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Macroscopic models of superconductivity

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Abstract

A hierarchy of models for type-II superconductors is presented. Through appropriate asymptotic limits the mesoscopic Ginzburg-Landau model is connected to the London model, to vortex-density models, and finally to macroscopic critical-state models such as the Bean model. The basic building block in deriving this hierarchy is the superconducting vortex which is a thin core of nonsuperconducting material circled by a superconducting electric current. Similar line singularities are found in other systems, for example, line vortices in an inviscid fluid, or Volterra dislocations in an elastic crystal. Analogous hierarchies for these systems are briefly discussed.

1 Introduction

One of the interesting aspects of superconductivity is the variety of models available to describe the phenomenon at different lengthscales, ranging from the microscopic theory of Bardeen, Cooper & Schreiffer through the mesoscopic theories of London and Ginzburg & Landau, to the macroscopic Critical State theories such as the Bean model. Through a series of appropriate asymptotic limits we will derive a hierarchy of models connecting the Ginzburg-Landau model to the London model with isolated superconducting vortices as line singularities, to vortex-density models, and finally to macroscopic critical-state models. The basic building block in deriving this hierarchy is the superconducting vortex, which is a thin core of nonsuperconducting material circled by a superconducting electric current. Similar line singularities are found in other systems, for example, line vortices in an inviscid fluid, or Volterra dislocations in an elastic crystal. This raises the interesting question of whether analogous hierarchies of models exist in each of these other systems, and whether similar connections can be established between them.

2 Macroscopic models of superconductivity

We begin by describing the basic phenomena of superconductivity. The first of these is perfect conductivity, discovered by Kamerlingh-Onnes in 1911 shortly

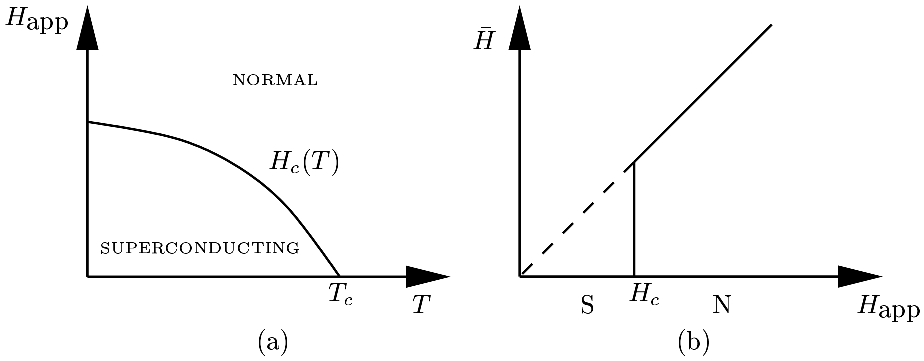

FIG. 1. (a) The phase diagram of a type I superconductor. (b) The average magnetic field, $\bar{H}$, inside a type I superconductor as a function of the applied magnetic field, H_{app}.

after he succeeded in liquefying helium [14]. When a superconducting material is cooled below a critical temperature, T_c, the resistivity drops sharply to zero, so that the material can carry an electric current without an associated electric field.

The second hallmark of superconductivity is sometimes known as perfect diamagnetism, and was discovered by Meissner & Ochsenfeld in 1933 [18]. They found that a material in the superconducting state will exclude a magnetic field from its interior, except in thin boundary layers whose thickness is known as the penetration depth, and which is usually denoted by λ. This exclusion of the magnetic field is now known as the Meissner effect.

The energy penalty associated with the exclusion of the magnetic field leads to the existence of a critical magnetic field, $H_c(T)$, above which superconductivity is destroyed and the sample reverts to the nonsuperconducting (normal) state. Thus the phase diagram and response of a superconductor to an applied magnetic field are as depicted in Fig. 1.

In fact superconductors with the properties described thus far are known as type I superconductors. There is a second class, known as type II superconductors, for which the exclusion of magnetic field is only partial over a certain range of applied fields. The single critical field of type I superconductors is replaced by a lower critical field $H_{c_1}(T)$ and an upper critical field $H_{c_2}(T)$. For $H > H_{c_2}$ the material is in the normal state, while for $H < H_{c_1}$ the material is in the superconducting state (see Fig. 2). For $H_{c_1} < H < H_{c_2}$ the superconductor is in a mixed phase in which the magnetic field penetrates the superconductor by means of "flux tubes". These are one-dimensional cores of normal material carrying an axial magnetic field and circled by a supercurrent vortex (see Fig. 3). Almost all technological applications are for type II superconductors in the mixed phase, since typically $H_{c_2} \gg H_c$, allowing high magnetic fields to be generated.

