ON THE LAWRENCE-DONIACH AND ANISOTROPIC GINZBURG-LANDAU MODELS FOR LAYERED SUPERCONDUCTORS

S. JONATHAN CHAPMAN†, QIANG DU‡, AND MAX D. GUNZBURGER§

Abstract. We consider two models, the anisotropic Ginzburg-Landau and the Lawrence-Doniach models, for layered superconductors such as the recently discovered high-temperature superconductors. We give a mathematical description of both models and derive existence results for their solution. We then relate the two models in the sense that we show that as the layer spacing tends to zero, the Lawrence-Doniach model reduces to the anisotropic Ginzburg-Landau model. Finally, we derive simplified versions of the models that can be used to accurately simulate high-temperature superconductors.

1. Introduction

Due to the recent discovery of high-T_c superconductors, i.e., materials that retain superconducting properties at relative high temperatures, there has been a tremendous resurgence of interest in superconductivity among the physics, material science, engineering, and mathematics communities. One of the features of high-T_c superconductors is their layered structure, comprising alternating layers of superconducting and non- (or weakly) superconducting materials. In planes parallel to the layers, the material is isotropic. However, there is a strong anisotropy present when one compares material properties parallel and perpendicular to the layers. One may consult [10] for a recent survey providing a lucid discussion of layered superconductors. (It should be noted that some of the low-T_c superconductors also possess a layered structure. One fortunate result of this is that scientists have been studying layered superconductors for a period of time substantially longer than that that has transpired since the discovery of high-T_c superconductors.)

A model proposed by Ginzburg and Landau [9] has become generally accepted as a macroscopic model for superconductivity in isotropic (and homogeneous) superconductors, e.g., in atomic metals. However, this model cannot account for the anisotropy of layered superconductors. In its place, alternative models have been proposed. One of these is the *anisotropic Ginzburg-Landau model* which is also referred to as the *effective mass model*. In this model, the effects of the microscopic layered structure are averaged out so that the anisotropic nature of the material appears only in the form of a mass tensor with unequal principal values. The model itself is only a slight variant of the Ginzburg-Landau model. Another model for layered superconductors is the *Lawerence-Doniach model* introduced in [14]; see also [4] and [12]. In this model, the material is treated as a stack of superconducting planes, each pair of which is separated by normal (nonsuperconducting) material. Furthermore, in this model, the coupling between the superconducting planes is similar to that that occurs in a Josephson junction. Again, one may consult [10] and the references cited therein for

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†Mathematical Institute, 24-29 St, Giles', University of Oxford, Oxford, OX1 3LB, UK.

‡Department of Mathematics, Michigan State University, East Lansing, MI 48224, USA.

§Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA. Supported in part by the Air Force Office of Scientific Research under grant number AFOSR-93-1-0061 and in part by the Los Alamos National Laboratory of the Department of Energy.

a complete discussion of these models and the physical circumstances necesssary for their validity. (Other models have also been proposed, e.g., the anisotropic London model, which we do not consider here.)

Our goal here is to examine some mathematical properties of the two models for layered superconductors and in particular to rigorously establish the connection between the two models as the spacing between the layers tends to zero. We also examine some simplications that can be effected in the models in the case of high-T_c superconductors. These simplifications are of importance since they can result, for example, in substantial savings in the cost of computer simulations of physical phenomena.

In the remainder of this section, we introduce some notation that will be used in the sequel. In §2 we briefly consider the isotropic Ginzburg-Landau (GL) model, to establish some terminology which will be used later. In §3 we consider the anisotropic Ginzburg-Landau, or effective mass (EM), model and provide some results concerning the model. We do likewise for the Lawrence-Doniach (LD) model in §4. In §5 and §6 we respectively make the rigorous connection between the two models and discuss the simplications that may be effected in the high-T_c case, both of which were alluded to above.

1.1. Notation

Throughout, we will denote three-vectors by $(\vec{\cdot})$ and two-vectors by bold face notation. Thus, A, $\mathbf{A}$, and $\vec{A}$ denote a scalar, a two-vector, and a three-vector, respectively. We will often have occasion to partition a three-vector $\vec{A}$ into the form

$$\vec{A} = \begin{pmatrix} \mathbf{A} \\ A_z \end{pmatrix}$$

so that here A_z denotes the third component of $\vec{A}$. The same notational convention will be used for operators. For example, we will denote the gradient operator with respect to the x and y coordinates by **grad** while the three-component gradient will be denoted by $\vec{\text{grad}}$.

Throughout, for any non-negative integer k and domain $\mathcal{D} \subset \mathbb{R}^3$, $H^k(\mathcal{D})$ will denote the Sobolev space of real-valued functions having square integrable derivatives of order up to k. The corresponding spaces of complex-valued functions will be denoted by $\mathcal{H}^k(\mathcal{D})$. Corresponding spaces of three-vector-valued functions, each of whose three components belong to $H^k(\mathcal{D})$, will be denoted by $\vec{H}^k(\mathcal{D})$, i.e., $\vec{H}^k(\mathcal{D}) = [H^k(\mathcal{D})]^3$. Norms of functions belonging to $H^k(\mathcal{D})$, $\vec{H}^k(\mathcal{D})$, and $\mathcal{H}^k(\mathcal{D})$ will all be denoted, without any possible ambiguity, by $\|\cdot\|_k$. We use $\vec{H}(\mathbb{R}^3)$ to denote the completion of $\vec{C}_0^\infty(\mathbb{R}^3)$ under the norm

$$\|\vec{Q}\|_*^2 = \int_{\mathbb{R}^3} |\vec{\text{grad}}\vec{Q}|^2 \, d\vec{x}\,.$$

For details concerning these spaces, one may consult [2].

When $\mathcal{D}$ corresponds to a layered material sample, we will assume that the layers are perpendicular to the z axis and that $\mathcal{D} = \Omega \times [0, S]$, where $\Omega \subset \mathbb{R}^2$ is a planar domain and S is the z-thickness of the material sample. The boundary of Ω will be denoted by Γ. The region exterior to $\mathcal{D}$ will be denoted by $\mathcal{D}_e$, i.e., $\mathcal{D}_e = \mathbb{R}^3/\mathcal{D}$. The interface between $\mathcal{D}$ and $\mathcal{D}_e$ will be denoted by $\partial\mathcal{D}$.

