On the rôle of Stokes lines in the selection of Saffman-Taylor fingers with small surface tension

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Small surface tension is known to select a discrete family from the continuum of Saffman-Taylor finger solutions with zero surface tension. Here a method developed recently in [2] is employed in which a naive perturbation expansion in powers of the surface tension parameter is optimally truncated to show exponentially small terms being switched on across Stokes lines. These terms are responsible for finger selection.

1 Introduction

In 1958 Saffman & Taylor [12] demonstrated a family of solutions to the Hele-Shaw equations with zero surface tension which describe a finger of inviscid fluid displacing a viscous fluid in a Hele-Shaw channel, with the relative width of the inviscid finger λ being arbitrary. In experiments [12] they found that λ was always close to $1/2$, and that the experimental finger profile and the theoretical profile for $\lambda = 1/2$ were very close.

It was conjectured that the presence of a small surface tension, of dimensionless size ϵ^2, would select the solution $\lambda = 1/2$ from the continuum, but a naive perturbation expansion in powers of the surface tension still allowed arbitrary λ. Several authors then showed that exponentially small terms beyond all orders of the algebraic expansion for small surface tension are responsible for the selection of a discrete set of possible values for λ [3, 7, 13], and the lowest eigenvalues were then computed in [6, 14]. Following this a complete solution to the problem for λ close to $1/2$ has been given in [4]. In [7, 13] an ad hoc linearisation of the problem is employed, followed by a derivation of the solvability condition using a Fredholm Alternative condition. This gives the correct scaling, but is not accurate quantitatively. In [3, 4] the method developed by Kruskal &

Segur [8] is used. The problem is first cast as an integrodifferential equation on the whole real line, for which the Saffman-Taylor finger corresponds to an antisymmetric solution. A naive perturbation expansion in powers of ϵ is performed, whose terms are observed to have singularities in the complex plane near which the expansion breaks down. An inner analysis is performed in the vicinity of these singularities. The far field of this inner expansion contains exponentialy small terms. When these are evaluated on the imaginary axis, the symmetry condition gives a condition for solvability. In [6] the inner problem is solved numerically to determine the solvability condition, while in [4] the technique of Borel summation is used. In [14] the fluid domain is mapped to a semicircle rather than the upper half-plane. Techniques similar to those of [4] are then employed, although the details differ because of the difference in formulation. Note, though, that unlike the other references the analysis in [14] is not restricted to symmetric solutions.

The aim of the present paper is to show how these results may be obtained by applying the method developed recently in [2]. The far field of the inner expansion exhibits Stokes phenomenon, with the exponentially small terms being switched on across certain lines in the complex plane. These Stokes lines continue into the outer solution, away from the singularities. The method of [2] is to

(i) Expand the solution naively as an algebraic asymptotic series, $\phi \sim \sum_{n=0}^{\infty} \epsilon^{2n} \phi_n$.

(ii) Apply a WKBJ-type ansatz to the equation for ϕ_n to find the behaviour of ϕ_n as $n \to \infty$.

(iii) Optimally truncate the algebraic series and observe the subdominant exponential being switched on across a Stokes line.

By using this method we will be able to see the region in the complex plane in which the subdominant exponentials are present, and get an understanding of the global behaviour of the solution. The present paper is an extension of the work in [2] in two respects. Firstly, the problem we consider is an integrodifferential equation, in contrast to the ordinary differential equations considered in [2]. However, we will see that the integral plays no part in the procedure except on the real axis. Secondly, in Section 5, we will be forcing an expansion in powers of $\epsilon^{2/3}$ when the equation naturally gives an expansion in powers of ϵ^2. This means that the late terms in the expansion behave as a sum of three independent series, so that we have in effect three coincident Stokes lines.

In [3] the limit $\lambda \to 1/2$ of the problem in which $\lambda - 1/2$ is assumed to be $O(1)$ as $\epsilon \to 0$ was considered. The results implied that for the principal solution branch the correct scalings should in fact be $\lambda - 1/2 = O(\epsilon^{4/3})$ as $\epsilon \to 0$, and this is the case which was considered in [4]. Our method is closely related to that of [4], though we use the technique of optimal truncation rather than Borel summation to identify the Stokes lines and to compute the exponentially small terms. However, we do this not only locally but globally, in the outer solution away from the inner regions near the singularities in the complex plane. Thus we are able to see the global behaviour of the Stokes lines and the subdominant exponentials in the solution in the whole complex plane.

The analyses of [3, 4, 6] are restricted to the case in which λ is close to $1/2$, where the singularities in the complex plane lie close together, and the solvability condition is determined by the inner region. Here we treat the case $\lambda - 1/2 = O(1)$, in which the singularities in the complex plane are well separated and which requires some sort of global analysis since the solvability condition is not determined solely by the inner region. We will see that our formulation of the problem is such that for λ not close to $1/2$ we can determine the solvability condition without needing to perform an analysis of the inner region, and that this condition arises easily and naturally in the form of requiring the pole of an integrand to have zero residue. This makes it much easier to study the effects of other types of perturbation to the zero surface tension problem (such as anisotropic surface tension) although we do not go into such matters here.

Our approach is also equally applicable to nonsymmetric solutions, since our solvability condition is determined by a global analysis and is not dependent on the complex singularity lying close to a symmetry line. However, we restrict ourselves to the selection of symmetric fingers in this paper.

Having considered the case $\lambda - 1/2 = O(1)$, we will then consider the case $\lambda - 1/2 = O(\epsilon^{4/3})$. Here the results we obtain are not new, and agree with those in [4]. We include this case though, since it is interesting to see how the condition for a solution arises as a result of the Stokes multipliers for three coincident Stokes lines summing to zero, and also to examine the transition of the Stokes line structure from the cases $\lambda < 1/2$ to $\lambda > 1/2$.

2 Formulation of the problem

The formulation of the problem we will use is that presented by McLean and Saffman in [9]. We begin by briefly recalling some of the steps required to obtain this formulation.

We consider a Hele-Shaw channel of width $2a$ and thickness b, with $b \ll a$, along which an incompressible fluid of viscosity μ is being pushed by a nonmiscible second

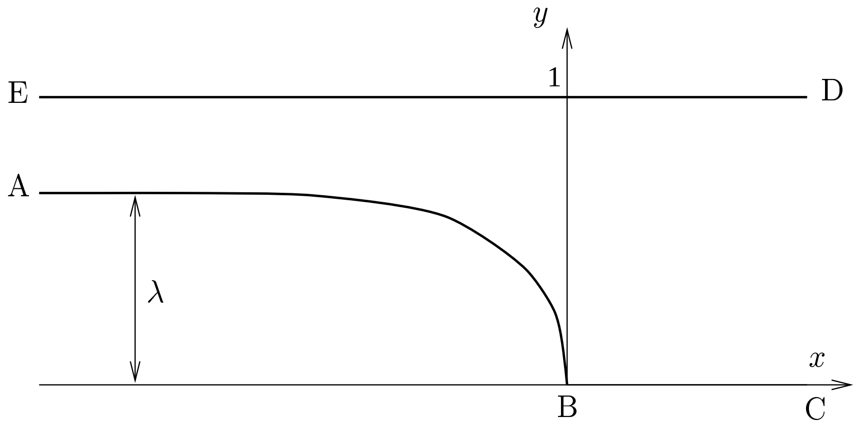

FIGURE 1. The flow region in the physical plane showing the upper half of the finger.

fluid of negligible viscosity. We suppose that a travelling wave has formed in which a single finger of the inviscid fluid with width $2\lambda a$ as $x \to -\infty$ is propagating at a velocity U (see Fig. 1). Nondimensionalising lengths with a, velocity with $(1-\lambda)U$, and pressure with $12\mu a(1-\lambda)U/b^2$, the dimensionless velocity $\boldsymbol{u}$ of the viscous fluid averaged across the cell satisfies

$$\boldsymbol{u} = (u, v) = -\nabla p = \nabla \bar{\phi}, \tag{2.1}$$

where p is the dimensionless pressure and $\bar{\phi}$ is the dimensionless velocity potential. We work in a frame moving with the finger by setting $\bar{\phi} = x/(1-\lambda) + \phi$. Since the fluid is incompressible we have

$$\nabla^2 \phi = 0. \tag{2.2}$$

The kinematic condition on the free boundary gives

$$\frac{\partial \phi}{\partial n} = 0, \tag{2.3}$$

where $\boldsymbol{n}$ is the unit normal to the interface. The dynamic condition on the free boundary gives

$$\phi = \phi_0 + C\kappa - \frac{x}{1-\lambda}, \tag{2.4}$$

where ϕ_0 is the constant value of the potential in the inviscid fluid region, κ is the dimensionless curvature of the interface, and

$$C = \frac{Tb^2}{12\mu a^2 U(1-\lambda)}$$

is the dimensionless surface tension parameter. On the fixed boundaries $y = \pm 1$ we have

$$\frac{\partial \phi}{\partial y} = 0. \tag{2.5}$$

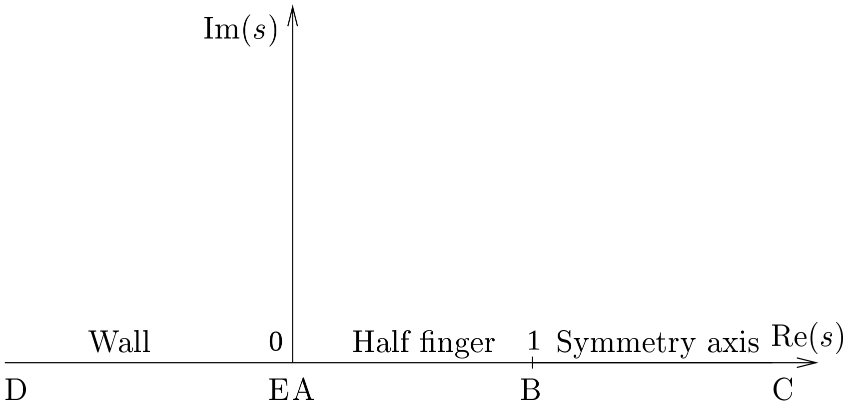

FIGURE 2. The flow region in the s plane.

