a new asymptotic expansion after choosing a new time variable

$$\epsilon t' = (\tau - \tau_\infty),$$

and treating Ψ as order one. The blow-up of the above solution would then provide a condition at $t' = -\infty$ for this solution. However, once Ψ becomes $O(1)$ we are faced with solving the one-dimensional Ginzburg-Landau equations in their entirety. We conjecture that the solution would evolve into a superconducting region with $\Psi \approx 1$, separated from the surrounding normal region by two propagating phase boundaries, as described in a previous paper [4].

Finally, we comment on the case $\kappa = 1/\sqrt{2}$. In this case $p = 0$, and so

$$\frac{dC}{d\tau} = qC.$$

For $h^{(1)} < 0$, $q > 0$ there is exponential growth, and for $h^{(1)} > 0$, $q < 0$ there is exponential decay. Thus as h is reduced through h_{c_2} ($= H_c$) there will be an abrupt transition to a wholly superconducting state which we conjecture to proceed by means of propagating phase boundaries (in this case with no supercooling). However, when the field is exactly at h_{c_2} any of the steady one-dimensional 'mixed states', as shown in fig. 4, is possible, and indeed no one is preferred to any other.

We note that the existence of the vertical bifurcation when $\kappa = 1/\sqrt{2}$ is dependent upon the superconductor being of infinite extent. When the superconductor is finite it is not possible to find solutions of the reduced equations that satisfy the boundary conditions. Thus it is the geometry of the sample that determines it's behaviour, that is whether the bifurcation is sub or supercritical, and which will select one of the mixed state solutions should the bifurcation be supercritical.

3.3 Surface superconductivity

Short-time The leading order solution is

$$f_s^{(0)}(x,t) = \sum_{n=-\infty}^{\infty} C_n e^{\sigma_n t}\theta_n(x), \tag{130}$$

where σ_n are the eigenvalues of

$$-\left(\frac{\alpha\sigma}{\kappa^2}\right)\theta + \frac{1}{\kappa^2}\frac{d^2\theta}{dx^2} = -\theta + (h_{c_3}x + c^{(0)})^2\theta, \tag{131}$$

$$\frac{d\theta}{dx}(0) = -\frac{\kappa}{d}\theta(0), \qquad \frac{d\theta}{dx} \to 0 \text{ as } x \to \infty. \tag{132}$$

with corresponding normalised eigenfunctions θ_n, and

$$C_n = \int_0^\infty f_s^{(0)}(x,0)\theta_n(x)\,dx. \tag{133}$$

Long-time The leading-order long-time solution is

$$f_l^{(0)}(x,\tau) = C(\tau)\theta_0(x), \tag{134}$$

where C satisfies equation (117) and boundary condition (114) with

$$p = -\int_0^\infty \theta_0^4\,dx + 2\int_0^\infty (h_{c_3}x + c^{(0)})\theta_0 \int_0^x (x-\xi)(h_{c_3}\xi + c^{(0)})\theta_0(\xi)^2\,d\xi\,dx, \tag{135}$$

$$q = -\frac{2h^{(1)}}{h_{c_3}}\int_0^\infty (h_{c_3}x + c^{(0)})^2\theta_0(x)^2\,dx. \tag{136}$$

3.4 Mixed State

The short-time solution here is

$$\psi_s^{(0)} = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} C_{n,m}e^{\sigma_m t}e^{inky}\psi_{n,m}(x), \tag{137}$$

where $\sigma_m = -2m\kappa^2/\alpha$, with the corresponding set of eigenfunctions

$$\psi_{n,m} = \frac{1}{(2^m m!)^{1/2}}\exp\left\{-\frac{\kappa^2}{2}\left(x - \frac{kn}{\kappa^2}\right)^2\right\}\mathrm{H}_m\left(\sqrt{2}\,\kappa\left(x - \frac{kn}{\kappa^2}\right)\right),$$

where H_m is the Hermite polynomial, and where the eigenfunctions have been scaled so that

$$\int_{-\infty}^{\infty} \psi_{n,m}\psi_{n,p}\, dx = \begin{cases} 0 & \text{if } m \neq p, \\ \sqrt{\pi}/\kappa & \text{if } m = p. \end{cases}$$

$C_{n,m}$ is given by

$$C_{n,m} = \frac{k\kappa}{2\pi^{3/2}} \int_{-\infty}^{\infty} \int_0^{2\pi/k} e^{-inky} \psi_{n,m}(x) \psi_s^{(0)}(x, y, 0)\, dy\, dx. \tag{138}$$

Note that $C_{n+N} = C_n$ for all n, as required.