We will make use of the subspaces

$$\vec{H}^1_n(\mathbb{R}^3) = \Big\{ \vec{Q} = \begin{pmatrix} \mathbf{Q} \\ Q_z \end{pmatrix} \in \vec{H}(\mathbb{R}^3) \ : \ \vec{Q}|_{\mathcal{D}} \in \vec{H}^1(\mathcal{D}) \,, \\ \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma\times[0,S]\,, \quad Q_z = 0 \text{ on } (\Omega\times\{0\})\cup(\Omega\times\{S\}) \Big\}$$

and

$$\vec{H}^1_n(\mathrm{div}\,;\mathbb{R}^3) = \{ \vec{Q} \in \vec{H}^1_n(\mathbb{R}^3) \ : \ \mathrm{div}\,\vec{Q} = 0 \text{ in } \mathbb{R}^3 \} \,.$$

2. The isotropic Ginzburg-Landau model

For isotropic and homogeneous superconducting materials, Ginzburg and Landau postulated [9] that the Gibbs free energy is given by

$$\begin{aligned} \mathcal{G}_{GL}(\psi,\vec{A}) = &\int_{\mathcal{D}} \left(\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m_s}\left| \left(i\hbar \vec{\mathrm{grad}} + \frac{e_s}{c}\vec{A}\right)\psi\right|^2 \right) d\vec{x} \\ &+ \frac{1}{8\pi}\int_{\mathcal{D}} |\vec{h}-\vec{H}|^2\,d\vec{x} + \frac{1}{8\pi}\int_{\mathcal{D}_e} |\vec{h}-\vec{H}|^2\,d\vec{x}\,. \end{aligned} \tag{2.1}$$

Here, ψ is the (complex-valued) order parameter, $\vec{A}$ is the magnetic potential, $\vec{h} = \vec{\mathrm{curl}}\,\vec{A}$ is the magnetic field, α and β are constants (with respect to the space variable $\vec{x}$) whose values depend on the temperature, c is the speed of light, e_s and m_s are the charge and mass, respectively, of the superconducting charge-carriers, $2\pi\hbar$ is Planck's constant, and $\vec{H}$ is the applied field, which throughout this paper is assumed to be a constant vector. (Actually, $\mathcal{G}_{GL}$ is the difference between the free energy and the free energy due to the applied field $\vec{H}$ in the absence of the superconductor. It is important to employ this energy difference since it is a finite quantity even on unbounded domains.) The basic postulate of the Ginzburg-Landau theory of superconductivity is that the Gibbs free energy (2.1) is minimized.

The coefficient α changes sign at the critical temperature T_c, with $\alpha < 0$ for $T < T_c$. If the temperature of the sample is lower than T_c, the sample is in the superconducting state; if $T > T_c$, then the sample is in the normal, i.e., nonsuperconducting, state.

There are two important length scales associated with changes in the order parameter and the magnetic field. These are the *coherence length*

$$\xi = \left(-\frac{\hbar^2}{2m_s\alpha} \right)^{1/2} \tag{2.2}$$

and the *penetration depth*

$$\lambda = \left(-\frac{\beta m_s c^2}{4\pi\alpha e_s^2} \right)^{1/2} \tag{2.3}$$

which measure distances over which the order parameter and the magnetic field, respectively, undergo appreciable change. The nondimensional ratio $\kappa = \lambda/\xi$ is known as the *Ginzburg-Landau parameter*. One should especially note the dependence of ξ and λ on m_s.

One may consult, e.g., [1], [6], [7], [8], [13], [15], [16], or [17], for details about the material of this section.

3. The anisotropic Ginzburg-Landau, or effective mass, model

An anisotropic form of the functional (2.1) may be obtained by replacing the scalar mass m_s by a mass tensor M_s so that the anisotropic Gibbs free energy difference is now given by

$$\begin{aligned}\mathcal{G}_{EM}(\psi,\mathbf{A}) = &\int_{\mathcal{D}} \left(\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{8\pi}|\vec{h}-\vec{H}|^2\right) d\vec{x} + \frac{1}{8\pi}\int_{\mathcal{D}_e} |\vec{h}-\vec{H}|^2\, d\vec{x} \\ &+ \frac{1}{2}\int_{\mathcal{D}} \left(-i\hbar \vec{\mathrm{grad}} + \frac{e_s}{c}\vec{A}\right)\psi^* \cdot M_s^{-1} \cdot \left(i\hbar \vec{\mathrm{grad}} + \frac{e_s}{c}\vec{A}\right)\psi\, d\vec{x}\,. \end{aligned} \tag{3.1}$$

We assume that the coordinate axes are aligned with the principal directions of M_s so that

$$M_s = \begin{pmatrix} m_x & 0 & 0 \\ 0 & m_y & 0 \\ 0 & 0 & m_z \end{pmatrix}. \tag{3.2}$$

Furthermore, we assume that planes parallel to the z-axis are isotropic so that $m_x = m_y = m_\| \neq m_z = m_\perp$. (For layered high-temperature superconductors, $m_\perp > m_\|$, in some cases by a large margin.) As a result, we may define two coherence lengths and two penetration depths, i.e.,

$$\xi_\| = \left(-\frac{\hbar^2}{2m_\|\alpha}\right)^{1/2} \quad \text{and} \quad \xi_\perp = \left(-\frac{\hbar^2}{2m_\perp\alpha}\right)^{1/2}$$

and

$$\lambda_\| = \left(-\frac{\beta m_\| c^2}{4\pi\alpha e_s^2}\right)^{1/2} \quad \text{and} \quad \lambda_\perp = \left(-\frac{\beta m_\perp c^2}{4\pi\alpha e_s^2}\right)^{1/2}.$$

For the layered superconductors described in §1, the anisotropic Gibbs free energy (3.1) is valid if the z-axis coherence length $\xi_\perp$ is large compared to the layer spacing. Although, for high temperature superconductors, this is usually true only near the critical temperature T_c, it turns out that many (but not all) properties of layered superconductors can be reasonably simulated using (3.1) and (3.2). Again, see [10] and the references cited therein.

We now introduce the usual nondimensionalizations. Lengths are nondimensionalized by $\lambda_\|$, magnetic fields by $\sqrt{2}H_c$, where $H_c = \sqrt{4\pi\alpha^2/\beta}$, the magnetic potential by $\sqrt{2}\lambda_\| H_c$, the order parameter by $\sqrt{-\alpha/\beta}$, free energy densities by α^2/β, and the Gibbs free energy by $\alpha^2\lambda_\|^3/\beta$. Denoting the nondimensionalized variables by the same symbols used for the corresponding dimensional ones, we have that the nondimensional Gibbs free energy is given by

$$\begin{aligned}\mathcal{G}_{EM}(\psi,\vec{A}) = &\int_0^S\int_\Omega \left(\frac{1}{2}\left(|\psi|^2-1\right)^2 + |\vec{\mathrm{curl}}\,\vec{A}-\vec{H}|^2\right) d\mathbf{x}dz \\ &+ \int_0^S\int_\Omega \left(\left|\left(\frac{i}{\kappa}\mathbf{grad} + \mathbf{A}\right)\psi\right|^2 + \rho\left|\left(\frac{i}{\kappa}\frac{\partial}{\partial z} + A_z\right)\psi\right|^2\right) d\mathbf{x}dz \\ &+ \int_{\mathcal{D}_e} |\vec{\mathrm{curl}}\,\vec{A}-\vec{H}|^2\, d\vec{x}\,, \end{aligned} \tag{3.3}$$

where

$$\kappa = \frac{\lambda_\|}{\xi_\|} \quad \text{and} \quad \rho = \frac{m_\|}{m_\perp} = \left(\frac{\lambda_\|}{\lambda_\perp}\right)^2 = \left(\frac{\xi_\perp}{\xi_\|}\right)^2 .$$

Note that in (3.3) we have made use of the notation introduced in §1.1 and that we have added an unimportant constant to the free energy in order to write it as a sum of squares.

Many of the results of, e.g., [6] and [8], concerning the isotropic Ginzburg-Landau model hold for the anisotropic model based on the minimization of (3.3). For example, one may obtain the following result using the same techniques as those employed in [8].