The fluid far behind the tip of the finger is moving with speed $-1/(1-\lambda)$, while far ahead of the tip it is moving with speed -1, giving

$$\phi \sim -\frac{x}{1-\lambda} \quad \text{as } x \to -\infty, \quad \lambda < |y| < 1, \tag{2.6}$$

$$\phi \sim -x \quad \text{as } x \to +\infty, \quad -1 < y < 1. \tag{2.7}$$

As usual, we define the complex potential by $w = \phi + i\psi$, where ψ is the streamfunction, and we let $z = x+iy$. Then the conformal transformation $z \to w$ maps the flow region into an infinite strip of unit width in the potential plane. The second conformal transformation

$$w \to s = e^{-(w-\phi_0)\pi} \tag{2.8}$$

finally maps the flow region into the upper half s-plane. The interface AB is mapped to the real segment $0 < s < 1$, with $s = 1$ corresponding to the finger tip. Figure 2 shows the s plane indicating the positions of various points of interest.

It is most convenient to work in terms of the complex velocity

$$\frac{dw}{dz} = u - iv.$$

At the interface this velocity must be tangential, and may be written as $\hat{q}e^{-i\hat{\theta}}$ where $\hat{\theta}$ is the angle between the tangent and the x direction. Making use of the analyticity of $\log\hat{q} - i\hat{\theta}$ in the upper half s plane, it can be shown using standard Hilbert transform results that

$$\log \hat{q} = -\frac{1}{\pi}\int_0^1 \frac{\hat{\theta}(s') - \pi}{s' - s}\, ds', \tag{2.9}$$

with the constants fixed by the condition that $\hat{q} \to 1$ as $\phi \to -\infty$. The curvature of the interface is given by

$$\kappa = \frac{d\hat{\theta}}{dS} = \frac{\partial\hat{\theta}}{\partial\phi}\frac{d\phi}{dS},$$

where S is arc length along the interface. Noting that

$$\frac{\partial x}{\partial S} = \cos\hat{\theta}, \qquad \frac{d\phi}{dS} = \hat{q}, \qquad \frac{d\phi}{ds} = -\frac{1}{\pi s}, \tag{2.10}$$

we find that differentiating equation (2.4) with respect to arclength leads to

$$C\pi^2(1-\lambda)\hat{q}s\frac{d}{ds}\left(\hat{q}s\frac{d\hat{\theta}}{ds}\right) - (1-\lambda)\hat{q} = \cos\hat{\theta}, \tag{2.11}$$

with boundary conditions

$$\hat{\theta}(0) = \pi, \qquad \hat{q}(0) = 1/(1-\lambda), \tag{2.12}$$

$$\hat{\theta}(1) = \pi/2, \;\; \hat{q}(1) = 0. \tag{2.13}$$

Finally, the dependence on λ may be simplified by setting

$$\theta = \hat{\theta} - \pi, \qquad q = (1-\lambda)\hat{q}, \tag{2.14}$$

to give

$$\epsilon^2 qs\frac{d}{ds}\left(qs\frac{d\theta}{ds}\right) - q = -\cos\theta, \tag{2.15}$$

$$\log q = -\frac{s}{\pi}\int_0^1 \frac{\theta(s')\,ds'}{s'(s'-s)}, \tag{2.16}$$

with boundary conditions

$$\theta(0) = 0, \qquad q(0) = 1, \tag{2.17}$$

$$\theta(1) = -\pi/2, \;\; q(1) = 0, \tag{2.18}$$

where

$$\epsilon^2 = \frac{C\pi^2}{(1-\lambda)} = \frac{Tb^2\pi^2}{12\mu U a^2(1-\lambda)^2} \tag{2.19}$$

is the dimensionless surface tension. Once a solution of (2.15)-(2.18) is known the corresponding value of λ can be deduced from the relation [9]

$$\log(1-\lambda) = \frac{1}{\pi}\int_0^1 \frac{\theta(s')}{s'}\,ds', \tag{2.20}$$

obtained from (2.9) and (2.12). The Saffman and Taylor solutions are given by

$$q_0 = \sqrt{\frac{1-s}{1+\alpha s}}, \tag{2.21}$$

$$\theta_0 = \cos^{-1}\sqrt{\frac{1-s}{1+\alpha s}}, \tag{2.22}$$

where

$$\alpha = \frac{2\lambda - 1}{(1-\lambda)^2} \tag{2.23}$$

is arbitrary.

Equation (2.16), involving a principal value integral, holds only when s is real. We will need the analytic continuation of (2.15), (2.16) into the complex s-plane, which by the Plemelj formulae is

$$\epsilon^2 qs\frac{d}{ds}\left(qs\frac{d\theta}{ds}\right) - q = -\cos\theta, \tag{2.24}$$

$$\log q = -\frac{s}{\pi}\int_0^1 \frac{\theta(s')\,ds'}{s'(s'-s)} + i\theta(s). \tag{2.25}$$

We could work directly with the variable s in what follows, but this would involve working in a slit plane, since the top and bottom branches of the finger map to the top and bottom of the slit $0 < s < 1$ respectively. We prefer to change variables by setting

$$y = -\sqrt{\frac{1-s}{s}}, \tag{2.26}$$

(note that y here is a new complex variable, and is not to be confused with the physical coordinate) giving

$$\frac{\epsilon^2 q(1+y^2)}{4y}\frac{d}{dy}\left(\frac{q(1+y^2)}{y}\frac{d\theta}{dy}\right) - q = -\cos\theta, \tag{2.27}$$

$$\log q - i\theta = \frac{2}{\pi}\int_{-\infty}^{0}\frac{\bar{y}\theta(\bar{y})}{y^2-\bar{y}^2}\,d\bar{y}, \tag{2.28}$$

$$\theta(-\infty) = 0, \tag{2.29}$$

$$\theta(0) = -\pi/2. \tag{2.30}$$

This maps the top branch of the finger to the negative real axis, the bottom branch to the positive real axis, and the symmetry line to the imaginary axis (see Fig.3). We may now replace the condition at the origin with the condition

$$\theta(\infty) = -\pi \tag{2.31}$$

if we wish.

3 Analysis with α of order one as $\epsilon \to 0$

We are interested in the limit in which $\epsilon \to 0$. We assume in this section that α is of order one. We begin by expanding in powers of ϵ:

$$\theta \sim \sum_{n=0}^{\infty} \epsilon^{2n}\theta_n, \tag{3.1}$$

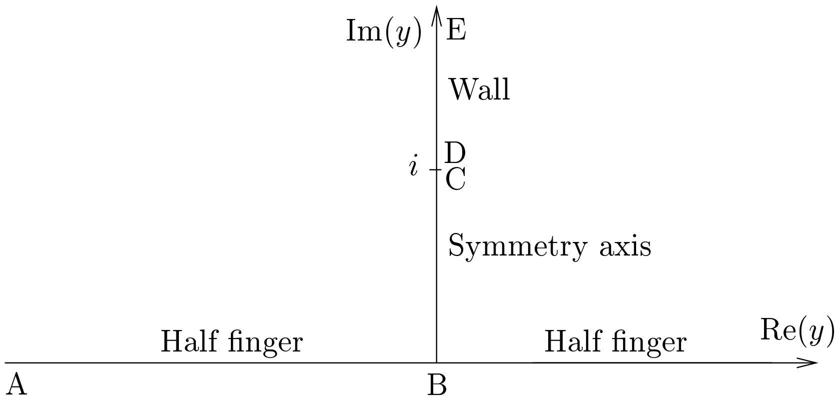

FIGURE 3. The flow region in the y plane.

$$q \sim \sum_{n=0}^{\infty} \epsilon^{2n} q_n, \tag{3.2}$$

giving

$$q_0 = \cos\theta_0, \tag{3.3}$$

$$\log q_0 - i\theta_0 = \frac{2}{\pi}\int_{-\infty}^{0} \frac{\bar{y}\theta_0(\bar{y})}{y^2 - \bar{y}^2}\, d\bar{y}, \tag{3.4}$$

$$\begin{aligned} &\frac{q_0(1+y^2)}{4y}\frac{d}{dy}\left(\frac{q_{n-1}(1+y^2)}{y}\frac{d\theta_0}{dy}\right) \\ &\quad + \frac{q_{n-1}(1+y^2)}{4y}\frac{d}{dy}\left(\frac{q_0(1+y^2)}{y}\frac{d\theta_0}{dy}\right) \\ &\quad\quad + \frac{q_0(1+y^2)}{4y}\frac{d}{dy}\left(\frac{q_0(1+y^2)}{y}\frac{d\theta_{n-1}}{dy}\right) + \cdots \\ &\quad\quad\quad - q_n = \theta_n \sin\theta_0 + \cdots, \qquad n \geq 1, \end{aligned} \tag{3.5}$$

$$\frac{q_n}{q_0} + \cdots - i\theta_n = \frac{2}{\pi}\int_{-\infty}^{0} \frac{\bar{y}\theta_n(\bar{y})}{y^2 - \bar{y}^2}\, d\bar{y}, \qquad n \geq 1, \tag{3.6}$$

with boundary conditions

$$\theta_0(-\infty) = 0, \tag{3.7}$$

$$\theta_0(\infty) = -\pi, \tag{3.8}$$

$$\theta_n(\pm\infty) = 0, \qquad n \geq 1. \tag{3.9}$$

The terms on the left-hand side of (3.5) come from expanding the triple product in q and θ, those on the right-hand side come from expanding $\cos\theta$, and those on the left-hand side of (3.6) come from expanding $\log q$. The leading order solution is a Saffman-Taylor solution

$$q_0 = \cos\theta_0 = -\frac{y}{(1+\alpha+y^2)^{1/2}}, \tag{3.10}$$

$$\sin\theta_0 = -\frac{(1+\alpha)^{1/2}}{(1+\alpha+y^2)^{1/2}}. \tag{3.11}$$

In order to truncate the expansions (3.1), (3.2) optimally we need to know the behaviour of θ_n and q_n as $n \to \infty$. From equation (3.5) we see that the set of singular points of θ_n will be the same as that of θ_0 for all n (i.e. $y = \pm i\sqrt{1+\alpha}$), with possibly additional singular points at $y = \pm i$. However, since each time we calculate a new term in the asymptotic expansion we differentiate the previous term twice, if θ_{n-1} has a singularity of strength p then θ_n will have a singularity of strength $p+2$. Thus we expect there to be factorial/power divergence in the large n behaviour of θ_n and q_n. Following [2] we therefore seek a solution as $n \to \infty$ of the form

$$\theta_n \sim \Lambda\frac{\Gamma(2n+\gamma+1)}{u^{2n+\beta}}, \tag{3.12}$$

where u, γ and β are possibly functions of y but are independent of n, and Λ is a constant. Note that (3.12) is not an approximation as $u \to 0$, but as $n \to \infty$, and is supposed to be valid for general y; we merely motivate the factorial/power ansatz by noting that θ_n will have such a form close to singularities of θ_0.