The long-time solution is

$$\psi_l^{(0)} = \sum_{-\infty}^{\infty} C_n(\tau) e^{inky} \psi_n(x), \tag{139}$$

where

$$\psi_n(x) = \exp\left\{ -\frac{\kappa^2}{2} \left(x - \frac{nk}{\kappa^2} \right)^2 \right\}, \tag{140}$$

as before, and $C_{n+N}(\tau) = C_n(\tau)$. The $C_n(\tau)$ are found to satisfy

$$\frac{\alpha}{\kappa^2}\frac{dC_n}{d\tau} = \frac{1}{\sqrt{2}} \left(\frac{1}{2\kappa^2} - 1 \right) \sum_{r,m} C_{n-r+m} C_m^* C_r e^{\left\{ -\frac{k^2}{2\kappa^2}\left[(r-n)^2 + (r-m)^2 \right] \right\}} - \frac{C_n h^{(1)}}{\kappa}, \tag{141}$$

with $C_n(0) = C_{n,0}$. We multiply equation (141) by C_n^* and sum over n. The resulting equation is

$$\frac{\alpha}{2\kappa^2} \frac{d\overline{|\psi_l^{(0)}|^2}}{d\tau} = \left(\frac{1}{2\kappa^2} - 1 \right) \overline{|\psi_l^{(0)}|^4} - \frac{h^{(1)}}{\kappa} \overline{|\psi_l^{(0)}|^2}, \tag{142}$$

or

$$\frac{\alpha}{2\kappa^2} \frac{d\overline{|\psi_l^{(0)}|^2}}{d\tau} = \beta \left(\frac{1}{2\kappa^2} - 1 \right) \left(\overline{|\psi_l^{(0)}|^2} \right)^2 - \frac{h^{(1)}}{\kappa} \overline{|\psi_l^{(0)}|^2}, \tag{143}$$

where $\beta = \overline{|\psi_l^{(0)}|^4} / \left(\overline{|\psi_l^{(0)}|^2} \right)^2$ as before. If β were constant this equation would have the same form as in the one-dimensional case. Even for varying β we have that β is independent of the amplitude of $\psi_l^{(0)}$ and depends only on the relative sizes of the coefficients C_n. We see that $\beta \geq 1$ and so (143) will have the same qualitive features as in one dimension.

We note that equation (141) can be written as

$$\frac{\alpha}{\kappa^2} \frac{dC_n}{d\tau} = \frac{Nk}{2\kappa\sqrt{\pi}} \left(\frac{1}{2\kappa^2} - 1 \right) \left(\overline{|\psi^{(0)}|^2} \right)^2 \frac{\partial \beta}{\partial C_n^*}. \tag{144}$$

Hence

$$\begin{aligned} \frac{d\beta}{d\tau} &= \sum_{n=0}^{N-1} \frac{\partial \beta}{\partial C_n^*} \frac{dC_n^*}{d\tau} + \frac{\partial \beta}{\partial C_n} \frac{dC_n}{d\tau}, \\ &= \frac{8\alpha\kappa\sqrt{\pi}}{\left(\overline{|\psi^{(0)}|^2} \right)^2 Nk(1 - 2\kappa^2)} \sum_{n=0}^{N-1} \left| \frac{dC_n}{d\tau} \right|^2. \end{aligned}$$

Hence

$$\frac{d\beta}{d\tau} \leq 0, \qquad \text{for } \kappa > 1/\sqrt{2}.$$

Thus we see that β decreases along trajectories, in keeping with the previous minimization of β in the steady state.

To proceed further we need to specify our choice of N. For $N = 1$ we have $C_n = C\ \forall n$, and

$$\frac{\alpha}{\kappa^2}\frac{dC}{d\tau} = \frac{1}{\sqrt{2}}\left(\frac{1}{2\kappa^2} - 1\right) C \mid C \mid^2 S_{1,0}^2 - \frac{Ch^{(1)}}{\kappa}. \tag{145}$$

This equation differs from that in one dimension simply by the constant $S_{1,0}^2$, which can be removed by a suitable scaling of C, and so behaves in exactly the same way.