THEOREM 3.1. *$\mathcal{G}_{EM}$ has a least one minimizer belonging to $\mathcal{H}^1(\Omega\times[0,S]) \times \vec{H}(\mathbb{R}^3)$. Moreover,*

$$\min_{\mathcal{H}^1(\Omega\times[0,S])\times\vec{H}(\mathbb{R}^3)} \mathcal{G}_{EM} = \min_{\mathcal{H}^1(\Omega\times[0,S])\times\vec{H}^1_n(\mathrm{div}\,;\mathbb{R}^3)} \mathcal{G}_{EM} . \quad \square$$

One may also easily deduce the *anisotropic Ginzburg-Landau equations* that result from the minimization of the functional (3.3). These are given by

$$\left(\frac{i}{\kappa}\mathbf{grad} + \mathbf{A}\right)^2 \psi + \rho\left(\frac{i}{\kappa}\frac{\partial}{\partial z} + A_z\right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \mathcal{D} = \Omega\times[0,S] ,$$

$$\vec{\mathrm{curl}}\,\vec{\mathrm{curl}}\,\vec{A} = -\begin{pmatrix} \frac{i}{2\kappa}(\psi^*\mathbf{grad}\,\psi - \psi\mathbf{grad}\,\psi^*) + |\psi|^2\mathbf{A} \\ \rho\left[\frac{i}{2\kappa}\left(\psi^*\frac{\partial\psi}{\partial z} - \psi\frac{\partial\psi^*}{\partial z}\right) + |\psi|^2 A_z\right] \end{pmatrix} \quad \text{in } \mathcal{D} = \Omega\times[0,S] ,$$

and

$$\vec{\mathrm{curl}}\,\vec{\mathrm{curl}}\,\vec{A} = 0 \quad \text{in } \mathcal{D}_e$$

along with the boundary conditions

$$(\frac{i}{\kappa}\mathbf{grad}\,\psi + \mathbf{A}\psi)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\times[0,S] ,$$

$$(\frac{i}{\kappa}\frac{\partial}{\partial z} + A_z)\psi = 0 \quad \text{on } (\Omega\times\{0\})\cup(\Omega\times\{S\}) ,$$

$$[\vec{A}\times\vec{n}] = \mathbf{0} \quad \text{on } (\Gamma\times[0,S])\cup(\Omega\times\{0\})\cup(\Omega\times\{S\}) ,$$

$$[\vec{\mathrm{curl}}\,\vec{A}\times\vec{n}] = \mathbf{0} \quad \text{on } (\Gamma\times[0,S])\cup(\Omega\times\{0\})\cup(\Omega\times\{S\}) ,$$

and

$$\vec{\mathrm{curl}}\,\vec{A} \to \vec{H} \quad \text{as } |\vec{x}|\to\infty .$$

Here, $[\cdot]$ denotes the jump in the argument across the interface.

4. The Lawrence-Doniach model for layered superconductors

If $\xi_\perp$ is less than the layer spacing, then it is necessary to account, in some way, for the discrete nature of the layered structure. In [14], Lawrence and Doniach proposed such a model. The layered superconductor is modeled as a stack of infinitely thin, planar superconducting layers separated by normal material. Moreover, there is a Josephson coupling between the superconducting layers.

The layers are again assumed perpendicular to the z-axis, and the z-thickness of material sample $\Omega \times [0, S]$ is again denoted by S. There are $N + 1$ superconducting planes, each having projection Ω on the (x, y)-plane, and each separated from its neighbors by a distance s; thus, $Ns = S$. We again adopt the notation of §1.1 with regards to scalars and two- and three-vectors.

4.1. The Lawrence-Doniach functional

The Lawrence-Doniach functional, which in the form below was first given in [12], is defined by

$$
\begin{aligned}
\mathcal{G}_{LD}(\psi_0, \psi_1, \dots, \psi_N, \vec{A}) &= \frac{1}{8\pi} \int_{\mathcal{D}_e} |\vec{h} - \vec{H}|^2 \, d\vec{x} \\
&+ s \sum_{n=0}^{N} \int_\Omega \left(\alpha |\psi_n|^2 + \frac{\beta}{2} |\psi_n|^4 + \frac{1}{2m} \left| (i\hbar \, \mathbf{grad} + \frac{e_s \mathbf{A_n}}{c}) \psi_n \right|^2 \right) d\mathbf{x} \\
&+ s \sum_{n=0}^{N-1} \int_\Omega \zeta \left| \psi_{n+1} \exp\left(-\frac{i e_s}{\hbar c} \int_{ns}^{(n+1)s} A_z \, dz \right) - \psi_n \right|^2 d\mathbf{x} \\
&+ \frac{1}{8\pi} \sum_{n=0}^{N-1} \int_\Omega \int_{ns}^{(n+1)s} |\vec{\mathrm{curl}}\, \vec{A} - \vec{H}|^2 \, dz \, d\mathbf{x} \, .
\end{aligned} \tag{4.1}
$$

Here, $\vec{A} = \vec{A}(x, y, z) = (\mathbf{A}, A_z)^T$ denotes the three-dimensional vector potential, A_z is its component in z direction, and $\mathbf{A_n} = \mathbf{A_n}(x, y) = \mathbf{A}(x, y, ns)$, $n = 0, 1, \dots, N$, is the restriction of $\mathbf{A}$ to the n-th superconducting plane.

The order parameter $\psi_n = \psi_n(x, y)$ is only defined on the n-th plane. However, it will be convenient to also view the order parameter ψ_n as the restriction, to the n-th plane, of a continuous function of z. Specifically, let $\mathcal{T}_s$ denote the uniform partition of the interval $[0, S]$ into N subintervals of length s. We then let $\psi^s(x, y, z)$ denote a piecewise linear function of z, with respect to $\mathcal{T}_s$, such that $\psi^s(x, y, ns) = \psi_n(x, y)$. We want to again emphasize that the extension of the order parameter to the layers between the superconducting planes is a matter of convenience and that, strictly speaking, the order parameter makes physical sense only on the superconducting planes.

The parameter ζ appearing in (4.1) will be specified later; note that ζ has the same dimension as α. We naturally associate the mass m in (4.1) with $m_\parallel$ of §3.

Using similar nondimensionalizations as were used in §3, we arrive at the following nondimensional form of the Lawrence-Doniach free energy:

$$
\begin{aligned}
\mathcal{G}_{LD}(\psi^s, \vec{A}) &= s \sum_{n=0}^{N} \int_\Omega \left(\frac{1}{2} (|\psi_n|^2 - 1)^2 + \left| (\frac{i}{\kappa} \mathbf{grad} + \mathbf{A_n}) \psi_n \right|^2 \right) d\mathbf{x} \\
&+ s \sum_{n=0}^{N-1} \int_\Omega \sigma \left| \psi_{n+1} \exp\left(-i\kappa \int_{ns}^{(n+1)s} A_z \, dz \right) - \psi_n \right|^2 d\mathbf{x} \\
&+ \sum_{n=0}^{N-1} \int_\Omega \int_{ns}^{(n+1)s} |\vec{\mathrm{curl}}\, \vec{A} - \vec{H}|^2 \, dz \, d\mathbf{x} + \int_{\mathcal{D}_e} |\vec{\mathrm{curl}}\, \vec{A} - \vec{H}|^2 \, d\vec{x} \, ,
\end{aligned} \tag{4.2}
$$

where $\sigma = -\zeta / \alpha$. We have again added an unimportant constant to the free energy in order to write it as a sum of squares.