Now, in (3.6) we need to estimate the relative sizes of θ_n and

$$I = \int_{-\infty}^{0} \frac{\bar{y}\theta_n(\bar{y})}{y^2-\bar{y}^2}\,d\bar{y}. \tag{3.13}$$

Substituting (3.12) into (3.13) we have

$$I \sim \Lambda\Gamma(2n+\gamma+1)\int_{-\infty}^{0} \frac{\bar{y}}{(y^2-\bar{y}^2)u^\beta}e^{-2n\log u}\,d\bar{y}, \qquad \text{as } n \to \infty. \tag{3.14}$$

We will verify a posteriori that the dominant contribution to the integral comes from the end point. Then I will be exponentially subdominant to θ_n wherever $u(y)$ is smaller than $u(0)$. This will certainly be true near the singularities in the complex plane (where $u = 0$), and in fact we will find that it is true throughout the region in the complex plane with which we are concerned. Thus we have

$$q_n \sim iq_0\theta_n \text{ as } n \to \infty. \tag{3.15}$$

Substituting (3.12) and (3.15) into (3.5) we find that at leading order as $n \to \infty$

$$(u')^2 = \frac{4y^2(iq_0+\sin\theta_0)}{q_0^2(1+y^2)^2}, \tag{3.16}$$

giving

$$u = 2i \int_{i(1+\alpha)^{1/2}}^{y} \frac{\left((1+\alpha)^{1/2} + i\bar{y}\right)^{3/4} \left((1+\alpha)^{1/2} - i\bar{y}\right)^{1/4}}{1+\bar{y}^2} \, d\bar{y}. \tag{3.17}$$

The integral can be evaluated explicitly, but the resulting expression is rather cumbersome; it is given in the appendix for completeness.

To evaluate γ and β we need to go to the next order in n. The easiest way to proceed is first to use u as the independent variable [2]. Equations (2.24) and (2.25) become

$$\frac{\epsilon^2 q(1+y(u)^2)}{4y(u)} u' \frac{d}{du}\left(\frac{q(1+y(u)^2)}{y(u)} u' \frac{d\theta}{du}\right) - q = -\cos\theta, \tag{3.18}$$

$$\log q - i\theta = \frac{2}{\pi} \int_{-\infty}^{0} \frac{\bar{y}\theta(\bar{y})}{y(u)^2 - \bar{y}^2} \, d\bar{y}, \tag{3.19}$$

while (3.5) becomes

$$\begin{aligned} &\frac{q_0^2(1+y^2)^2}{4y^2} \frac{d\theta_0}{dy} u' \frac{dq_{n-1}}{du} \\ &\quad + \frac{q_0^2(1+y^2)^2}{4y^2}\left(u'' \frac{d\theta_{n-1}}{du} + (u')^2 \frac{d^2\theta_{n-1}}{du^2}\right) \\ &\quad\quad + \frac{q_0(1+y^2)}{4y} \frac{d}{dy}\left(\frac{q_0(1+y^2)}{y}\right) u' \frac{d\theta_{n-1}}{du} + \cdots \\ &\quad\quad\quad - q_n = \sin\theta_0 \theta_n + \cdots, \end{aligned} \tag{3.20}$$

where u' is $u'(y(u))$, etc. Substituting in the ansatz (3.12) we find that the terms of order $n \log n$ give $\gamma' = 0$, so that γ is constant. Equating the terms of order n we find

$$\beta + u \log u \frac{d\beta}{du} = \gamma + 1 + \frac{u'' u}{2(u')^2} + \frac{iu}{2u'} \frac{d\theta_0}{dy} + \frac{yu}{2(1+y^2) q_0 u'} \frac{d}{dy}\left(\frac{q(1+y^2)}{y}\right). \tag{3.21}$$

Hence

$$\beta = \gamma + 1 + \frac{\log u'}{2\log u} + \frac{i\theta_0}{2\log u} + \frac{\log(q_0(1+y^2)/y)}{2\log u}. \tag{3.22}$$

Now from (3.17) and (3.10), (3.11), as $y \to i(1+\alpha)^{1/2}$,

$$u \sim -\frac{2^{13/4} e^{i7\pi/8}(1+\alpha)^{1/8}}{7\alpha}(y - i(1+\alpha)^{1/2})^{7/4}, \tag{3.23}$$

$$\theta_0 \sim (i/2)\log(y - i(1+\alpha)^{1/2}). \tag{3.24}$$

Since θ_0 is logarithmic in u as $u \to 0$ we therefore have that $\beta \to 0$ as $u \to 0$ (i.e. as $y \to i(1+\alpha)^{1/2}$), and equation (3.22) then implies

$$\gamma = -1 - 3/14 + 1/7 + 1/7 = -13/14. \tag{3.25}$$

Thus

$$\theta_n \sim \frac{\Lambda\Gamma(2n+\gamma+1)e^{-i\theta_0/2}}{u^{2n+\gamma+1}(u')^{1/2}}\left(\frac{y}{q_0(1+y^2)}\right)^{1/2} = \frac{\Lambda\Gamma(2n+1/14)}{u^{2n+\gamma+1}}\frac{i}{\sqrt{2}}\left(\frac{(1+\alpha)^{1/2}+iy}{(1+\alpha)^{1/2}-iy}\right)^{1/8}. \tag{3.26}$$

Having found the asymptotic behaviour of θ_n as $n\to\infty$, the next step is to truncate the asymptotic series and study the behaviour of the remainder [2]. In this way we aim to observe the switching on of subdominant exponentials across Stokes lines directly, exactly as in the case of linear ordinary differential equations [1, 10, 11]. Truncating after N terms we have

$$\theta = \sum_{n=0}^{N-1}\epsilon^{2n}\theta_n + R_N, \tag{3.27}$$

where

$$\begin{aligned}
&\frac{i\epsilon^2 q_0^2(1+y^2)^2}{4y^2}\frac{d\theta_0}{dy}u'\frac{dR_N}{du}\\
&\quad+\frac{\epsilon^2 q_0^2(1+y^2)^2}{4y^2}\left(u''\frac{dR_N}{du}+(u')^2\frac{d^2R_N}{du^2}\right)\\
&\quad\quad+\frac{\epsilon^2 q_0(1+y^2)}{4y}\frac{d}{dy}\left(\frac{q_0(1+y^2)}{y}\right)u'\frac{dR_N}{du}+\cdots\\
&\quad\quad\quad-iq_0R_N-\sin\theta_0 R_N = -(iq_0+\sin\theta_0)\,\theta_N+\cdots.
\end{aligned} \tag{3.28}$$

Using our expression for u' we find that as $\epsilon\to 0$ and $N\to\infty$

$$\begin{aligned}
&\frac{i\epsilon^2}{u'}\frac{d\theta_0}{dy}\frac{dR_N}{du}+\epsilon^2\frac{u''}{(u')^2}\frac{dR_N}{du}+\epsilon^2\frac{d^2R_N}{du^2}\\
&\quad+\frac{\epsilon^2 y}{u'q_0(1+y^2)}\frac{d}{dy}\left(\frac{q_0(1+y^2)}{y}\right)\frac{dR_N}{du}-R_N\sim-\epsilon^{2N}\theta_N.
\end{aligned} \tag{3.29}$$

The homogeneous version of this equation has solutions with the following asymptotic behaviour as $\epsilon\to 0$:

$$R_N \sim \frac{e^{-u/\epsilon}e^{-i\theta_0/2}}{(u')^{1/2}}\left(\frac{y}{q_0(1+y^2)}\right)^{1/2}, \qquad R_N \sim \frac{e^{u/\epsilon}e^{-i\theta_0/2}}{(u')^{1/2}}\left(\frac{y}{q_0(1+y^2)}\right)^{1/2}. \tag{3.30}$$

Following Dingle [5] we expect there to be Stokes lines whenever late successive terms the expansion of θ have the same phase, that is, whenever u is real. We aim to show that a multiple of the first term in (3.30) is switched on across the Stokes line where u is real and positive. We define the Stokes multiplier A by setting

$$R_N = \frac{e^{-u/\epsilon}e^{-i\theta_0/2}A}{(u')^{1/2}}\left(\frac{y}{q_0(1+y^2)}\right)^{1/2}, \tag{3.31}$$

which in (3.29) gives

$$2\epsilon e^{-u/\epsilon}\frac{dA}{du}+\cdots\sim\frac{\epsilon^{2N}\Lambda\Gamma(2N+\gamma+1)}{u^{2N+\gamma+1}}. \tag{3.32}$$