Let us now examine the case $N = 2$. In this case there are two possible steady-state solutions. Since the response is qualitively the same as in one dimension, the question of interest is which of the two steady-state solutions will be approached in the case when $\kappa > 1/\sqrt{2}$ and the applied magnetic field is slightly less than κ. We rescale C_0, C_1, τ for simplicity by setting

$$\tau' = \frac{\kappa h^{(1)}}{\alpha}\tau, \qquad C_i' = \left[\frac{\kappa}{h^{(1)}\sqrt{2}}\left(1 - \frac{1}{2\kappa^2}\right)\right]^{1/2} C_i, \quad i = 0, 1.$$

Then, dropping the primes, (141) becomes

$$\frac{dC_0}{d\tau} = -\left[\mid C_0\mid^2 C_0 S_{2,0}^2 + C_0^* C_1^2 S_{2,1}^2 + 2\mid C_1\mid^2 C_0 S_{2,0}S_{2,1}\right] + C_0, \tag{146}$$

$$\frac{dC_1}{d\tau} = -\left[\mid C_1\mid^2 C_1 S_{2,0}^2 + C_1^* C_0^2 S_{2,1}^2 + 2\mid C_0\mid^2 C_1 S_{2,0}S_{2,1}\right] + C_1. \tag{147}$$

The two steady states are given by

$$\begin{array}{lll} \text{(a)} & C_1 = C_0, & \mid C_0\mid = \frac{1}{S_{1,0}^2}, \\ \text{(b)} & C_1 = iC_0, & \mid C_0\mid = \frac{1}{(S_{1,0}^2 - 2S_{2,1}^2)^{1/2}}. \end{array}$$

Letting $C_1 = \eta C_0$, multiplying (146) by η, and subtracting from (147), we obtain

$$\frac{d\eta}{d\tau} = \mid C_0\mid^2\left[\eta(1 - \mid\eta\mid^2)S_{2,0}^2 + (\eta^3 - \eta^*)S_{2,1}^2 + 2\eta(\mid\eta\mid^2 - 1)S_{2,0}S_{2,1}\right]. \tag{148}$$

The steady-state solutions are $\eta = \pm 1, \pm i$. We let $\eta = re^{i\vartheta}$. Then

$$\frac{d\vartheta}{d\tau} = \mid C_0\mid^2 S_{2,1}^2(1 + r^2)\sin 2\vartheta, \tag{149}$$

$$\frac{dr}{d\tau} = \mid C_0\mid^2 r(1 - r^2)\left[S_{2,0}(S_{2,0} - 2S_{2,1}) - S_{2,1}^2\cos 2\vartheta\right]. \tag{150}$$

Firstly we note that $S_{2,0} \gg S_{2,1}$ and so

$$\frac{dr}{d\tau} < 0, \text{ if } r > 1, \qquad \frac{dr}{d\tau} > 0, \text{ if } r < 1.$$

Hence we expect that $r \to 1$, as $\tau \to \infty$. We also note that

$$\begin{aligned}
\frac{d\vartheta}{d\tau} &> 0 \text{ for } 0 < \vartheta < \pi/2, \ \pi < \vartheta < 3\pi/2, \\
\frac{d\vartheta}{d\tau} &< 0 \text{ for } \pi/2 < \vartheta < \pi, \ 3\pi/2 < \vartheta < 2\pi/, \\
\frac{d\vartheta}{d\tau} &= 0 \text{ for } \vartheta = 0, \ \pi.
\end{aligned}$$

Thus we expect

$$\begin{aligned}
\vartheta &\to \pi/2, \text{ as } \tau \to \infty, \text{ if } 0 < \vartheta_0 < \pi, \\
\vartheta &\to 3\pi/2, \text{ as } \tau \to \infty, \text{ if } \pi < \vartheta_0 < 2\pi, \\
\vartheta &\equiv 0, \ \pi, \text{ if } \vartheta_0 = 0, \ \pi.
\end{aligned}$$

Thus we see that only if the initial data has $C_1 = \pm C_0$ will this solution be approached. Any other initial data will result in the solutions $C_1 = \pm i C_0$ being approached as $\tau \to \infty$.