The simplest model of the mixed state is the London model in which vortices

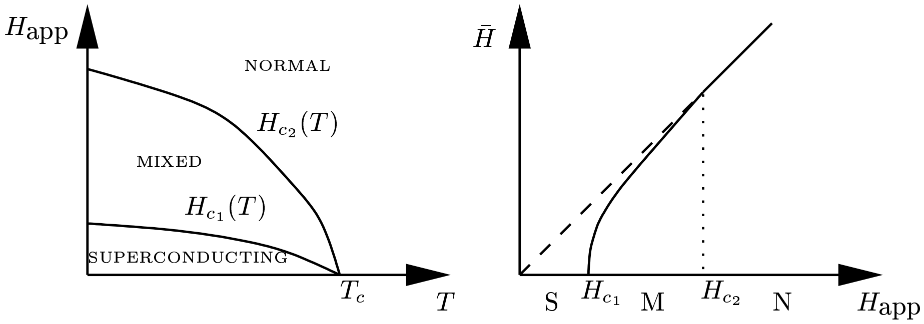

FIG. 2. (a) The phase diagram of a type II superconductor. (b) The average magnetic field, $\bar{H}$, inside a type II superconductor as a function of the applied magnetic field, Happ.

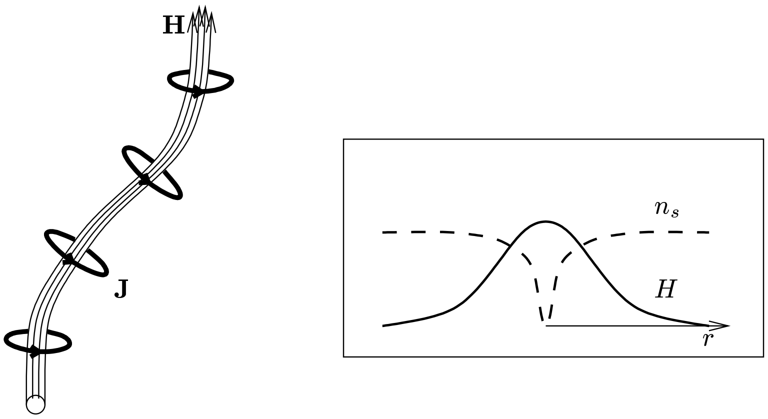

FIG. 3. Schematic diagram of a superconducting vortex. The inset shows the variation of magnetic field H and superconducting electron density n_s across the core of the vortex.

appear as δ-functions in the equation for the magnetic field. For a single vortex in an infinite superconductor the dimensionless equations are

$$(\mathrm{curl})^2\mathbf{H} + \mathbf{H} = 2\pi\delta_\Gamma, \tag{2.1}$$

$$\mathrm{div}\ \mathbf{H} = 0, \tag{2.2}$$

where

$$\delta_\Gamma = \int_\Gamma \delta(x-x')\delta(y-y')\delta(z-z')\,\mathbf{dx}', \tag{2.3}$$

and δ is the usual Dirac δ-function. Equation (2.1) was written down by London & London in 1935 [17; 16] (without the δ-functions) to describe Meissner effect.

It results in an exponential decay of the magnetic field away from the boundary of the material (length in the equations has been nondimensionalised with the penetration depth λ). The effect of vortices on the London equation was first proposed by Abrikosov in 1957 [1].

At this point it is very useful to draw an analogy between line vortices in the London equation and line vortices in an inviscid fluid.