Following [2] we let $u = re^{i\vartheta}$, and write

$$\frac{d}{du} = -\frac{ie^{-i\vartheta}}{r}\frac{d}{d\vartheta}.$$

The right-hand side of (3.32) is minimal for $N \sim r/(2\epsilon)$. We therefore set $N = r/(2\epsilon)+\alpha$, where α is bounded as $\epsilon \to 0$. Then, as $\epsilon \to 0$, (3.32) becomes

$$\frac{dA}{d\vartheta} \sim \frac{i\sqrt{\pi}\, e^{re^{i\vartheta}/\epsilon}\Lambda e^{-r/\epsilon}r^{1/2}}{\sqrt{2}\,\epsilon^{\gamma+3/2}e^{i\vartheta r/\epsilon}e^{i\vartheta(2\alpha+\gamma)}}. \tag{3.33}$$

The right-hand side is exponentially small except on the Stokes line $\vartheta = 2\pi k$, $k \in \mathbb{Z}$, where it is algebraic in ϵ. Hence A will undergo a rapid transition near $\vartheta = 2\pi k$. To examine this transition we introduce the local scaling $\vartheta = 2\pi k + \delta\bar{\vartheta}$, giving

$$\frac{dA}{d\bar{\vartheta}} \sim \frac{i\sqrt{\pi}\,\delta r^{1/2}\Lambda e^{-r\delta^2\bar{\vartheta}^2/(2\epsilon)}}{\sqrt{2}\,\epsilon^{\gamma+3/2}e^{2\pi ik\gamma}}. \tag{3.34}$$

(Note that there is no term $e^{4\pi ik\alpha}$ since $r/(2\epsilon)+\alpha$ is an integer.) We see that the correct scalings are $\vartheta = \epsilon^{1/2}\bar{\vartheta}$, $A = \hat{A}/\epsilon^{\gamma+1}$, giving

$$\frac{d\hat{A}}{d\bar{\vartheta}} \sim \frac{i\sqrt{\pi}\,r^{1/2}\Lambda e^{-r\bar{\vartheta}^2/2}}{\sqrt{2}\,e^{2\pi ik\gamma}}. \tag{3.35}$$

Thus

$$\hat{A} = a + \frac{i\sqrt{\pi}\,\Lambda}{\sqrt{2}\,e^{2\pi ik\gamma}}\int_{-\infty}^{\vartheta\sqrt{r}} e^{-x^2/2}\,dx. \tag{3.36}$$

Matching with the outer solution away from the Stokes line there is a jump in A given by

$$A(\vartheta = 2\pi k+) - A(\vartheta = 2\pi k-) = \frac{i\pi\Lambda e^{-2\pi ik\gamma}}{\epsilon^{\gamma+1}}. \tag{3.37}$$

A similar calculation shows that a multiple of the second solution of the homogeneous equation is switched on across the Stokes lines on which u is real and negative.

Let us now examine the implications of these Stokes lines for the problem (2.27)-(2.30).

Let us consider first the case $\alpha < 0$, corresponding to $\lambda < 1/2$. The Stokes lines are given by u real. There is a Stokes line down the imaginary axis from $y = i\sqrt{1+\alpha}$ on which the argument of u is zero and across which

$$-\frac{\pi\Lambda e^{-u(y)/\epsilon}}{\sqrt{2}\,\epsilon^{\gamma+1}}\left(\frac{(1+\alpha)^{1/2}+iy}{(1+\alpha)^{1/2}-iy}\right)^{1/8} \tag{3.38}$$

is switched on. Similarly, or by symmetry, there is a Stokes line up the imaginary axis from $y = -i\sqrt{1+\alpha}$ across which

$$-\frac{\pi\bar{\Lambda}e^{-\bar{u}(y)/\epsilon}}{\sqrt{2}\,\epsilon^{\gamma+1}}\left(\frac{(1+\alpha)^{1/2}-iy}{(1+\alpha)^{1/2}+iy}\right)^{1/8} \tag{3.39}$$

is switched on (see Fig. 4; note that $y = \pm i\sqrt{1+\alpha}$ are branch points of θ, so that we have inserted branch cuts between here and $\pm i\infty$.). Hence, on the real axis,

$$-\mathrm{Re}\left(\frac{\sqrt{2}\,\pi\Lambda e^{-u/\epsilon}}{\epsilon^{\gamma+1}}\left(\frac{(1+\alpha)^{1/2}+iy}{(1+\alpha)^{1/2}-iy}\right)^{1/8}\right) \tag{3.40}$$

is switched on at $y=0$. Note that there are other Stokes lines from $y=\pm i\sqrt{1+\alpha}$, corresponding to the argument of u being a nonzero multiple of 2π, but that these do not meet the real axis, as shown in Figure 4.

Evaluating at $y=0$ we find that

$$\theta \sim -\pi/2 - \frac{\pi R\cos\Phi}{\sqrt{2}\,\epsilon^{\gamma+1}}\exp\left(-\frac{2}{\epsilon}\int_0^{\sqrt{1+\alpha}}\frac{\left(\sqrt{1+\alpha}-x\right)^{3/4}\left(\sqrt{1+\alpha}+x\right)^{1/4}}{1-x^2}\,dx\right), \tag{3.41}$$

where $\Lambda = Re^{i\Phi}$ (note the factor of $1/2$ due to the fact that we are evaluating exactly on the Stokes line). Hence there is no solution in this case.

Let us now consider the case $\alpha > 0$, corresponding to $\lambda > 1/2$. In this case typical Stokes lines are shown in Fig. 5.

There are three Stokes lines emerging from $y = i\sqrt{1+\alpha}$, one down the imaginary axis which ends at $y=i$, and two either side of this axis which end at $y=-i$. There are three similar Stokes lines emerging from $y=-i\sqrt{1+\alpha}$.

On the left-hand Stokes line emerging from $y=i\sqrt{1+\alpha}$ the argument of u is -2π, and as we cross it

$$-\frac{\pi\Lambda e^{-u(y)/\epsilon}}{\sqrt{2}\,e^{-2\pi i(\gamma+1)}\epsilon^{\gamma+1}}\left(\frac{(1+\alpha)^{1/2}+iy}{(1+\alpha)^{1/2}-iy}\right)^{1/8} \tag{3.42}$$

is switched on. Similarly, as we cross the left-hand Stokes line emerging from $y = -i\sqrt{1+\alpha}$

$$-\frac{\pi\bar{\Lambda} e^{-\bar{u}(y)/\epsilon}}{\sqrt{2}\,e^{2\pi i(\gamma+1)}\epsilon^{\gamma+1}}\left(\frac{(1+\alpha)^{1/2}-iy}{(1+\alpha)^{1/2}+iy}\right)^{1/8} \tag{3.43}$$

is switched on (see Fig. 5). On the real axis where these Stokes lines meet the exponentials are comparable, and

$$-\mathrm{Re}\left(\frac{\sqrt{2}\,\pi\Lambda e^{-u/\epsilon}}{e^{-2\pi i(\gamma+1)}\epsilon^{\gamma+1}}\left(\frac{(1+\alpha)^{1/2}+iy}{(1+\alpha)^{1/2}-iy}\right)^{1/8}\right) \tag{3.44}$$

is switched on. On the right-hand Stokes lines the argument of u is zero, and as we cross the point at which they meet the real axis,

$$-\mathrm{Re}\left(\frac{\sqrt{2}\,\pi\Lambda e^{-u/\epsilon}}{\epsilon^{\gamma+1}}\left(\frac{(1+\alpha)^{1/2}+iy}{(1+\alpha)^{1/2}-iy}\right)^{1/8}\right) \tag{3.45}$$

is switched on. The condition for a solution is that these two functions exactly cancel as $y\to\infty$. We must remember that the branch of the function u in (3.44) is not the same as that in (3.45), since the first is integrated down the left-hand Stokes line and then along the real axis, while the second is integrated along the right-hand Stokes line, and the difference is the integral over a contour which circles the branch point at $y = i$. In fact, if we represent the first branch of u by u_1 and the second by u_2, then

$$u_1 - u_2 = 2\pi i \alpha^{1/4}\left((1+\alpha)^{1/2} - 1\right)^{1/2}, \tag{3.46}$$

since, from (3.17) the difference is exactly the residue due to the pole at $y = i$. Hence, for a solution to exist,

$$\frac{u_1}{\epsilon} - 2\pi i(\gamma+1) - \frac{u_2}{\epsilon} = (2N+1)\pi i, \tag{3.47}$$

where $N \in \mathbb{Z}$, i.e.

$$\alpha^{1/4}\left((1+\alpha)^{1/2} - 1\right)^{1/2} = \epsilon(N + 1/2 + \gamma + 1) = \epsilon(N + 4/7), \tag{3.48}$$

remembering that $\gamma = -13/14$.

Alternatively, rather than requiring that the function switched on across the first Stokes line is switched off across the second, we may use the symmetry condition at the origin, giving the condition for a solution as $\theta(0) = -\pi/2$. This gives

$$\cos\left(\Phi - \frac{\mathrm{Im}(u_1(0))}{\epsilon} + 2i\pi(\gamma+1)\right) = 0 = \cos\left(\Phi - \frac{\mathrm{Im}(u_2(0))}{\epsilon}\right). \tag{3.49}$$

Since

$$\mathrm{Im}(u_1(0)) = \pi\alpha^{1/4}\left((1+\alpha)^{1/2} - 1\right)^{1/2}, \tag{3.50}$$

a solution can exist only if

$$\alpha^{1/4}\left((1+\alpha)^{1/2} - 1\right)^{1/2} = \epsilon\left(N + \frac{1}{2} + \frac{\Phi}{\pi} + \frac{1}{7}\right) = \epsilon\left(N + \frac{9}{14} + \frac{\Phi}{\pi}\right), \tag{3.51}$$

where $N \in \mathbb{Z}$. For this to be consistent with (3.48) we must have $\Phi + \pi/14 \in \mathbb{Z}$. In the next section we will show how to obtain the constant Λ from matching with an inner region in the vicinity of the singularity at $y = i\sqrt{1+\alpha}$, and we will see that Φ is equal to $13\pi/14$. However, we note that we have been able to obtain the condition for a solution (3.48) without this inner analysis.