4 Conclusion

We found in I that as the magnitude of the external magnetic field is varied there is a series of bifurcations from the normal state solution to superconducting solution branches. The largest positive eigenvalue is known as the upper critical magnetic field, and is denoted by h_{c_2}. Here we found that the normal state solution is linearly stable for fields of magnitude greater than h_{c_2}, and linearly unstable for fields of magnitude less than h_{c_2}. All the superconducting solution branches other than the one bifurcating at h_{c_2} are linearly unstable near the bifurcation points. The stability of this solution branch depends on the value of κ and the geometry of the sample.

When the sample dimensions are large compared with the penetration depth, the sample may be considered infinite when we are using the penetration depth as our lengthscale, and this was our first example. In this case the upper critical field is equal to κ, the Ginzburg-Landau parameter. For type I superconductors we find that $h_{c_2} < H_c$, the thermodynamic critical field (at which the free energy of the superconducting state becomes lower than that of the normal state), and the one-dimensional bifurcation at $h = h_{c_2}$ is subcritical, that is, the superconducting solution exists for values of the field slightly greater than h_{c_2}

and is linearly unstable near the bifurcation point. For type II superconductors we find that $h_{c_2} > H_c$, and the bifurcation at $h = h_{c_2}$ is supercritical, that is, the superconducting solution exists for values of the field slightly less than h_{c_2} and is linearly stable near the bifurcation point. A weakly-nonlinear analysis of an initial value problem shows that for type I superconductors the solution will grow given any initial data for $h < h_{c_2}$. For $h > h_{c_2}$ the solution will decay to zero given small initial data, but for sufficiently large initial data the solution will grow. Thus, under well controlled conditions it ought to be possible to keep a type I superconductor in the normal state for fields down to $h = h_{c_2}$, but any large fluctuation will cause the conversion to a superconducting state to occur.

For type II superconductors given any initial data the solution will decay to zero for $h > h_{c_2}$, and will tend to the steady state superconducting solution branch for $h < h_{c_2}$.

However, even for a large superconducting body things may be different near the surface of the sample. Our second example considered the effects of a surface on the nucleation of superconductivity. If the body is again large compared to the penetration depth, and we again use the penetration depth as our lengthscale, with lengths measured from the boundary, then the body appears as a halfspace, and we are considering surface boundary layers. It was found in I that the nucleation field, here known as h_{c_3}, is increased above that for bulk nucleation. Hence for any real sample superconductivity will nucleate first at the surface in the form of a superconducting sheath (known as surface superconductivity). Furthermore the value of κ at which the switch from a subcritical to supercritical bifurcation occurs is no longer $1/\sqrt{2}$, but is given by figure 7. The criterion for the observation of surface superconductivity is usually taken to be whether h_{c_3} is greater than or less than H_c. In actual fact surface superconductivity will be observed whenever the bifurcation is supercritical. Figure 2 shows the value of h_{c_3} at the point where the bifurcation switches from being subcritical to supercritical, which can be seen to be less than H_c.

Our final example considered two-dimensional mixed state solutions. The stability of the mixed state solutions was found to be qualitatively the same as in one dimension, with the bifurcation being subcritical for type I superconductors and supercritical for type II superconductors. Hence for fields $h < h_{c_2}$ there will be a partial expulsion of the magnetic field from a type II superconductor, as it forms a mixed state, in agreement with fig. 1b. In some cases there is more than one possible mixed state solution, and in these cases an examination of the linear and weakly-nonlinear stability of the solutions led the the selection of a particular solution, which was found to be in agreement with the free energy arguments of Abrikosov [1].

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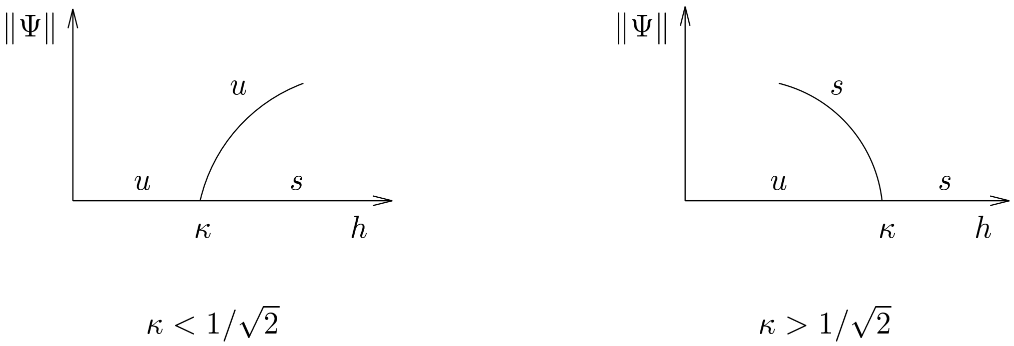

Figure 1: Stability of the solution branches in the response diagrams of type I and type II superconductors.