Analogy with fluid vortices. A line vortex in an otherwise irrotational, incompressible, inviscid fluid satisfies

$$\begin{aligned} \text{curl}\ \mathbf{u} &= 2\pi\gamma\delta_\Gamma, && (2.4)\\ \text{div}\ \mathbf{u} &= 0, && (2.5)\end{aligned}$$

where γ is the vortex strength. Equation (2.5) implies the existence of a stream-function $\mathbf{\Omega}$ such that

$$\begin{aligned} \text{curl}\ \mathbf{\Omega} &= \mathbf{u}, && (2.6)\\ \text{div}\ \mathbf{\Omega} &= 0, && (2.7)\\ (\text{curl})^2\mathbf{\Omega} = -\nabla^2\mathbf{\Omega} &= 2\pi\gamma\delta_\Gamma. && (2.8)\end{aligned}$$

Now using the Green's function for the Laplacian gives

$$\mathbf{\Omega}(\mathbf{x}) = \frac{\gamma}{2}\int_\Gamma \frac{1}{s}\,\mathbf{dx}', \qquad \mathbf{s} = \mathbf{x} - \mathbf{x}', \quad s = |\mathbf{s}|. \tag{2.9}$$

$$\mathbf{u}(\mathbf{x}) = \frac{\gamma}{2}\int_\Gamma \nabla\left(\frac{1}{s}\right)\wedge\mathbf{dx}' = -\frac{\gamma}{2}\int_\Gamma \frac{\mathbf{s}\wedge\mathbf{dx}'}{s^3}, \tag{2.10}$$

which is the familiar Biot-Savart law. As the point $\mathbf{x}$ approaches the vortex

$$\mathbf{u} \sim \frac{\gamma}{r}\mathbf{e}_\theta - \frac{\gamma C\log r}{2}\mathbf{b} + O(1), \tag{2.11}$$

where r, θ are local polar coordinates, C is the curvature of Γ, $\mathbf{e}_\theta$ is the unit azimuthal vector, and $\mathbf{b}$ is the unit binormal. The law of motion of fluid vortices states that they are convected with the fluid. The asymptotic behaviour (2.11) would then seem to imply that the vortex moves with infinite speed in the binormal direction.

Of course, eqn (2.4) ceases to be valid as $r \to 0$, i.e. in the vortex core. They arise as the leading-order term in expansion of the Navier-Stokes eqns as the Reynolds number $\text{Re} \to \infty$. In the core velocity gradients are large, so that there is an inner region in which viscous forces are not negligible. Matching between this core region and the outer inviscid solution gives the velocity of the vortex as

$$\mathbf{v} \sim -\frac{\gamma C\log\epsilon}{2}\mathbf{b} + \mathbf{u}_{\text{reg}}, \tag{2.12}$$

where ϵ is the vortex core radius (which is $\text{Re}^{-1/2}$), and $\mathbf{u}_{\text{reg}}$ is the regular part of the fluid velocity evaluated at the vortex, which includes the contributions from the "arms" of the vortex (the order-one term in (2.11)) as well as contributions from other vortices, boundaries, and any background irrotational flow. See, for example, [4; 15; 19].

In the analogy between superconductors and fluids $\mathbf{H}$ corresponds to $\mathbf{\Omega}$ while $\mathbf{J}$ corresponds to $\mathbf{u}$. In comparing (2.8) and (2.1) the first point to note is that for superconducting vortices γ is not arbitrary but equal to unity. This is because superconducting vortices have quantised strengths, with stable vortices containing exactly one quantum. Using the Green's function for the modified Helmholtz eqn on (2.8) gives

$$\mathbf{H}(\mathbf{x}) = \frac{1}{2}\int_\Gamma \frac{e^{-s}}{s}\,\mathbf{dx}', \tag{2.13}$$

$$\mathbf{J}(\mathbf{x}) = \frac{1}{2}\int_\Gamma \nabla\left(\frac{e^{-s}}{s}\right)\wedge \mathbf{dx}' \tag{2.14}$$

with s as before. As the point $\mathbf{x}$ approaches the vortex

$$\mathbf{J} \sim \frac{1}{r}\mathbf{e}_\theta - \frac{C\log r}{2}\mathbf{b} + O(1). \tag{2.15}$$

However, the law of motion for superconducting vortices states that $\mathbf{v} = \mathbf{J}\wedge\mathbf{t}$, where $\mathbf{t}$ is the tangent to Γ. This can be thought of as balancing the "Lorentz" force $\mathbf{J}\wedge\mathbf{t}$ (remember the vortex carries a unit quantum of magnetic flux) with a "viscous dissipation" proportional to velocity as the vortex moves. Such a law of motion would seem to imply that the vortex moves with an infinite speed in the *normal* direction.