After defining the space

$$\mathcal{H}^1_s(\Omega\times[0,S]) = \{\ \psi^s \in\mathcal{H}^1(\Omega\times[0,S])\ :\ \psi^s \text{ is a piecewise linear function of } z \text{ with respect to } \mathcal{T}_s\ \},$$

we have that ψ^s and $\vec{A}$ are determined as solutions of the minimization problem

$$\min_{(\psi^s,\vec{A})\in\mathcal{H}^1_s(\Omega\times[0,S])\times\vec{H}(\mathbb{R}^3)} \mathcal{G}_{LD}(\psi^s,\vec{A})\,.$$

Note that if $\vec{A}\in\vec{H}(\mathbb{R}^3)$, then, by the trace theorem, $\mathbf{A}_n\in\mathbf{H}^{1/2}(\Omega)$. Also, $\psi^s\in\mathcal{H}^1_s(\Omega\times[0,S])$ implies $\psi_n\in\mathcal{H}^1(\Omega)$. Therefore, it follows from standard Sobolev imbedding results that the functional $\mathcal{G}_{LD}$ is well defined on $\mathcal{H}^1_s(\Omega\times[0,S])\times\vec{H}(\mathbb{R}^3)$. See, e.g., [2], for details.

4.2. Gauge invariance and the existence of minimizers

As is the case for the functionals (2.1) and (3.1), the Lawrence-Doniach functional (4.2) is invariant under a certain gauge transformation. For any smooth function g, let the linear transformation G_g be defined by

$$G_g(\psi^s,\vec{A}) = (\zeta^s,\vec{Q})$$

if

$$\zeta_n(x,y) = \psi_n(x,y)\exp\big(i\kappa g(x,y,ns)\big) \quad\text{and}\quad \vec{Q} = \vec{A} + \begin{pmatrix}\mathbf{grad}\, g\\ \frac{\partial g}{\partial z}\end{pmatrix}.$$

Note that if $(\zeta^s,\vec{Q}) = G_g(\psi^s,\vec{A})$, then $(\psi^s,\vec{A}) = G_{-g}(\zeta^s,\vec{Q})$.

Definition. *$(\psi^s,\vec{A})$ and $(\zeta^s,\vec{Q})$ are said to be* gauge equivalent *if and only if there exists a $g\in H^2(\mathbb{R}^3)$ such that $(\zeta^s,\vec{Q}) = G_g(\psi^s,\vec{A})$.*

Again, as is the case for the previous models, a possible choice of gauge for the Lawrence-Doniach model is given by $\operatorname{div}\vec{A}=0$ in $\mathbb{R}^3$ and $\vec{A}\cdot\vec{n}=0$ on the boundary $\partial\mathcal{D}$ of $\mathcal{D}=\Omega\times[0,S]$. We then have the following result.

LEMMA 4.1. *Any $(\zeta^s,\vec{Q})\in\mathcal{H}^1_s(\Omega\times[0,S])\times\vec{H}(\mathbb{R}^3)$ is gauge equivalent to an element of $\mathcal{H}^1_s(\Omega\times[0,S])\times\vec{H}^1_n(\operatorname{div};\mathbb{R}^3)$.* □

The next result follows from the obvious relation

$$\mathcal{G}_{LD}(\psi^s,\vec{A}) = \mathcal{G}_{LD}(G_g(\psi^s,\vec{A}))\,.$$

PROPOSITION 4.2. *For any $g\in H^2(\mathbb{R}^3)$, $\mathcal{G}_{LD}$ is invariant under the gauge transformation G_g.* □

Using a standard variational argument, we then may obtain the following existence result.

LEMMA 4.3. *The functional $\mathcal{G}_{LD}$ defined in (4.2) has a least one minimizer belonging to $\mathcal{H}^1_s(\Omega\times[0,S])\times\vec{H}^1_n(\operatorname{div};\mathbb{R}^3)$.* □

This lemma, along with Lemma 4.1 and Proposition 4.2, yield the following result.

THEOREM 4.4. *The functional $\mathcal{G}_{LD}$ defined in (4.2) has a least one minimizer belonging to $\mathcal{H}^1_s(\Omega\times[0,S])\times\vec{H}(\mathbb{R}^3)$. Moreover,*

$$\min_{\mathcal{H}^1_s(\Omega\times[0,S])\times\vec{H}(\mathbb{R}^3)} \mathcal{G}_{LD} = \min_{\mathcal{H}^1_s(\Omega\times[0,S])\times\vec{H}^1_n(\operatorname{div};\mathbb{R}^3)} \mathcal{G}_{LD}\,. \quad \square$$

Thus, we have that the Lawrence-Doniach functional $\mathcal{G}_{LD}$ defined in (4.2) has a minimizer in $\mathcal{H}^1_s(\Omega \times [0, S]) \times \vec{H}(\mathbb{R}^3)$ and that we may locate such minimizers by finding minimizers in the constrained space $\mathcal{H}^1_s(\Omega \times [0, S]) \times \vec{H}^1_n(\mathrm{div}\,; \mathbb{R}^3)$.

4.3. The Lawrence-Doniach equations

Using standard techniques from the calculus of variations, we may deduce the Euler-Lagrange equations corresponding to the minimization of the Lawrence-Doniach functional (4.2). The result is the *Lawrence-Doniach equations* which are given by

$$
\begin{aligned}
&-\psi_n + |\psi_n|^2\psi_n + (\frac{i}{\kappa}\mathbf{grad} + \mathbf{A_n})^2\psi_n \\
&\quad + \sigma\Big[2\psi_n - \psi_{n+1}\exp(-i\phi_n^{n+1}) - \psi_{n-1}\exp(i\phi_{n-1}^{n})\Big] = 0 \\
&\qquad\qquad \text{in } \Omega \text{ and for } n = 0, 1, \ldots, N\,,
\end{aligned}
\tag{4.3}
$$

where

$$
\phi_n^{n+1} = \kappa \int_{ns}^{(n+1)s} A_z\, dz \quad \text{for } n = 0, \ldots, N-1\,,
\tag{4.4}
$$

and

$$
\vec{\mathrm{curl}}\,\vec{\mathrm{curl}}\,\vec{A} = \begin{pmatrix} \mathbf{j} \\ j_z \end{pmatrix} \quad \text{in } \mathbb{R}^3\,,
\tag{4.5}
$$

where

$$
\mathbf{j} = s\sum_{n=0}^{N}\left(\frac{i}{\kappa}(\psi_n \mathbf{grad}\,\psi_n{}^* - \psi_n{}^*\mathbf{grad}\,\psi_n) - |\psi_n|^2\mathbf{A_n}\right)\delta(z - ns) \quad \text{in } \mathbb{R}^3\,,
\tag{4.6}
$$

$$
\begin{aligned}
j_z &= -i\kappa\sigma s\Big[\psi_n\exp(i\phi_n^{n+1})\psi^*_{n+1} - \psi^*_n\exp(-i\phi_n^{n+1})\psi_{n+1}\Big] \\
&\qquad \text{in } \Omega \times [ns, (n+1)s],\ n = 0, \ldots, N-1\,,
\end{aligned}
\tag{4.7}
$$

and

$$
j_z = 0 \quad \text{in } \mathcal{D}_e = \mathbb{R}^3/\mathcal{D}\,.
\tag{4.8}
$$