Thus we see that for any fixed N, i.e. for any particular solution branch, $\alpha \downarrow 0$ as $\epsilon \downarrow 0$, corresponding to $\lambda \downarrow 1/2$. However, for each fixed ϵ there are infinitely many solution branches, which cluster around $\alpha = \infty$, corresponding to $\lambda = 1$. The first ten solution branches are shown in Figure 6.

For $\alpha \to 0$ we find

$$\alpha^{3/4} = \sqrt{2}\,\epsilon(N + 4/7), \tag{3.52}$$

confirming the result of [4]. Now, for the lower solution branches, (and in particular for the lowest solution branch $N = 0$, which is the one we might expect to observe in practice) $\alpha = O(\epsilon^{4/3})$, while our analysis has been performed under the assumption that $\alpha = O(1)$ as $\epsilon \to 0$. This means in particular that (3.48) is valid only in the limit $N \to \infty$ so that ϵN is order one. In Section 5 we will consider the case $\alpha = O(\epsilon^{4/3})$.

Finally, having found u we can now verify that the dominant contribution to I does indeed arise from the endpoint when we deform to the steepest descent contour. In the region which contains the Stokes lines this endpoint contribution is subdominant to θ_n.

Let us remark here on an alternative method of deriving the relationship (3.15) between q_n and θ_n, which does not rely on being able to find the relationship (2.28) between q and θ. We have that $q\exp(-i\theta)$ is analytic in the fluid region, and therefore

$$\nabla^2 qe^{-i\theta} = 0. \tag{3.53}$$

Explicitly, this gives

$$q_{XX} + q_{YY} - iq\theta_{XX} - iq\theta_{YY} - 2i\theta_X q_X - 2i\theta_Y q_Y - q\theta_X^2 - q\theta_Y^2 = 0, \tag{3.54}$$

where X and Y are cartesian coordinates. With the ansatz (3.12) for θ_n, and a similar ansatz

$$q_n \sim \hat{\Lambda}\frac{\Gamma(2n+\hat{\gamma}+1)}{\hat{u}^{2n+\hat{\beta}}}, \tag{3.55}$$

for q_n, we find that the leading terms in (3.54) are given by

$$\hat{\Lambda}\frac{\Gamma(2n+\hat{\gamma}+3)}{\hat{u}^{2n+\hat{\beta}+2}}\left(\hat{u}_X^2+\hat{u}_Y^2\right) - iq_0\Lambda\frac{\Gamma(2n+\gamma+3)}{u^{2n+\beta+2}}\left(u_X^2+u_Y^2\right) = 0. \tag{3.56}$$

The only sensible balance occurs if $u = \hat{u}$, in which case (3.56) implies that

$$q_n \sim iq_0\theta_n \text{ as } n \to \infty, \tag{3.57}$$

as required.

Thus we are able to obtain the relationship (3.15) between the late terms of the series for q and θ without the need for an explicit relationship between q and θ. For linear problems this is not so important, since the Green's function may always be used to give q in terms of θ. However, the method working directly with the equation as above, as well as being simpler, will work as easily with nonlinear field equations, for which the Green's function approach is not available.

We note also that this is an advantage of being able to find the condition for a solution without matching to the inner problem in the vicinity of the singularity at $y = i\sqrt{1+\alpha}$;

in order to formulate this inner problem we need to know the dependence of q on θ in the inner region.

4 The constant Λ

We are left with the problem of evaluating Λ. To evaluate Λ we may write down the full Laurent expansion for θ_n, and compare with our asymptotic behaviour (3.12). Since we require just one number, we need to compare for just one value of y. We therefore choose the simplest value of s to make this comparison, namely at the singularity $y = i\sqrt{1+\alpha}$. To evaluate the Laurent expansion of θ as $y \to i\sqrt{1+\alpha}$ we take advantage of matched asymptotics. In that language we require one term in the inner limit of the outer expansion. It is therefore enough for us to have the outer limit of the one term inner expansion. The correct scaling for the inner problem is

$$y = i(1+\alpha)^{1/2} + \epsilon^{4/7} z, \tag{4.1}$$

giving

$$q_0 = \cos\theta_0 \sim \frac{e^{-3i\pi/4}(1+\alpha)^{1/4}}{2^{1/2} z^{1/2} \epsilon^{2/7}}, \tag{4.2}$$

$$\sin\theta_0 \sim -\frac{e^{-i\pi/4}(1+\alpha)^{1/4}}{2^{1/2} z^{1/2} \epsilon^{2/7}}, \tag{4.3}$$

$$\theta_0 \sim \frac{i}{2}\log z - i\log\left(\frac{2^{1/2}(1+\alpha)^{1/4} e^{-3i\pi/4}}{\epsilon^{2/7}}\right). \tag{4.4}$$

Writing (2.28) in inner coordinates gives

$$\log q - i\theta = \frac{2}{\pi}\int_{-\infty}^{0} \frac{\bar{y}\theta(\bar{y})}{y^2 - \bar{y}^2}\, d\bar{y}, \tag{4.5}$$

$$\sim -\log 2 - \frac{i\epsilon^{4/7} z}{2(1+\alpha)^{1/2}}. \tag{4.6}$$

Therefore, we define the inner variable ϕ by

$$q = \frac{e^{-3i\pi/4}(1+\alpha)^{1/4}}{2^{1/2}\epsilon^{2/7} z^{1/2}}\phi, \tag{4.7}$$

giving, to leading order in the inner region,

$$-\frac{\alpha^2 e^{i\pi/4}}{2\sqrt{2}(1+\alpha)^{1/4} z}\frac{d^2}{dz^2}\left(\frac{\phi}{z^{1/2}}\right) = -1 + \frac{1}{\phi^2}. \tag{4.8}$$

The rescaling

$$z = \left(\frac{\alpha^2 e^{i\pi/4}}{2\sqrt{2}(1+\alpha)^{1/4}}\right)^{2/7} \zeta, \tag{4.9}$$

then gives

$$-\frac{1}{\zeta}\frac{d^2}{d\zeta^2}\left(\frac{\phi}{\zeta^{1/2}}\right) = -1 + \frac{1}{\phi^2}. \tag{4.10}$$

This is equation (31) of [4] (their $g = \phi/\zeta^{1/2}$). We require the outer limit, that is, we need to expand ϕ as $\zeta \to \infty$. We find

$$\phi = \sum_{n=0}^{\infty} \frac{A_n}{\zeta^{7n/2}} = \sum_{n=0}^{\infty} \frac{A_n}{z^{7n/2}} \left(\frac{\alpha^2 e^{i\pi/4}}{2\sqrt{2}(1+\alpha)^{1/4}}\right)^n, \tag{4.11}$$

where the A_n satisfy the recurrence relation

$$\sum_{m=1}^{n} \left(\frac{7m}{2} - 3\right)\left(\frac{7m}{2} - 2\right) A_{m-1} \sum_{k=0}^{n-m} A_{n-m-k} A_k = \sum_{m=0}^{n} A_{n-m} A_m, \quad n \geq 1, \tag{4.12}$$

with

$$A_0 = 1. \tag{4.13}$$

Returning to the outer expansion, since $q_n \sim i\theta_n q_0$ as $n \to \infty$, we have

$$\epsilon^{2n} q_n \sim \frac{i\epsilon^{2n}\Lambda q_0 \Gamma(2n+\gamma+1)}{u^{2n+\gamma+1}} \frac{i}{\sqrt{2}} \left(\frac{(1+\alpha)^{1/2} + iy}{(1+\alpha)^{1/2} - iy}\right)^{1/8} \qquad \text{as } n \to \infty \tag{4.14}$$

$$\sim -\frac{\Lambda\Gamma(2n+\gamma+1)(7\alpha)^{1/14}(1+\alpha)^{5/28} e^{-19i\pi/28}}{\epsilon^{2/7} z^{1/2} 2^{19/14}} \left(\frac{\alpha^2 e^{i\pi/4}}{(1+\alpha)^{1/4} 2\sqrt{2}} \frac{49}{32}\right)^n \frac{1}{z^{7n/2}} \tag{4.15}$$

in the inner variable as $\epsilon \to 0$. Comparing (4.7), (4.11) and (4.15) we see

$$\Lambda = \frac{2^{6/7} e^{13i\pi/14}(1+\alpha)^{1/14}}{(7\alpha)^{1/14}} \lim_{n\to\infty} \left(\frac{32}{49}\right)^n \frac{A_n}{\Gamma(2n+\gamma+1)}. \tag{4.16}$$

From (4.12) we find that A_n is real and positive as $n \to \infty$. Hence, as we expected,

$$\Phi = \frac{13\pi}{14}. \tag{4.17}$$

Iterating (4.12) to large values of n (~ 300) we find that

$$\Lambda \approx 0.46048 \frac{e^{13i\pi/14}(1+\alpha)^{1/14}}{\alpha^{1/14}}. \tag{4.18}$$

5 Analysis with α order $\epsilon^{4/3}$ as $\epsilon \to 0$

In the previous section we found that for N small, $\alpha = O(\epsilon^{4/3})$ as $\epsilon \to 0$. However, the analysis assumed that α was order one, so that the singularities at $y = i$ and $y = i\sqrt{1+\alpha}$ were well separated in the outer region. In this section we consider the case $\alpha = O(\epsilon^{4/3})$ via a similar Stokes line analysis. We set

$$\alpha = \epsilon^{4/3} a. \tag{5.1}$$

The expansion of θ and q now proceeds in powers of $\epsilon^{2/3}$:

$$q \sim \sum_{n=0}^{\infty} \epsilon^{2n/3} q_n, \qquad \theta \sim \sum_{n=0}^{\infty} \epsilon^{2n/3} \theta_n. \tag{5.2}$$