Again, however, the London equations are not valid in the vortex core. They are first term in an expansion of the Ginzburg-Landau (GL) model as the Ginzburg-Landau parameter $\kappa\to\infty$. Matching the outer London solution with an inner core region in which the GL model is solved gives the law of motion

$$\mathbf{v} \sim -\frac{C\log\epsilon}{2}\mathbf{n} + \mathbf{J}_{\text{reg}}\wedge\mathbf{t}, \tag{2.16}$$

where $\mathbf{n}$ is the normal, $\epsilon = 1/\kappa$ is the core radius, and $\mathbf{J}_{\text{reg}}$ is the regular part of the current density evaluated at the vortex line, which includes contributions from the "arms" of the vortex, other vortices and boundary effects (see, for example, [20; 9; 13; 12]).

The law of motion for superconducting vortices implies that for rectilinear vortices like vortices will repel each other while unlike vortices will attract each other, in contrast to fluid vortices where like vortices will circle about the mid-point of the line if centres and unlike vortices will translate uniformly maintaining a constant separation.

Homogenisation For typical magnetic fields the separation of vortices in a superconductor is $< 0.1\,\mu$m. Thus except in very small samples there will be many millions. Since we are mainly interested in bulk variables, it becomes attractive to seek an averaged model in which the individual vortices have been smeared into a vorticity.

Here the analogy with fluid vortices again becomes useful: Can we derive the Euler equations by averaging a system of line vortices in an inviscid fluid?

Let us start with a system of equal strength vortices, so that

$$\text{curl } \mathbf{u} = 2\pi\gamma \sum_{i=1}^{N} \delta_{\Gamma_i}(\mathbf{x}). \tag{2.17}$$

Following [7] we define formally

$$\omega_\eta = \int_{|\mathbf{x}-\mathbf{x}'|<\eta} \frac{3\gamma}{2\eta^3} \sum_i \delta_{\Gamma_i}(\mathbf{x}')\, \mathbf{dx}, \tag{2.18}$$

and let $\eta \to 0$, $\gamma \to 0$, $N \to \infty$, and the separation of vortices $d \to 0$, with $d \ll \eta$ such that γ/d^2 is finite; then $\omega = \lim_{\eta\to 0} \omega_\eta$ is finite, and formally

$$\begin{aligned} \text{curl } \mathbf{u} = 2\pi\gamma \sum_i \delta_{\Gamma_i}(\mathbf{x}) \quad &\to \quad \text{curl } \mathbf{u} = \omega, \\ \text{div } \mathbf{u} = 0 \quad &\to \quad \text{div } \mathbf{u} = 0. \end{aligned}$$

Now individual vortices are automatically conserved, but after averaging we need to add an equation to describe vortex conservation. To derive this equation we consider all the vortices intersecting an arbitrary open surface S, and let S be convected with the same velocity field as the vortices. Then the number of vortices crossing S is constant. Thus

$$\frac{d}{dt}\left(2\pi\gamma \int_S \sum_i \delta_{\Gamma_i} \cdot \mathbf{n}\, dS\right) = 0,$$

so that on averaging

$$\frac{d}{dt}\left(\int_S \omega \cdot \mathbf{n}\, dS\right) = 0.$$

Since S was arbitrary there is an equivalent differential conservation law,

$$\frac{\partial \omega}{\partial t} + \text{curl}\,(\omega \wedge \mathbf{v}) = 0, \tag{2.19}$$

where $\mathbf{v}$ is the velocity field of the vortices. Finally, to close the model, we need to specify the velocity field $\mathbf{v}$. In the law of motion (2.12) $\mathbf{u}_{\text{reg}} \sim \gamma/d^2$, so if $1/d^2 \gg |\log \epsilon|$ the motion is dominated by the background fluid velocity, and the self-induced term is negligible. Hence $\mathbf{v} \sim \mathbf{u}_{\text{reg}} \to \mathbf{u}$ in the limit.