One also obtains the boundary conditions

$$
(\frac{i}{\kappa}\mathbf{grad}\,\psi_n + \mathbf{A_n}\psi_n)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma \text{ and for } n = 0, 1, \ldots, N\,,
\tag{4.9}
$$

$$
[\vec{A} \times \vec{n}] = \mathbf{0} \quad \text{and} \quad [\vec{\mathrm{curl}}\,\vec{A} \times \vec{n}] = \mathbf{0} \quad \text{on the boundary } \partial\mathcal{D} \text{ of } \Omega \times [0, S]\,,
\tag{4.10}
$$

and

$$
\vec{\mathrm{curl}}\,\vec{A} \to \vec{H} \quad \text{as } |\vec{x}| \to \infty\,.
\tag{4.11}
$$

If we choose the London gauge, then also

$$\operatorname{div} \vec{A} = 0 \quad \text{in } \mathbb{R}^3 ,$$

$$\mathbf{A} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times [0, S] , \quad \text{and} \quad A_z = 0 \quad \text{on } (\Omega \times \{0\}) \cup (\Omega \times \{1\}) .$$

The Lawrence-Doniach equations (4.3)-(4.11) where first given in [4]. In [14], (4.3) was given but the magnetic potential was assumed to be that corresponding to a constant external field in the absence of the superconductor. (In §6, we will see that this is not an unreasonable simplification.) Also, the version of (4.3) given in [14] was not in a gauge invariant form. However, the two most important features of the system (4.3)-(4.11) were first given in [14]. These are the discrete nature of the order parameter as a function of z, and, foremost, the Josephson coupling between the superconducting planes.

4.4. Bound on the order parameter

In the current nondimensionalization, $|\psi_n| = 1$ corresponds to the superconducting state. Thus, we naturally expect that $|\psi^s| \leq 1$ everywhere in the superconducting sample.

PROPOSITION 4.5. *If $(\psi^h, \vec{A})$ is a continuous solution of the Lawrence-Doniach equations* (4.3)-(4.11)*, then $|\psi^s| \leq 1$ in $\Omega \times [0, S]$.*

Proof. Note that for any fixed x and y,

$$|\psi^s(x, y, z)| \leq \max\{ |\psi_n(x, y)|, |\psi_{n+1}(x, y)| \} \qquad \text{for } z \in [ns, (n+1)s],\ n = 0, 1, \ldots, N-1 ,$$

so that we may assume that the maximum value of $|\psi^s|$ is achieved by $|\psi_k(x_k, y_k)|$ for some integer k between 0 and N and for some point $(x_k, y_k) \in \overline{\Omega}$. We now apply a technique introduced in [11]. If $|\psi_k(x_k, y_k)| \geq \beta > 1$, using a gauge transformation, we may assume, without loss of generality, that in a neighbood of (x_k, y_k), the solution ψ_k is real valued, positive, and greater than unity. In this case, the real part of equation (4.3) is given, near (x_k, y_k), by

$$(|\psi_k|^2 - 1)\psi_k + (-\kappa^{-2}\Delta\psi_k) + \sigma\left[2\psi_k - \Re\{\psi_{k+1}\exp(-i\phi_k^{k+1}) - \psi_{k-1}\exp(i\phi_{k-1}^k)\}\right] = 0 .$$

Since $\psi_k(x_k, y_k) \geq \beta > 1$, the first term on the left-hand side is positive and bounded away from zero in a sufficiently small neighborhood of (x_k, y_k). Since ψ_k has a maximum at (x_k, y_k), the second term is nonnegative, perhaps in a smaller neighborhood. Finally, in perhaps an even smaller neighborhood, the third term is larger than an negative number of arbitrarily small magnitude. Therefore, the left-hand side is positive. This contradiction implies $|\psi_k(x_k, y_k)| \leq 1$. □

5. The relation between the anisotropic Ginzburg-Landau and Lawrence-Doniach models

It is commonly accepted that if the layer thickness s is very small, then the Lawrence-Doniach model reduces to the anisotropic Ginzburg-Landau model. Here, we give a rigorous demonstration of this fact.

The various nondimensional parameters appearing in the two models are to be related by

$$\sigma s^2 = \rho \kappa^{-2} \,. \tag{5.1}$$

Substituting their dimensional counterparts, i.e., $\sigma = -\zeta/\alpha$, $\kappa = \lambda_\parallel/\xi_\parallel$, $\rho = m_\parallel/m_\perp$, and $s \to s/\lambda_\parallel$, we have that

$$\zeta = -\alpha \frac{\xi_\perp^2}{s^2} = \frac{\hbar^2}{2m_\perp s^2} \,, \tag{5.2}$$

where now all parameters, including s are dimensional. Below, we relate the nondimensional forms of the anisotropic Ginzburg-Landau and Lawrence-Doniach free energies by letting the nondimensional layer spacing s tend to zero. If we were to examine this relation in the dimensional setting, it can be seen from (5.2) that one must let the dimensional spacing s tend to zero relative to the perpendicular coherence length $\xi_\perp$.

Given the uniform partition of step length s of the interval $[0, S]$ in the z axis, we let I^s denote the piecewise linear interpolation operator in z with respect to the partition.

The main result that relates the anisotropic Ginzburg-Landau model to the Lawrence-Doniach model is given in the following theorem.

THEOREM 5.1. *Let the functionals $\mathcal{G}_{EM}$ and $\mathcal{G}_{LD}$ be defined by* (3.3) *and* (4.2), *respectively, which are in nondimensional form. Let the parameters appearing in* (3.3) *and* (4.2) *be related by* (5.1). *For given $s > 0$, let $(\psi_0^s, \vec{A}_0)$ be a minimizer of $\mathcal{G}_{LD}(\psi^s, \vec{A})$ in $\mathcal{H}_s^1(\Omega \times [0,S]) \times \vec{H}_n^1(\mathrm{div}\,; \mathbb{R}^3)$. Then, as $s \to 0$, $\{(\psi_0^s, \vec{A}_0)\}$ forms a minimizing sequence of $\mathcal{G}_{EM}$ and thus converges to some minimizer of $\mathcal{G}_{EM}$.* □

We shall divide the proof into several lemmas. The first three are easy to verify.

LEMMA 5.2. *Let $(\psi^s, \vec{A}) \in \mathcal{H}_s^1(\Omega \times [0,S]) \times \vec{H}_n^1(\mathrm{div}\,; \mathbb{R}^3)$. If $\{\mathcal{G}_{LD}(\psi^s, \vec{A})\}$ is uniformly bounded as $s \to 0$, then $\{(\psi^s, \vec{A})\}$ is uniformly bounded in $\mathcal{H}_s^1(\Omega \times [0,S]) \times \vec{H}_n^1(\mathrm{div}\,; \mathbb{R}^3)$.* □

LEMMA 5.3. *Let $(\psi^s, \vec{A}) \in \mathcal{H}_s^1(\Omega \times [0,S]) \times \vec{H}_n^1(\mathrm{div}\,; \mathbb{R}^3)$. If $\{\mathcal{G}_{EM}(\psi^s, \vec{A})\}$ is uniformly bounded as $s \to 0$, then $\{(\psi^s, \vec{A})\}$ is uniformly bounded in $\mathcal{H}_s^1(\Omega \times [0,S]) \times \vec{H}_n^1(\mathrm{div}\,; \mathbb{R}^3)$.* □

LEMMA 5.4. *Let $\psi \in \mathcal{C}^\infty(\Omega \times [0,S])$ and $\vec{A} \in \vec{H}_n^1(\mathrm{div}\,; \mathbb{R}^3)$. Then, $\{(I^s\psi, \vec{A})\}$ is uniformly bounded in $\mathcal{H}_s^1(\Omega \times [0,S]) \times \vec{H}_n^1(\mathrm{div}\,; \mathbb{R}^3)$.* □

The next lemma is proved in §5.1.