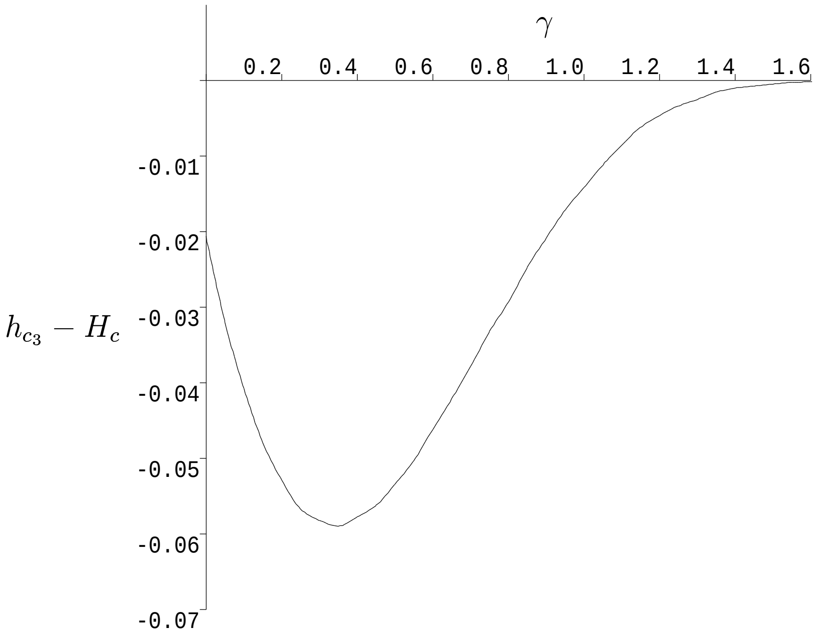

Figure 2: The value of h_{c_3} at which the superconducting solution becomes stable minus the thermodynamic critical field H_c, as a function of γ.

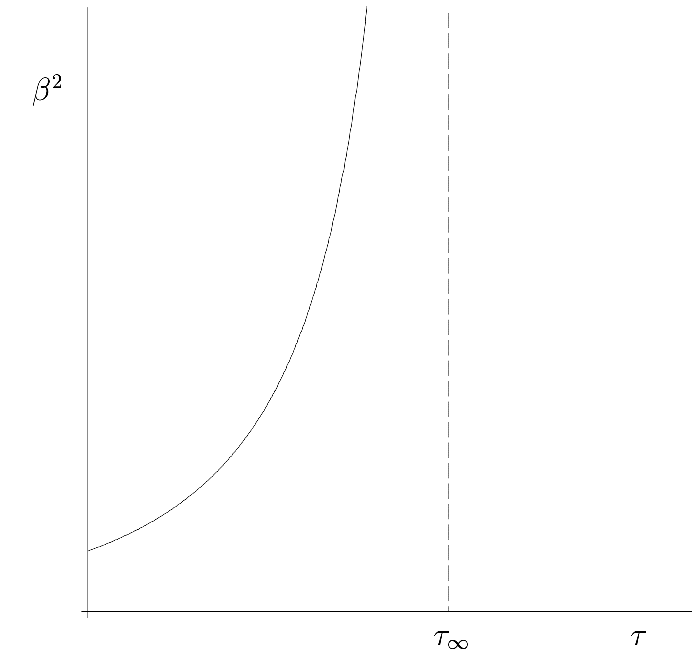

Figure 3: Response of a type I superconductor with $h < \kappa$.