Thus we arrive at the Euler equations

$$\begin{aligned} \text{curl } \mathbf{u} &= \omega, \\ \text{div } \mathbf{u} &= 0, \\ \omega_t + \text{curl}(\omega \wedge \mathbf{u}) &= 0. \end{aligned}$$

FLUID	SUPERCONDUCTOR
$(\mathrm{curl})^2\boldsymbol{\Omega} = \omega$	$(\mathrm{curl})^2\mathbf{H} + \mathbf{H} = \omega$
$\mathrm{div}\ \boldsymbol{\Omega} = 0$	$\mathrm{div}\ \mathbf{H} = 0$
$\omega_t + \mathrm{curl}(\omega \wedge \mathbf{v}) = 0$	$\omega_t + \mathrm{curl}(\omega \wedge \mathbf{v}) = 0$
$\mathbf{v} = \mathrm{curl}\ \boldsymbol{\Omega}$	$\mathbf{v} = \mathrm{curl}\ \mathbf{H} \wedge \frac{\omega}{\lvert\omega\rvert}$

FIG. 4. A comparison of the vortex-density models for inviscid fluids and superconductors.

Let us now do the same for superconducting vortices. For superconductors we cannot let $\gamma \to 0$, but we can simply rescale $\mathbf{H}$ and ω with $1/d^2$. Then

$$\begin{aligned} (\mathrm{curl})^2\mathbf{H} + \mathbf{H} = 2\pi \sum_i \delta_{\Gamma_i}(\mathbf{x}) \quad &\to \quad (\mathrm{curl})^2\mathbf{H} + \mathbf{H} = \omega, \\ \mathrm{div}\ \mathbf{H} = 0 \quad &\to \quad \mathrm{div}\ \mathbf{H} = 0. \end{aligned}$$

Conservation of vortices is identical to the fluid case, so that again

$$\omega_t + \mathrm{curl}(\omega \wedge \mathbf{v}) = 0.$$

Finally, in the law of motion the self-induced term is again negligible, giving

$$\mathbf{v} = \mathbf{J} \wedge \frac{\omega}{\lvert\omega\rvert}, \qquad \mathbf{J} = \mathrm{curl}\ \mathbf{H}.$$

However, there is a proviso here due to the possibility of instability in superconducting vortices, which we will return to later.

In Fig. 4 we show a comparison of the vortex-density models for inviscid fluids and superconductors. There are two important differences to note. Firstly, the rotation of the vortex velocity in the superconducting case means that the motion is dissipative. The repulsion of like vortices leads to diffusive-type behaviour. Secondly, the decay of magnetic field away from vorticity is exponential rather than algebraic. This means that there is a lengthscale in the superconducting problem, unlike the Euler equations. This lengthscale is λ, the penetration depth. Rescaling length with the size of the domain L gives

$$\frac{\lambda^2}{L^2}(\mathrm{curl})^2\mathbf{H} + \mathbf{H} = \omega, \tag{2.20}$$

$$\mathrm{div}\,\mathbf{H} = 0, \tag{2.21}$$

$$\omega_t + \mathrm{curl}\,(\omega \wedge \mathbf{v}) = 0, \tag{2.22}$$

$$\mathbf{v} = \mathrm{curl}\,\mathbf{H} \wedge \frac{\omega}{\lvert\omega\rvert}. \tag{2.23}$$

Typically λ is very small ($\sim 1\,\mu$m). Therefore it makes sense to consider the limit $\lambda/L \to 0$. Then, except in thin boundary layers, we find that $\mathbf{H} = \omega$ and

$$\mathbf{H}_t + \mathrm{curl}\,\left(\mathbf{H} \wedge \left(\mathrm{curl}\,\mathbf{H} \wedge \hat{\mathbf{H}}\right)\right) = \mathbf{0}, \tag{2.24}$$

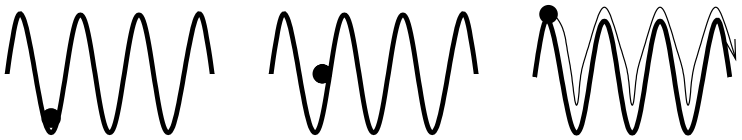

FIG. 5. The effect of pinning on the motion of vortices.

where $\hat{\mathbf{H}} = \mathbf{H}/|\mathbf{H}|$. This equation was first written down by Brandt [3]. In two dimensions with $\mathbf{H} = (0, 0, H(x, y, t))$ it becomes

$$H_t = \text{div } (|H|\nabla H)$$

showing the nonlinear diffusion more clearly.

Pinning Vortex motion dissipates energy, creates an electric field, and therefore leads to an effective resistivity. In practice attempts are made to pin vortices by introducing inhomogeneities into the material. Vortices can be pinned by anything which can destroy superconductivity locally, such as impurities, grain boundaries and dislocations.