The first three terms in this expansion may be obtained by expanding (3.10), (3.11) in powers of ϵ after substituting in (5.1), since the coupling between terms will not appear until $O(\epsilon^2)$:

$$q_0 = \cos\theta_0 = -\frac{y}{(1+y^2)^{1/2}}, \tag{5.3}$$

$$q_1 = 0, \tag{5.4}$$

$$q_2 = \frac{ay}{2(1+y^2)^{3/2}}, \tag{5.5}$$

$$\sin\theta_0 = -\frac{1}{(1+y^2)^{1/2}}, \tag{5.6}$$

$$\theta_1 = 0, \tag{5.7}$$

$$\theta_2 = -\frac{ay}{2(1+y^2)}. \tag{5.8}$$

Now, the equations (2.27), (2.28) lead naturally to an expansion in powers of ϵ^2, while we have forced them into a series in powers of $\epsilon^{2/3}$ by giving a non-zero term of order $\epsilon^{4/3}$. As $n \to \infty$, with factorial/power divergence, there will be coupling only between every third term in the series. Thus our large n behaviour ansatz becomes

$$\theta_{3n+k} \sim \Lambda_k \frac{\Gamma(2n+\gamma_k+1)}{u_k^{2n+\beta_k}}, \qquad k = 0, 1, 2. \tag{5.9}$$

At leading order as $n \to \infty$ we find

$$u_k' = 2i(1+iy)^{-1/4}(1-iy)^{-3/4}, \qquad k = 0, 1, 2, \tag{5.10}$$

giving

$$u_k = u = 2i \int_i (1+iy)^{-1/4}(1-iy)^{-3/4}\, dy, \qquad k = 0, 1, 2. \tag{5.11}$$

We may evaluate the integral to give

$$u = \sqrt{2}\left(2\arctan(1-\sqrt{2z}) - 2\arctan(1+\sqrt{2z}) + \log\left(\frac{1+\sqrt{2z}+z}{1-\sqrt{2z}+z}\right)\right).$$

where

$$z = \left(\frac{1+iy}{1-iy}\right)^{1/2}. \tag{5.12}$$

At next order we find that $\gamma_k' = 0$ for each k, and the equation for β_k is

$$\beta_k + u\log u \frac{d\beta_k}{du} = \gamma_k + 1 + \frac{iu}{2u'}\frac{d\theta_0}{dy} + \frac{uu''}{2(u')^2} + \frac{yu}{2u' q_0(1+y^2)}\frac{d}{dy}\left(\frac{q_0(1+y^2)}{y}\right), \tag{5.13}$$

giving

$$\beta_k = \gamma_k + 1 + \frac{\log u'}{2\log u} + \frac{i\theta_0}{2\log u} + \frac{\log(q_0(1+y^2)/y)}{2\log u}. \tag{5.14}$$

Now, as $y \to i$,

$$u \sim \frac{2^{9/4}e^{3i\pi/8}(y-i)^{3/4}}{3}, \tag{5.15}$$

$$\theta_0 \sim (i/2)\log(y-i), \tag{5.16}$$

$$\theta_2 \sim -\frac{a}{4(y-i)}, \tag{5.17}$$

$$\theta_4 \sim \frac{K}{(y-i)^2}, \tag{5.18}$$

for some constant K. From (5.15) and (5.16) we have $\beta_0 \to 0$ as $y \to i$, giving, from (5.14)

$$\gamma_0 = -1 + 1/6 + 1/3 - 1/3 = -5/6. \tag{5.19}$$

Similarly from (5.17) and (5.18), as $y \to i$, $\beta_1 \to 2/3$, $\beta_2 \to 4/3$, giving $\gamma_1 = -1/6$, $\gamma_2 = 1/2$. Thus

$$\theta_{3n+k} \sim \frac{\Lambda_k \Gamma(2n+\gamma_k+1)}{u^{2n+\gamma_k+1}} \frac{i}{\sqrt{2}} \left(\frac{1+iy}{1-iy}\right)^{1/8}. \tag{5.20}$$

The analysis of the Stokes line now follows exactly as before, with each value of k giving a series in powers of ϵ^2 which may be treated independently. Thus, since the Stokes lines are given by u real and positive, there is a Stokes line down the imaginary axis from $y = i$ across which

$$-\frac{\pi\Lambda_0 e^{-u/\epsilon}}{\sqrt{2}\,\epsilon^{\gamma_0+1}}\left(\frac{1+iy}{1-iy}\right)^{1/8} - \frac{\pi\Lambda_1 e^{-u/\epsilon}}{\sqrt{2}\,\epsilon^{\gamma_1+1/3}}\left(\frac{1+iy}{1-iy}\right)^{1/8} - \frac{\pi\Lambda_2 e^{-u/\epsilon}}{\sqrt{2}\,\epsilon^{\gamma_2-1/3}}\left(\frac{1+iy}{1-iy}\right)^{1/8} \tag{5.21}$$

is switched on, and an analogous result for the Stokes line up the real axis from $y = -i$ (see Fig. 7). Remembering that $\gamma_0 = \gamma_1 - 2/3 = \gamma_2 - 4/3$ we see that on the real axis

$$-\mathrm{Re}\left((\Lambda_0 + \Lambda_1 + \Lambda_2)\frac{\pi e^{-u/\epsilon}}{\sqrt{2}\,\epsilon^{\gamma_0+1}}\left(\frac{1+iy}{1-iy}\right)^{1/8}\right) \tag{5.22}$$

is switched on. The condition for a solution is that this is zero on $y = 0$ giving

$$R_0\cos(\Phi_0) + R_1\cos(\Phi_1) + R_2\cos(\Phi_2) = 0. \tag{5.23}$$

Let us comment here on the manner in which Figure 7 matches with the limits of Figures 4 and 5. As $\alpha \to 0$ from above the two intersection points of the Stokes lines with the real axis in Figure 5 move towards the origin, and these Stokes lines become almost vertical away from the singularity at $y = i(1+\alpha)^{1/2}$.

In the present case, for large positive a the Stokes multipliers for two of the Stokes lines down the imaginary axis remain finite, while the third tends to zero. Thus two of the

Stokes lines match into those of Figure 5, while the third disappears. For large negative a the third Stokes multiplier remains finite and the first two tend to zero. The Stokes line which survives matches into that down the imaginary axis in Figure 4.

Thus we see that the transition from Figure 5 to Figure 4 occurs through the gradual disappearance of the Stokes lines in Figure 5 and the gradual emergence of the Stokes line in Figure 4, with all three being present during the transition.

6 The constants Λ_0, Λ_1, Λ_2

As before, we determine the constants Λ_0, Λ_1, Λ_2 by comparing our ansatz (5.9) with the outer limit of the inner solution in the vicinity of the singularity, which is now at $y = i$. Defining the inner variable z by

$$y = i + \epsilon^{4/3} z, \tag{6.1}$$

we find

$$\cos\theta_0 \sim \frac{e^{-3i\pi/4}}{2^{1/2}\epsilon^{2/3}z^{1/2}}, \tag{6.2}$$

$$\sin\theta_0 \sim -\frac{e^{-i\pi/4}}{2^{1/2}\epsilon^{2/3}z^{1/2}}. \tag{6.3}$$

$$\log q - i\theta \sim -\log 2 - \epsilon^{4/3}\left(\frac{a}{4} + \frac{iz}{2}\right). \tag{6.4}$$

$$\theta_0 \sim \frac{i}{2}\log z - i\log\left(\frac{2^{1/2}e^{-3i\pi/4}}{\epsilon^{2/3}}\right). \tag{6.5}$$

We define the inner solution ϕ by

$$q = \frac{e^{-3i\pi/4}}{2^{1/2}\epsilon^{2/3}z^{1/2}}\phi, \tag{6.6}$$

giving, to leading order,

$$\sqrt{2}e^{-3i\pi/4}\frac{d}{dz}\left(z\frac{d}{dz}\left(\frac{\phi}{z^{1/2}}\right)\right) = 1 - \frac{1}{\phi^2} - \frac{ia}{2z}. \tag{6.7}$$

The change of variables

$$\zeta = \frac{iz}{2^{1/3}}, \tag{6.8}$$

gives

$$\frac{d}{d\zeta}\left(\zeta\frac{d}{d\zeta}\left(\frac{\phi}{\zeta^{1/2}}\right)\right) = 1 - \frac{1}{\phi^2} + \frac{a}{2^{4/3}\zeta}. \tag{6.9}$$

This is equivalent to equation (42) of [4] (their $y = \zeta^{1/2}$, their $G = \phi/\zeta^{1/2}$.) As before, we are interested in the outer limit of this inner expansion. Expanding as $\zeta \to \infty$ we find

$$\phi = \sum_{n=0}^{\infty} \frac{A_n}{\zeta^{n/2}} = \sum_{n=0}^{\infty} A_n \left(\frac{2^{1/3}}{iz}\right)^{n/2}, \tag{6.10}$$

where the A_n satisfy the recurrence relation

$$\sum_{m=3}^{n} \left(\frac{m}{2} - 1\right)^2 A_{m-3} \sum_{k=0}^{n-m} A_{n-m-k} A_k = \sum_{m=0}^{n} A_{n-m} A_m + \frac{a}{2^{4/3}} \sum_{m=2}^{n} A_{n-m} A_{m-2}, \quad n \geq 3, \tag{6.11}$$

with

$$A_0 = 1, \qquad A_1 = 0, \qquad A_2 = \frac{a}{2^{7/3}}. \tag{6.12}$$

From the outer expansion we have

$$\epsilon^{2n+2k/3} q_{3n+k} \sim -\frac{3^{1/6} e^{-3i\pi/4} \Lambda_k \, \Gamma(2n + \gamma_k + 1)}{2^{3/2} \epsilon^{2/3} z^{1/2}} \left(\frac{9}{2^{9/2} e^{3i\pi/4} z^{3/2}}\right)^{n+k/3}. \tag{6.13}$$