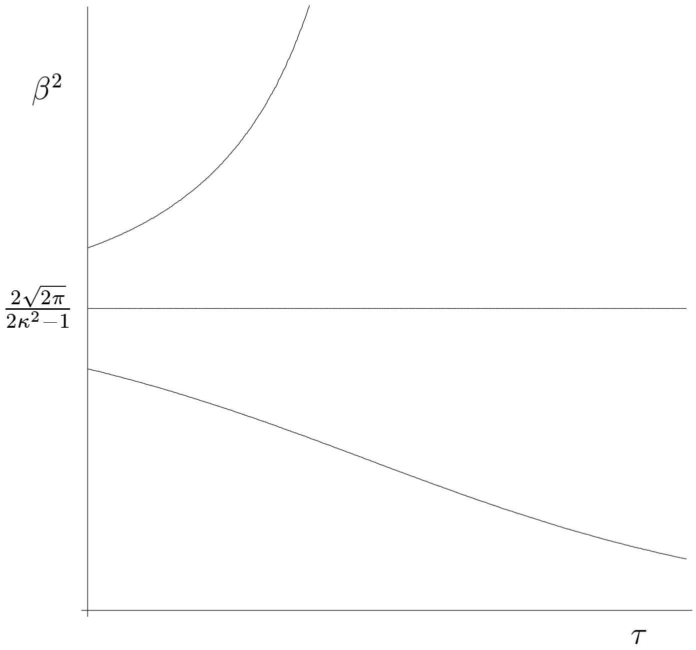

Figure 4: Response of a type I superconductor with $h > \kappa$.

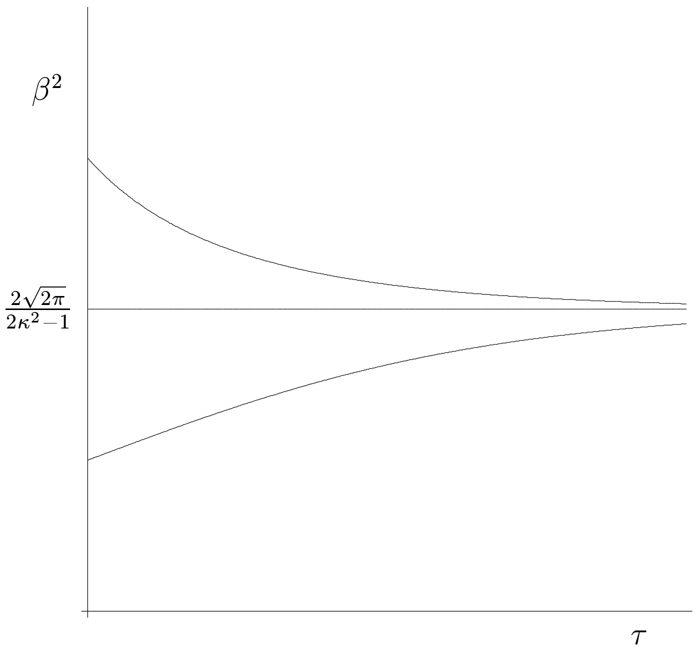

Figure 5: Response of a type II superconductor with $h < \kappa$.

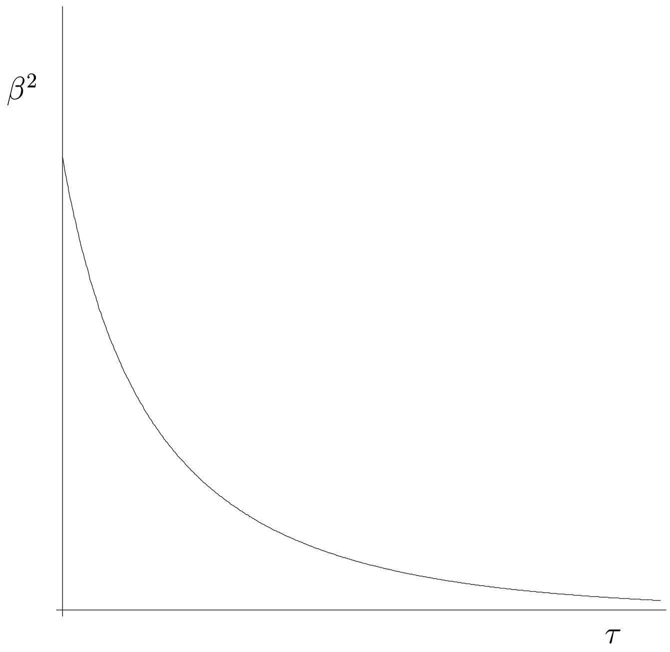

Figure 6: Response of a type II superconductor with $h > \kappa$.