Pinning can be modelled in the GL theory by allowing the equilibrium density of superconducing vortices to vary spatially. If the asymptotic analysis leading to the law of motion is repeated in this case, then the result is that this variation leads to a pinning potential Φ such that

$$\mathbf{v} = \text{curl}\,\mathbf{H} \wedge \hat{\omega} - \nabla\Phi$$

(see [10]). Vortices are attracted to the minima of Φ, which are the pinning sites.

Now suppose that the pinning sites are numerous and close together, i.e. that Φ varies rapidly. Then we can homogenise this potential (see Fig. 5). Under small enough forcing current the vortex will stay in its local potential well, and will only move out of this when the forcing reaches a critical value. In limit in which the separation of pinning sites tends to zero we end up with a modified "stick/slip" velocity law

$$\mathbf{v} = f(|\mathbf{J}|)\hat{\mathbf{J}} \wedge \hat{\omega}.$$

where f is as shown in Fig. 6(a) for example.

Now, if the magnetic field varies slowly in time compared to vortex motion then the graph of f may be approximated as in Fig. 6(b). In that case we have

$$\mathbf{v} = m\mathbf{J} \wedge \hat{\omega},$$

with

$$m \geq 0, \qquad |\mathbf{J}| \leq J_c, \qquad |\mathbf{J}| < J_c \Rightarrow m = 0.$$

Such models are known as critical state models.

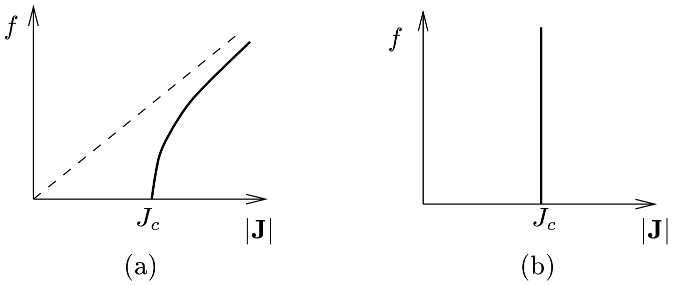

FIG. 6. Speed of a superconducting vortex as a function of the local current in the presence of homogenised pinning.

In two dimensions with $\mathbf{H} = (0, 0, H(x, y, t))$ the model simplifies considerably to

$$-\lambda^2 \nabla^2 H + H = \omega, \tag{2.25}$$

$$\omega_t = \operatorname{div}(m \nabla H), \tag{2.26}$$

$$m \geq 0, \qquad |\nabla H| \leq J_c, \qquad m(|\nabla H| - J_c) = 0. \tag{2.27}$$

With $\lambda \to 0$ we have $H = \omega$ and

$$H_t = \operatorname{div}(m \nabla H), \tag{2.28}$$

$$m \geq 0, \qquad |\nabla H| \leq J_c, \qquad m(|\nabla H| - J_c) = 0, \tag{2.29}$$

which is known as the Bean model [2]. Both (2.25)-(2.27) and (2.28)-(2.29) can be formulated as variational inequalities [21].

3 Conclusion

The hierarchy of models we have presented is summarised in Fig. 7. The presentation here has had to be brief, and there are many modelling difficulties which have not been addressed (see [8] for a fuller exposition). However, two important problems are worthy of note.

1. Boundary conditions For finite λ the system (2.20)-(2.23) is second-order elliptic coupled with first-order hyperbolic, i.e., we have two imaginary and one real characteristic. The usual Maxwell conditions on $\mathbf{H}$ on the boundary of the superconductor take care of the elliptic part. However, we need a boundary condition on the hyperbolic part if and only if the characteristic points inwards, i.e. if $\mathbf{v} \cdot \mathbf{n} < 0$. Hence the region over which the boundary condition is applied depends on the solution itself.