Comparing (6.6), (6.10) and (6.13) we find

$$\Lambda_0 = -\frac{2}{3^{1/6}} \lim_{n\to\infty} \frac{A_{3n} 2^{5n}}{\Gamma(2n + 1/6) 3^{2n}}, \tag{6.14}$$

$$\Lambda_1 = -\frac{2^{8/3}}{3^{5/6}} \lim_{n\to\infty} \frac{A_{3n+1} 2^{5n}}{\Gamma(2n + 5/6) 3^{2n}}, \tag{6.15}$$

$$\Lambda_2 = -\frac{2^{13/3}}{3^{3/2}} \lim_{n\to\infty} \frac{A_{3n+2} 2^{5n}}{\Gamma(2n + 3/2) 3^{2n}}. \tag{6.16}$$

Since all the Λ's are real, the condition (5.23) for a solution is simply

$$\Lambda_0 + \Lambda_1 + \Lambda_2 = 0. \tag{6.17}$$

This is equivalent to equation (67) in [4] since using (6.14)-(6.16) we find that their b's are given by

$$b_k = -\frac{\Gamma((k+1)/3)^2 3^{(4k+1)/6} \Lambda_k}{2^{(k+4)/3} \sqrt{2\pi}}, \qquad k = 0, 1, 2. \tag{6.18}$$

7 Conclusion

We have examined the rôle of Stokes lines in the selection of Saffman-Taylor fingers with small surface tension. By applying a WKB-type ansatz to the late terms in the asymptotic expansion of the free-boundary shape in terms of the surface tension, and optimally truncating this expansion we were able to identify the Stokes lines and observe the switching on of subdominant exponentials. These exponentials are responsible for finger selection, choosing a discrete family of finger widths from the continuum in the absence of surface tension. For any given value of the surface tension these solutions cluster near $\lambda = 1$; however, for any given solution branch $\lambda \to 1/2$ as surface tension tends to zero.

The condition (3.48) for a solution to exist was derived without the need for an inner analysis about singularities in the complex plane. While this inner analysis is not difficult in the present situation, the ability to avoid it may be more useful in more complicated problems, for example, in problems having a nonlinear field equation. In the limit as $\lambda \to 1/2$, (3.48) agrees with the results in [3] as it should.

Finally, for the lower solution branches a separate analysis is necessary, since two singuarities in the complex plane are no longer well separated. In this case an inner analysis is necessary to obtain the condition for a solution to exist, and the results we obtain are identical to those in [4]. However, we include this case as an interesting extension to the method, since the late terms in the series now form effectively three independent series, with three coincident Stokes lines. The condition for a solution to exist arises as the condition that the Stokes multipliers add up to zero.

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Appendix A Appendix 1

The expression (3.17) may be integrated explicitly to give

$$
\begin{aligned}
u = {} & \frac{1}{16\sqrt{2}\,(1-b)^{\frac{3}{4}}\,b^2}\Big(-4\,(1-b)^{\frac{3}{4}}\,\tan^{-1}(1-\sqrt{2t}) \\
& + 4\,(1-b)^{\frac{3}{4}}\,\tan^{-1}(1+\sqrt{2t}) + 2\,(1-b)\,(1+b)^{\frac{3}{4}}\,\tan^{-1}\left(1-\frac{(1-b)^{\frac{1}{4}}\sqrt{2t}}{(1+b)^{\frac{1}{4}}}\right) \\
& - 2\,(1+b)^{\frac{3}{4}}\,\tan^{-1}\left(1+\frac{(1-b)^{\frac{1}{4}}\sqrt{2t}}{(1+b)^{\frac{1}{4}}}\right) + 2\,b\,(1+b)^{\frac{3}{4}}\,\tan^{-1}\left(1+\frac{(1-b)^{\frac{1}{4}}\sqrt{2t}}{(1+b)^{\frac{1}{4}}}\right) \\
& + 2\,(1-b)^{\frac{3}{2}}\,(1+b)^{\frac{1}{4}}\,\tan^{-1}\left(1-\frac{(1+b)^{\frac{1}{4}}\sqrt{2t}}{(1-b)^{\frac{1}{4}}}\right) \\
& - 2\,(1-b)^{\frac{3}{2}}\,(1+b)^{\frac{1}{4}}\,\tan^{-1}\left(1+\frac{(1+b)^{\frac{1}{4}}\sqrt{2t}}{(1-b)^{\frac{1}{4}}}\right) \\
& + 2\,(1-b)^{\frac{3}{4}}\,\log(1-\sqrt{2t}+t) - 2\,(1-b)^{\frac{3}{4}}\,\log(1+\sqrt{2t}+t) \\
& + (-1+b)\,(1+b)^{\frac{3}{4}}\,\log(\sqrt{1+b}-\big(1-b^2\big)^{\frac{1}{4}}\sqrt{2t}+\sqrt{1-b}\,t) \\
& + (1-b)\,(1+b)^{\frac{3}{4}}\,\log(\sqrt{1+b}+\big(1-b^2\big)^{\frac{1}{4}}\sqrt{2t}+\sqrt{1-b}\,t) \\
& - (1-b)^{\frac{3}{2}}\,(1+b)^{\frac{1}{4}}\,\log(\sqrt{1-b}-\big(1-b^2\big)^{\frac{1}{4}}\sqrt{2t}+\sqrt{1+b}\,t) \\
& + (1-b)^{\frac{3}{2}}\,(1+b)^{\frac{1}{4}}\,\log(\sqrt{1-b}+\big(1-b^2\big)^{\frac{1}{4}}\sqrt{2t}+\sqrt{1+b}\,t)\Big)
\end{aligned}
$$

where

$$
\begin{aligned}
b &= \sqrt{1+\alpha}, \\
t &= \sqrt{\frac{b+i\,y}{b-i\,y}}.
\end{aligned}
$$

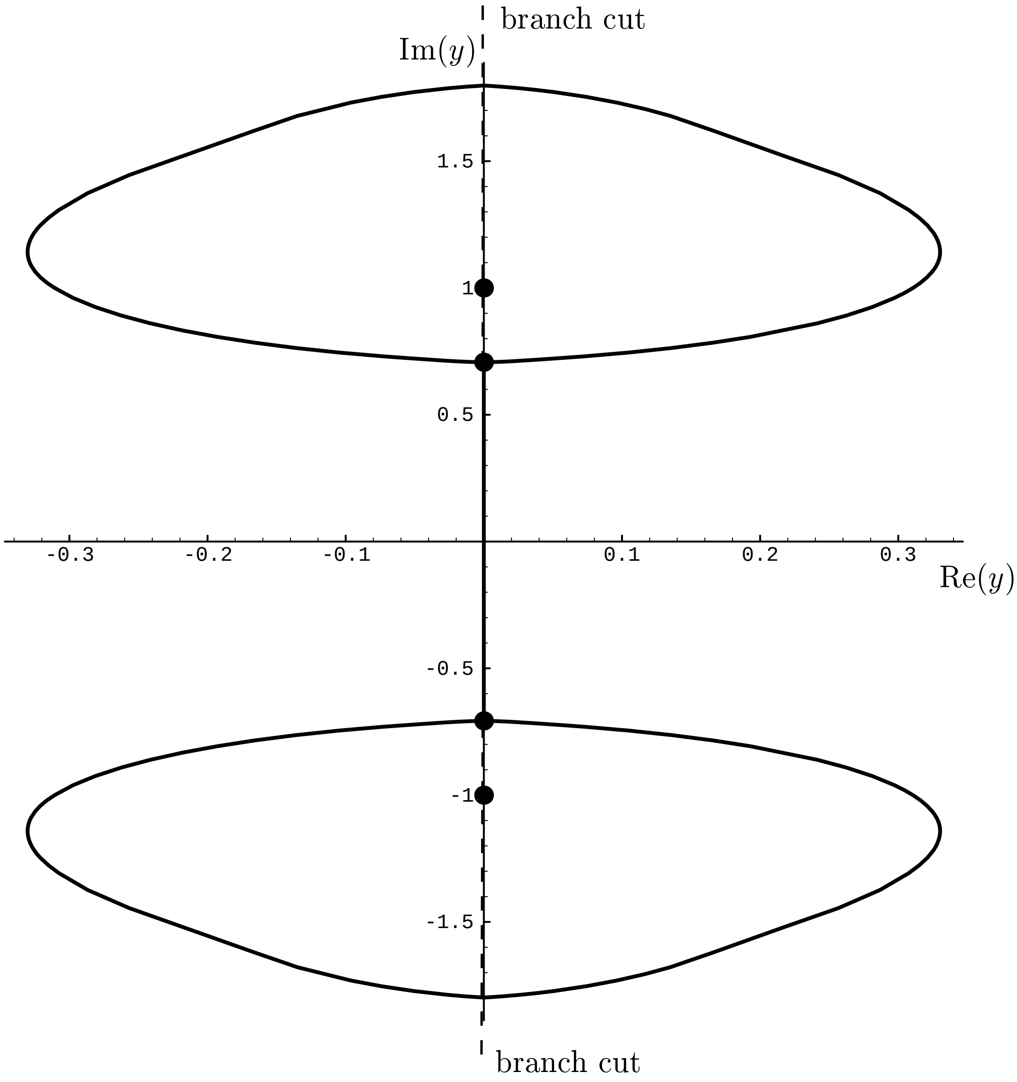

FIGURE 4. Stokes lines for $\alpha = -1/2$, which are given by $\mathrm{Im}(u) = 0$.