LEMMA 5.5. *Let $\{(\psi^s, \vec{A})\}$ be uniformly bounded in $\mathcal{H}_s^1(\Omega \times [0,S]) \times \vec{H}_n^1(\mathrm{div}\,; \mathbb{R}^3)$, and let*

$$\|\psi^s\|_\infty \le 1 \,. \tag{5.3}$$

Then

$$\lim_{s \to 0} \{\mathcal{G}_{EM}(\psi^s, \vec{A}) - \mathcal{G}_{LD}(\psi^s, \vec{A})\} = 0 \,. \quad \square$$

Let

$$f_s = \min_{\mathcal{H}_s^1(\Omega \times [0,S]) \times \vec{H}(\mathbb{R}^3)} \mathcal{G}_{LD}(\psi^s, \vec{A})$$

and

$$f = \min_{\mathcal{H}^1(\Omega \times [0,S]) \times \vec{H}(\mathbb{R}^3)} \mathcal{G}_{EM}(\psi, \vec{A}) \,.$$

Then, Theorems 3.1 and 4.4 and the above lemmas yield the following result from which Theorem 5.1 follows.

COROLLARY 5.6.

$$\lim_{s\to 0} f_s = f\,. \quad \square$$

5.1. Proof of Lemma 5.5

The magnetic energy terms in (3.3) and (4.2) are identical so that, upon subtraction, they cancel. Also, using the uniform boundedness (5.3), it is easy to show that

$$s\sum_{n=0}^{N}\left[\int_\Omega\left(\frac{1}{2}(|\psi_n|^2-1)^2+\left|(\frac{i}{\kappa}\nabla+\mathbf{A_n})\psi_n\right|^2\right)d\mathbf{x}\right] \\ -\int_0^1\left[\int_\Omega\left(\frac{1}{2}(|\psi|^2-1)^2+\left|(\frac{i}{\kappa}\nabla+\mathbf{A})\psi\right|^2\right)d\mathbf{x}\right]dz\to 0 \quad \text{as } s\to 0\,.$$

It remains to show that

$$\frac{1}{s}\sum_{n=0}^{N-1}\int_\Omega\left|\psi_{n+1}\exp\left(-i\kappa\int_{ns}^{(n+1)s}A_z\,dz\right)-\psi_n\right|^2 d\mathbf{x} \\ -\int_0^S\int_\Omega\left|\left(\frac{\partial}{\partial z}-i\kappa A_z\right)\psi\right|^2 d\mathbf{x}dz\to 0 \quad \text{as } s\to 0 \tag{5.4}$$

or, equivalently,

$$R_1-R_2\to 0 \quad \text{as } s\to 0\,,$$

where

$$R_1=\frac{1}{s}\sum_{n=0}^{N-1}\int_\Omega\left|(\psi_{n+1}-\psi_n)\exp(-i\phi_n^{n+1})-\psi_n(1-\exp(-i\phi_n^{n+1}))\right|^2 d\mathbf{x}$$

with

$$\phi_n^{n+1}=\kappa\int_{ns}^{(n+1)s}A_z\,dz$$

and

$$R_2=\frac{1}{s^2}\sum_{n=0}^{N-1}\int_{ns}^{(n+1)s}\int_\Omega\left|(\psi_{n+1}-\psi_n)(1-i\kappa(z-ns)A_z)-i\kappa sA_z\psi_n\right|^2 d\mathbf{x}\,dz\,.$$

We expand the quadratic terms of R_1 and R_2 and compare the corresponding terms.

First, we have that

$$
\begin{aligned}
|R_{11} - R_{21}| &= \Bigg| \frac{1}{s^2} \sum_{n=0}^{N-1} \int_\Omega \Big[s|\psi_{n+1} - \psi_n|^2 \\
&\qquad - \int_{ns}^{(n+1)s} |\psi_{n+1} - \psi_n|^2 (1 + \kappa^2 (z-ns)^2 {A_z}^2)\, dz \Big]\, d\mathbf{x} \Bigg| \\
&= \frac{\kappa^2}{s^2} \sum_{n=0}^{N-1} \int_\Omega \int_{ns}^{(n+1)s} \left(|\psi_{n+1} - \psi_n|^2 (z-ns)^2 {A_z}^2 \right)\, dz\, d\mathbf{x} \\
&\le \kappa^2 \sum_{n=0}^{N-1} \int_\Omega \left(|\psi_{n+1} - \psi_n|^2 \int_{ns}^{(n+1)s} {A_z}^2\, dz \right) d\mathbf{x} \\
&\le 2\kappa^2 \sum_{n=0}^{N-1} \int_\Omega \left(|\psi_{n+1} - \psi_n| \int_{ns}^{(n+1)s} {A_z}^2\, dz \right) d\mathbf{x} \\
&\le 2\kappa^2 \sum_{n=0}^{N-1} \left(\int_\Omega |\psi_{n+1} - \psi_n|^2\, d\mathbf{x} \right)^{1/2} \cdot \left[\int_\Omega \left(\int_{ns}^{(n+1)s} |A_z|^2\, dz \right)^2 d\mathbf{x} \right]^{1/2} \\
&\le 2\kappa^2 \left[\sum_{n=0}^{N-1} \int_\Omega |\psi_{n+1} - \psi_n|^2\, d\mathbf{x} \right]^{1/2} \cdot s^{1/2} \left[\sum_{n=0}^{N-1} \int_\Omega \int_{ns}^{(n+1)s} |A_z|^4\, dz\, d\mathbf{x} \right]^{1/2} \\
&\le 2\kappa^2 s^{1/2} \left[s \sum_{n=0}^{N-1} \int_\Omega \left(\frac{|\psi_{n+1} - \psi_n|}{s} \right)^2 d\mathbf{x} \right]^{1/2} \cdot s^{1/2} \cdot \left[\int_0^1 \int_\Omega |A_z|^4\, d\mathbf{x} dz \right]^{1/2} \\
&\le \mathit{constant} \cdot s^{1/2} \cdot s^{1/2} \to 0 \quad \text{as } s \to 0\,.
\end{aligned}
$$

Next,

$$
\begin{aligned}
|R_{12} - R_{22}| &= 2 \Bigg| \Re \left\{ \sum_{n=0}^{N-1} \int_\Omega \left(\frac{\psi_{n+1} - \psi_n}{s} \right) \psi_n^* \left(1 - \exp(-i\kappa \int_{ns}^{(n+1)s} A_z\, dz) \right) d\mathbf{x} \right\} \\
&\quad - \Re \left\{ \sum_{n=0}^{N-1} \int_\Omega \left(\frac{\psi_{n+1} - \psi_n}{s} \right) \psi_n^* \left(\int_{ns}^{(n+1)s} (i\kappa A_z + \kappa^2 (z-ns) A_z^2)\, dz \right) d\mathbf{x} \right\} \Bigg| \\
&= 2 |\Re \{ I_1 + I_2 \}|\,,
\end{aligned}
$$

where

$$
\begin{aligned}
|I_1| = \Bigg| \sum_{n=0}^{N-1} \int_\Omega \left(\frac{\psi_{n+1} - \psi_n}{s} \right) \psi_n^* \Bigg(1 - \exp\left(-i\kappa \int_{ns}^{(n+1)s} A_z\, dz \right) \\
- i\kappa \int_{ns}^{(n+1)s} A_z\, dz \Bigg) d\mathbf{x} \Bigg|
\end{aligned}
$$

and

$$
|I_2| = \left| \sum_{n=0}^{N-1} \int_\Omega \left(\frac{\psi_{n+1} - \psi_n}{s} \right) \psi_n^* \left(\kappa^2 \int_{ns}^{(n+1)s} (z-ns) A_z^2\, dz \right) d\mathbf{x} \right| .
$$