The extra boundary condition should determine the flux of vorticity through the boundary. Asymptotic analysis suggests that there should be no nucleation of vortices until $|\mathbf{J}|$ reaches a critical value [6]. Above this critical value it is an open question as to the correct form of the boundary condition, although

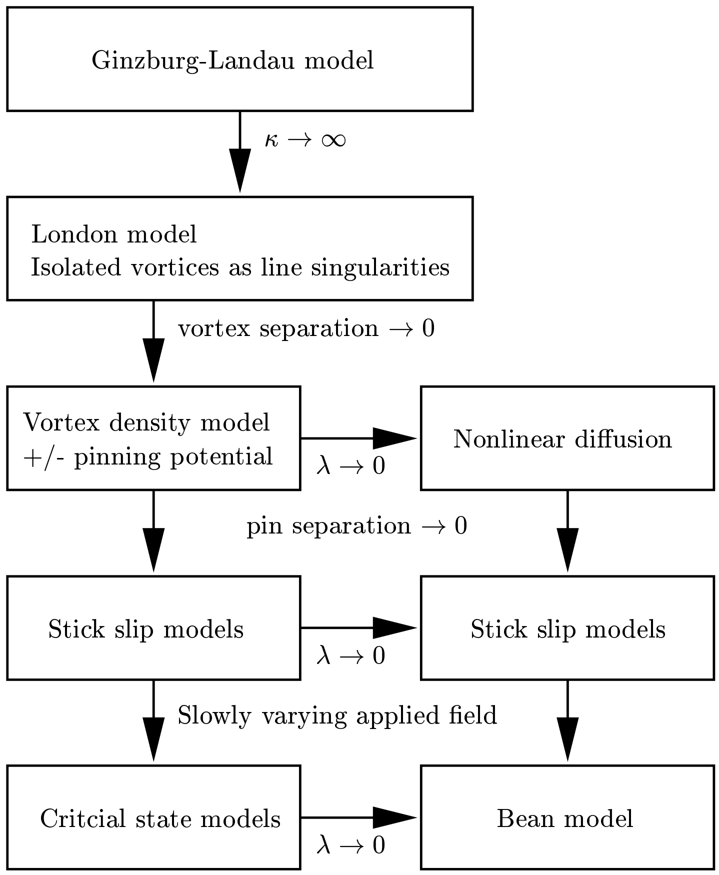

FIG. 7. The hierarchy of models for type II superconductors.

the model is probably not particularly sensitive to the particular form chosen, providing the flux increases with increasing current.

Once we let $\lambda \to 0$ the problem disappears, since we now have a diffusion equation, and it is sufficient to give $\mathbf{H}$ on the boundary. However, we might have expected the boundary condition on $\mathbf{H}$ to be the one we lose, since this is the singularly perturbed equation.

2. Instability/ill-posedness If $\mathbf{J} \cdot \mathbf{t} \neq 0$ then a single vortex is prone to a helical instability and the law $\mathbf{v} = \mathbf{J} \wedge \mathbf{t}$ is ill-posed without the regularising curvature term [22]. This is reflected in the vortex density model, which also seems to be ill-posed [23]. The nonlinear diffusion model and Bean model exhibit the same instability, so that all of our models are justified only if symmetry implies that $\omega \cdot \mathbf{J} \equiv 0$ (which includes the two-dimensional situations of a cylinder in an axial or transverse magnetic field, and three-dimensions with axial symme-

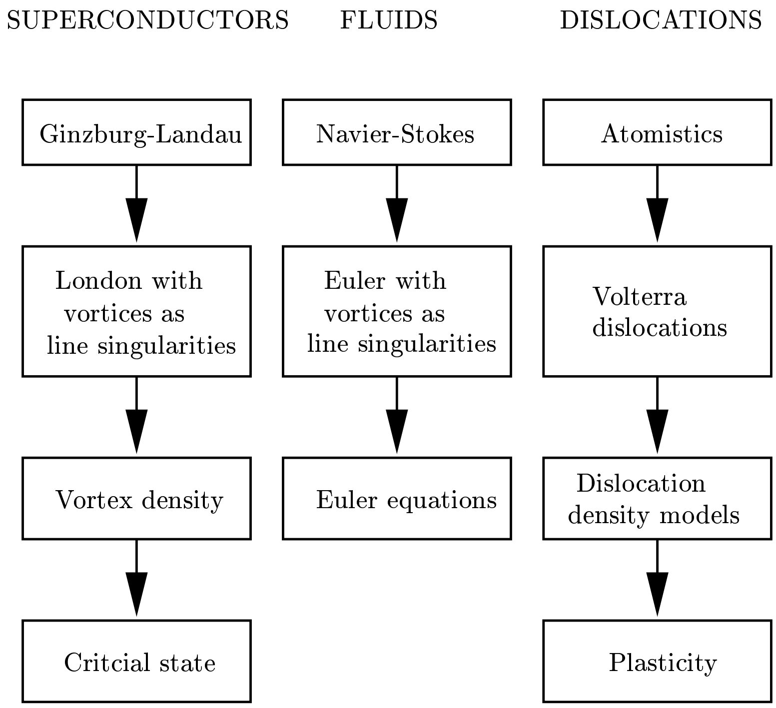

FIG. 8. Illustration of the analogies between the various models for superconducting vortices, vortices in inviscid fluids, and dislocations in elastic crystals.

try). As yet there is no generally accepted three-dimensional version of the Bean model.