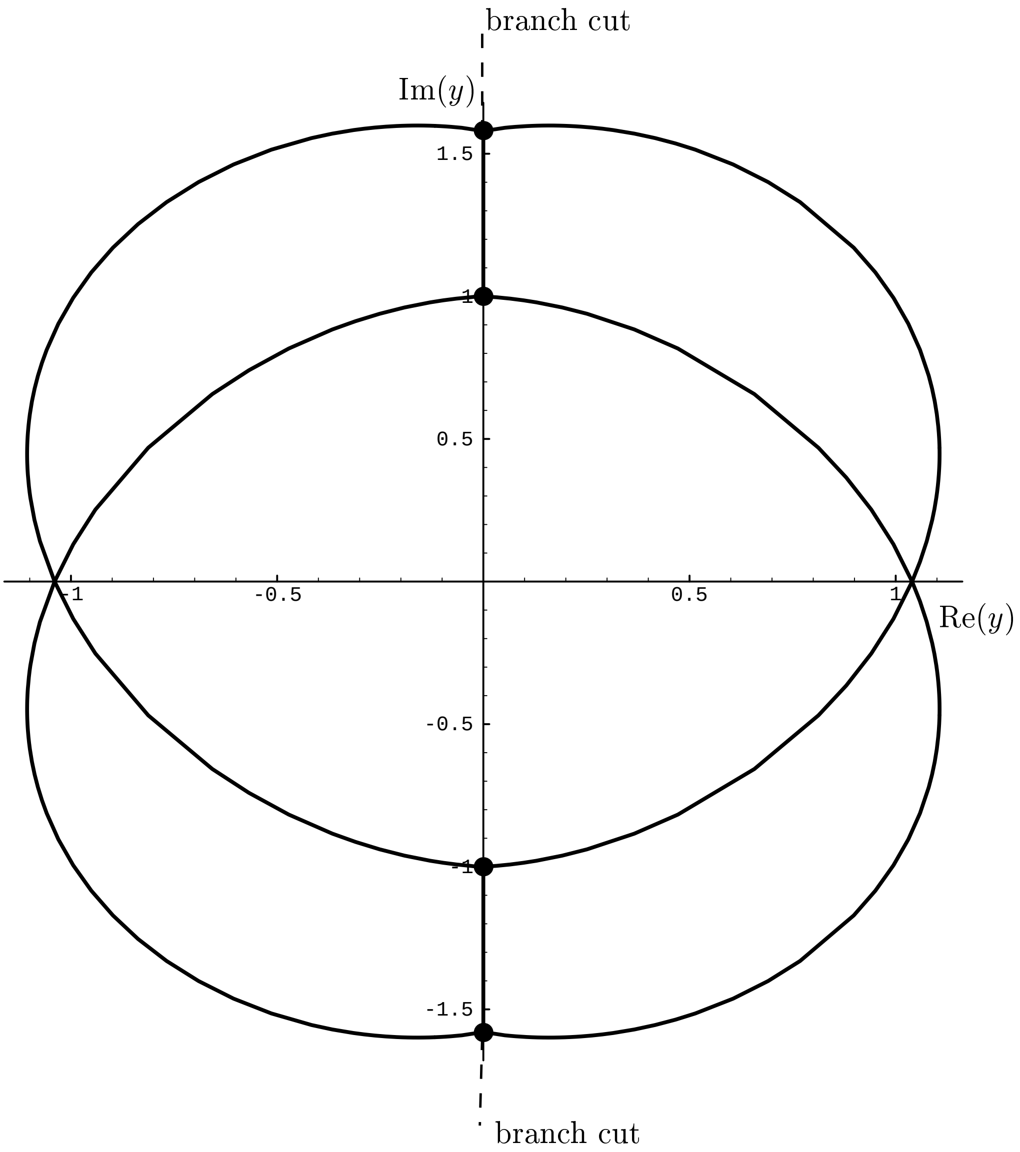

FIGURE 5. Stokes lines for $\alpha = 3/2$, which are given by $\mathrm{Im}(u) = 0$.

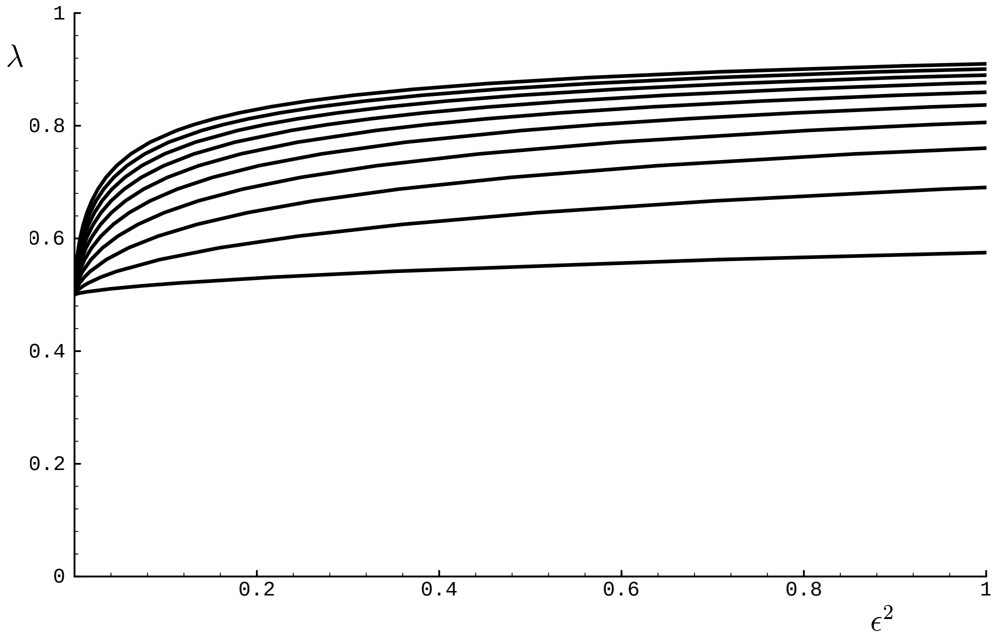

FIGURE 6. The relative finger width λ as a function of surface tension ϵ^2 for the first ten solution branches. The curves are valid asymptotically in the limit $\epsilon \to 0$, ϵN order one.

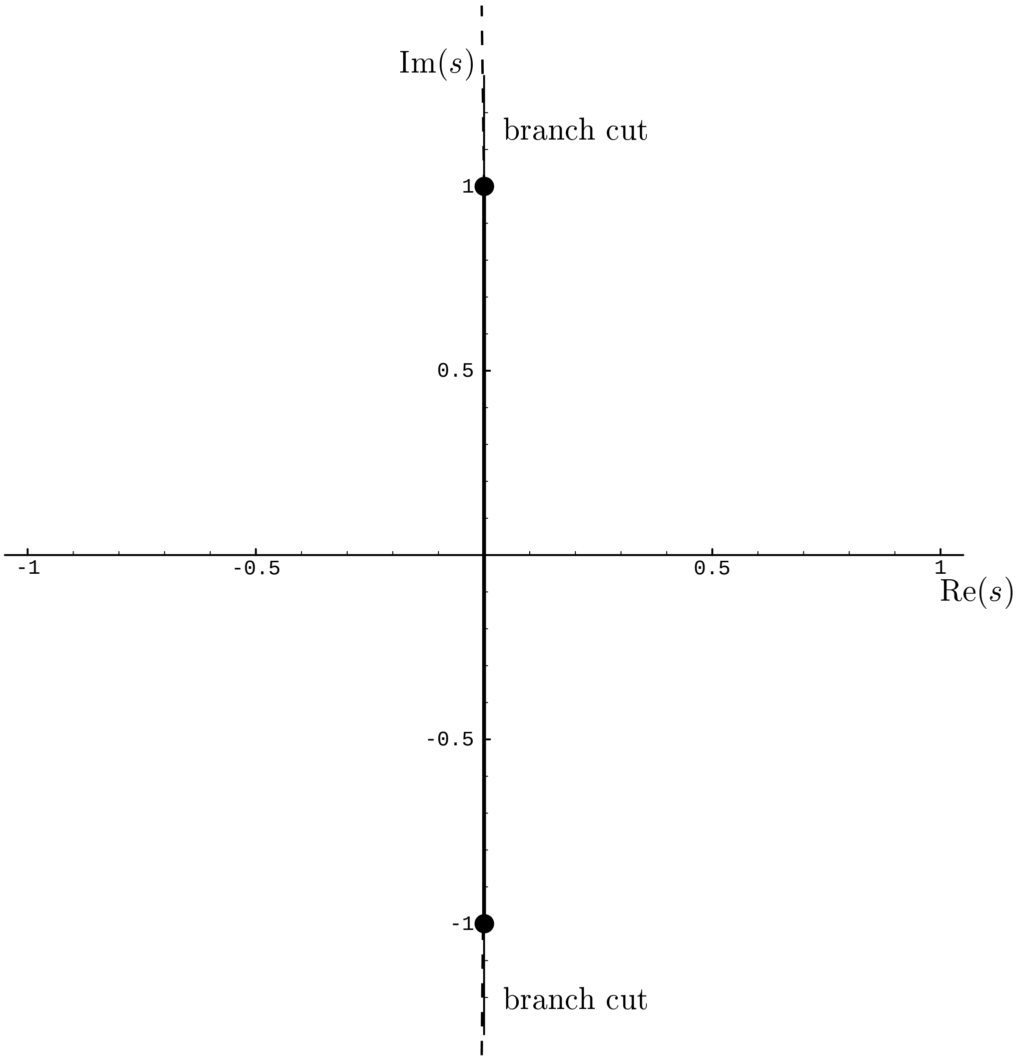

FIGURE 7. Stokes lines for $\alpha = O(\epsilon^{4/3})$.