Since

$$
|1 - iy - \exp(-iy)| \le |y|^2 \quad \forall\, y \in \mathbb{R}^1\,,
$$

we have

$$
\begin{aligned}
|I_1| &\le \kappa^2 \sum_{n=0}^{N-1} \int_\Omega \left| \frac{\psi_{n+1}-\psi_n}{s} \right| \left(\int_{ns}^{(n+1)s} A_z\, dz \right)^2 \, d\mathbf{x} \\
&\le \kappa^2 \sum_{n=0}^{N-1} \int_\Omega \left(|\psi_{n+1}-\psi_n| \int_{ns}^{(n+1)s} |A_z|^2\, dz \right) d\mathbf{x} \\
&\le constant \cdot s^{1/2} \cdot s^{1/2} \to 0 \quad \text{as } s \to 0\, .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_2| &\le \kappa^2 \sum_{n=0}^{N-1} \int_\Omega \left(|\psi_{n+1}-\psi_n| \int_{ns}^{(n+1)s} |A_z|^2\, dz \right) d\mathbf{x} \\
&\le constant \cdot s^{1/2} \cdot s^{1/2} \to 0 \quad \text{as } s \to 0\, .
\end{aligned}
$$

Therefore

$$
|R_{12} - R_{22}| \to 0 \quad \text{as } s \to 0\, .
$$

Finally,

$$
\begin{aligned}
|R_{13} - R_{23}| = \Bigg| &\frac{1}{s} \sum_{n=0}^{N-1} \int_\Omega |\psi_n|^2 \left| 1 - \exp(-i\kappa \int_{ns}^{(n+1)s} A_z\, dz) \right|^2 d\mathbf{x} \\
&- \sum_{n=0}^{N-1} \int_\Omega \left[|\psi_n|^2\, \kappa^2 \int_{ns}^{(n+1)s} |A_z|^2\, dz \right] d\mathbf{x} \Bigg| \\
= \Bigg| &\frac{4}{s} \sum_{n=0}^{N-1} \int_\Omega |\psi_n|^2 \sin^2 \left(\frac{\kappa}{2} \int_{ns}^{(n+1)s} A_z\, dz \right) d\mathbf{x} \\
&- \sum_{n=0}^{N-1} \int_\Omega \left[|\psi_n|^2 \kappa^2 \int_{ns}^{(n+1)s} |A_z|^2\, dz \right] d\mathbf{x} \Bigg| \\
= \Bigg| &\frac{4}{s} \sum_{n=0}^{N-1} \int_\Omega \left\{ |\psi_n|^2 \left[\sin^2 \left(\frac{\kappa}{2} \int_{ns}^{(n+1)s} A_z\, dz \right) - \left(\frac{\kappa}{2} \int_{ns}^{(n+1)s} A_z\, dz \right)^2 \right] \right\} d\mathbf{x} \\
&+ \frac{4}{s} \sum_{n=0}^{N-1} \int_\Omega \left\{ |\psi_n|^2 \left[\left(\frac{\kappa}{2} \int_{ns}^{(n+1)s} A_z\, dz \right)^2 - \frac{s\kappa^2}{4} \int_{ns}^{(n+1)s} |A_z|^2\, dz \right] \right\} d\mathbf{x} \Bigg| \, .
\end{aligned}
$$

Note that

$$
|\sin^2 y - y^2| \le \frac{1}{3} |y|^4 \quad \forall\, y \in \mathbb{R}^1\, ,
$$

so that

$$
\begin{aligned}
\frac{4}{s} \left[\sin^2 \left(\frac{\kappa}{2} \int_{ns}^{(n+1)s} A_z\, dz \right) - \left(\frac{\kappa}{2} \int_{ns}^{(n+1)s} A_z\, dz \right)^2 \right] &\le \frac{\kappa^4}{12 s} \left(\int_{ns}^{(n+1)s} A_z\, dz \right)^4 \\
&\le \frac{\kappa^4 s^2}{12} \int_{ns}^{(n+1)s} |A_z|^4\, dz\, .
\end{aligned}
$$

Also,

$$\frac{4}{s}\left[\left(\frac{\kappa}{2}\int_{ns}^{(n+1)s} A_z\,dz\right)^2 - \frac{s\kappa^2}{4}\int_{ns}^{(n+1)s} |A_z|^2\,dz\right]$$
$$= \kappa^2\int_{ns}^{(n+1)s}\left[A_z - \frac{1}{s}\int_{ns}^{(n+1)s} A_z\,dz\right]^2 dz$$
$$\leq \kappa^2\int_{ns}^{(n+1)s}\left[\left(\int_{ns}^{(n+1)s} |\frac{\partial A_z}{\partial z}|\,dz\right)\cdot\left|A_z - \frac{1}{s}\int_{ns}^{(n+1)s} A_z\,dz\right|\right] dz$$
$$\leq 2\kappa^2\int_{ns}^{(n+1)s}\left|\frac{\partial A_z}{\partial z}\right| dz\;\cdot\;\int_{ns}^{(n+1)s} |A_z|\,dz\,.$$

Hence,

$$|R_{13} - R_{23}| \leq \sum_{n=0}^{N-1}\int_\Omega\left[|\psi_n|^2\left(\frac{\kappa^4 s^2}{12}\int_{ns}^{(n+1)s} |A_z|^4\,dz\right.\right.$$
$$\left.\left. +2\kappa^2\int_{ns}^{(n+1)s}\left|\frac{\partial A_z}{\partial z}\right| dz\int_{ns}^{(n+1)s} |A_z|\,dz\right)\right] d\Omega$$
$$\leq \frac{\kappa^4 s^2}{12}\int_\Omega\int_0^1 |A_z|^4 dz\,d\mathbf{x} + 2\kappa^2 s\,\|\frac{\partial A_z}{\partial z}\|_{L^2(\Omega\times[0,1])}\cdot\|A_z\|_{L^2(\Omega\times[0,1])}$$
$$\to 0 \quad \text{as } s \to 0\,.$$

The combination of the above estimates yields (5.4), and thus we have proved Lemma 5.5. □

REMARK. The above proof may be modified to prove Lemma 5.5 without assuming the pointwise bound (5.3) on the magnitude of the order parameter.

6. Simplified models valid for large κ and fields of $O(\kappa)$

We now turn to simplifications that can be effected on the anisotropic Ginzburg-Landau and Lawerence-Doniach models in the case of large values of κ and applied fields of $O(\kappa)$. Both of these ranges are of interest since it is known that $\kappa > 50$ for high-temperature superconductors and since large applied fields are of technological interest. Note that the applied field being $O(\kappa)$ does not necessarily mean that it is close to the upper critical field H_{c2} (which is equal to κ for a bulk isotropic superconductor) since $|\vec{H}|$ may be, for example, $\kappa/2$. It does mean that however that $|\vec{H}| \gg H_{c1}$, the lower critical field, which is $O((1/\kappa)\log\kappa)$.