Of course, the law of motion can be regularised by retaining the curvature term, but this involved bringing a small lengthscale into the problem. Alternatively attempts have been made to build the effect of the instability on the small scale into the large scale model [8; 11].

Finally we note that another system in which analogous line singularities occur is that of Volterra dislocations in elastic crystals. The corresponding hierarchy of models is illustrated in Fig. 8, although the connections between the various models in the hierarchy are not so well determined as they are for fluids and superconductors (see, e.g. [5]).

Bibliography

1. A.A. Abrikosov, On the magnetic properties of superconductors of the second group, *Soviet Phys. JETP*, **5** (1957) 1174–1182.
2. C.P. Bean, *Phys. Rev. Lett.*, **9** (1962) 309.
3. E.H. Brandt, Flux diffustion in high-T_c superconductors, *Z. Phys. B*, **80**

(1990) 167–175.

4. A.J. Callegari & L. Ting, Motion of a curved vortex filament with decaying vortical core and axial velocity, *SIAM J. Appl. Math.*, **35** (1978) 148–175.
5. A. Carpio, S.J. Chapman, S.D. Howison & J.R. Ockendon, Dynamics of line singularities, *Phil. Trans. R. Soc. Lond. A*, **355** (1997) 2013–2024.
6. S.J. Chapman, Superheating field of type II superconductors, *SIAM J. Appl. Math.*, **55** (1995) 1233–1258.
7. S.J. Chapman, A mean-field model of superconducting vortices in three dimensions, *SIAM J. Appl. Math.*, **55** (1995) 1259–1274.
8. S.J. Chapman, A hierarchy of models of type II superconductors, to appear *SIAM Review* (2000).
9. S.J. Chapman & G.W. Richardson, Motion of vortices in type II superconductors, *SIAM J. Appl. Math.*, **55** (1995) 1275–1296.
10. S.J. Chapman & G.W. Richardson, Vortex pinning by inhomogeneities in type II superconductors, *Physica D*, **108** (1997) 397–407.
11. J.R. Clem & A. Perez-Gonzales, Theory of the double critical state in type II superconductors, *Proc of L-T 17*, Editors U. Eckern, A. Schmid, W. Weber & H. Wühl (1984).
12. A.T. Dorsey, Vortex motion and the Hall effect in type II superconductors, *Phys. Rev. B*, **46** (1992) 8376–8392.
13. W. E, Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity, *Physica D*, **77** (1994) 383–404.
14. H. Kamerlingh-Onnes, *Leiden Comm.* **139F** (1914).
15. R. Klein & L. Ting, Theoretical and experimental studies of slender vortex filaments, *Appl. Math. Lett.*, **8** (1995) 45–50.
16. F. London, *Superfluids*, Dover (1961) .
17. F. London & H. London, The electromagnetic equations of the supraconductor, *Proc. Roy. Soc. Lond. A*, **14** (1935) 71.
18. W. Meissner & R. Ochsenfeld, *Naturwissenschaffen*, **21** (1933) 787.
19. D.W. Moore & P.G. Saffman, The motion of a vortex filament with axial flow, *Phil. Trans. R. Soc. Lond. A*, **272** (1972) 403–429.
20. L. Peres & J. Rubinstein, Vortex dynamics in $U(1)$ Ginzburg-Landau models, *Physica D*, **64** (1993) 299–309.
21. L. Prigozhin, The Bean model in superconductivity: Variational formulation and numerical solution, *J. Comp. Phys.*, **129** (1996) 190–200.
22. G.W. Richardson, Instability of a superconducting vortex line, *Physica D*, **110** (1997) 139–153.
23. G.W. Richardson & B. Stoth, Ill-posedness of the mean-field model of superconducting vortices and the regularisation thereof, *Europ. J. Appl. Math.* , to appear (1999).