The analogous simplifications for the isotropic, homogeneous Ginzburg-Landau model were discussed in detail in [5]. Since most of that discussion applies to the current cases, here we will not provide details concerning the derivation or justification of the simplified models. We also note that, *in practice*, $\kappa \approx 5$ is a sufficiently "large" value of κ for the simplified models to yield accurate approximations of the corresponding full models; see [5] for details.

6.1. The simplified anisotropic Ginzburg-Landau model

In the high κ, high field regime, it is more convenient to nondimesionalize lengths by $\xi_{\|}$. We now nondimensionalize magnetic fields by $\sqrt{2}H_c$, the magnetic potential by $\sqrt{2}\xi_{\|}H_c$, the order parameter by $\sqrt{-\alpha/\beta}$, free energy densities by α^2/β, free energies by $\alpha^2\xi_{\|}^3/\beta$, and currents by $cH_c/(\pi\xi_{\|}\sqrt{8})$. In terms of this nondimensionalization scheme, the anisotropic Ginzburg-Landau equations are given by

$$(6.1)\quad \left(i\mathbf{grad} + \frac{1}{\kappa}\mathbf{A}\right)^2\psi + \rho\left(i\frac{\partial}{\partial z} + \frac{1}{\kappa}A_z\right)^2\psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \mathcal{D} = \Omega\times[0,S]\,,$$

$$(6.2)\quad \vec{\text{curl}}\,\vec{\text{curl}}\,\vec{A} = -\begin{pmatrix} \frac{i}{2\kappa}(\psi^*\mathbf{grad}\,\psi - \psi\mathbf{grad}\,\psi^*) + |\psi|^2\mathbf{A} \\ \rho\left[\frac{i}{2\kappa}\left(\psi^*\frac{\partial\psi}{\partial z} - \psi\frac{\partial\psi^*}{\partial z}\right) + |\psi|^2 A_z\right]\end{pmatrix} \quad \text{in } \mathcal{D} = \Omega\times[0,S]\,,$$

and

$$(6.3)\quad \vec{\text{curl}}\,\vec{\text{curl}}\,\vec{A} = 0 \quad \text{in } \mathcal{D}_e$$

along with the boundary conditions

$$(6.4)\quad (i\mathbf{grad}\,\psi + \frac{1}{\kappa}\mathbf{A}\psi)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\times[0,S]\,,$$

$$(6.5)\quad (i\frac{\partial}{\partial z} + \frac{1}{\kappa}A_z)\psi = 0 \quad \text{on } (\Omega\times\{0\})\cup(\Omega\times\{S\})\,,$$

$$(6.6)\quad [\vec{A}\times\vec{n}] = \mathbf{0} \quad \text{on } (\Gamma\times[0,S])\cup(\Omega\times\{0\})\cup(\Omega\times\{S\})\,,$$

$$(6.7)\quad [\vec{\text{curl}}\,\vec{A}\times\vec{n}] = \mathbf{0} \quad \text{on } (\Gamma\times[0,S])\cup(\Omega\times\{0\})\cup(\Omega\times\{S\})\,,$$

and

$$(6.8)\quad \vec{\text{curl}}\,\vec{A}\to\vec{H} \quad \text{as } |\vec{x}|\to\infty\,.$$

We assume that the applied field $\vec{H}$ is of $O(\kappa)$; specifically, we assume that $\vec{H} = \kappa\vec{H}_0$. We let $\vec{A}_0$ be a magnetic potential such that $\vec{\text{curl}}\times\vec{A}_0 = \vec{H}_0$. For example, if $\vec{H}_0$ is aligned with the z-axis, i.e., $\vec{H}_0 = (0,0,H_0)$ for some constant H_0, $\mathbf{A}_0 = H_0(0,x,0)$ satisfies $\vec{\text{curl}}\times\vec{A}_0 = \vec{H}_0$. Then, using the techniques of [5], it can be shown that there exists a solution $(\psi,\vec{A})$ of the anisotropic Ginburg-Landau equations (6.1)-(6.8) satisfying

$$(6.9)\quad \vec{A} = \kappa\left(\vec{A}_0 + 0\left(\kappa^{-2}\right)\right) \quad \text{and} \quad \psi = \psi_0 + 0\left(\kappa^{-2}\right)\,,$$

where ψ_0 satisfies

$$(6.10)\quad (i\mathbf{grad} + \mathbf{A}_0)^2\psi_0 + \rho\left(i\frac{\partial}{\partial z} + A_{0z}\right)^2\psi_0 - \psi_0 + |\psi_0|^2\psi_0 = 0 \quad \text{in } \mathcal{D} = \Omega\times[0,S]\,,$$

$$(i\mathbf{grad}\,\psi_0 + \mathbf{A}_0\psi_0)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\times[0,S]\,, \tag{6.11}$$

and

$$(i\frac{\partial}{\partial z} + A_{0z})\psi_0 = 0 \quad \text{on } (\Omega\times\{0\})\cup(\Omega\times\{S\})\,, \tag{6.12}$$

where, of course, $\vec{A}_0 = (\mathbf{A}_0, A_{0z})$.

Certainly, for $\kappa > 50$, we see from (6.9) that $\kappa\vec{A}_0$ and ψ_0 are accurate approximations to solutions of the anisotropic Ginzburg-Landau equations (6.1)-(6.8). Of course, the determination of $\vec{A}_0$ is trivial for the constant applied field $\vec{H} = \kappa\vec{H}_0$. The determination of ψ_0 is also greatly simplified since $\vec{A}_0$ is known in (6.10)-(6.12).

Of course, it is easy to continue with this asymptotic expansion to find the corrections to this leading order behaviour (see [5]).

In addition to the fact that, to leading order, the determination of the magnetic potential and order parameter are essentially uncoupled, there is another simplification that occurs in the high κ, high field regime. Following [3], we note that we can rescale certain variables so that the leading order anisotropic equations (6.10)-(6.12) reduce to corresponding isotropic equations. (The full equations (6.1)-(6.8) cannot be rescaled with such an effect.) Specifically, let $\tilde{z} = z/\sqrt{\rho}$, $\tilde{A}_{0z} = \sqrt{\rho}A_{0z}$, and $\tilde{\mathcal{D}} = \Omega\times[0,\tilde{S}]$, where $\tilde{S} = S/\sqrt{\rho}$. We do not scale x, y, $\mathbf{A}_0$, or Ω. However, the applied field $\vec{H} = (\mathbf{H}, H_z)$ is rescaled to $(\tilde{\mathbf{H}}, H_z)$, where $\tilde{\mathbf{H}} = \sqrt{\rho}\mathbf{H}$; the z-component is not scaled. With these changes of scale, (6.10)-(6.12) becomes

$$(i\mathbf{grad} + \mathbf{A}_0)^2\,\psi_0 + \left(i\frac{\partial}{\partial\tilde{z}} + \tilde{A}_{0z}\right)^2\psi_0 - \psi_0 + |\psi_0|^2\psi_0 = 0 \quad \text{in } \tilde{\mathcal{D}}\,, \tag{6.13}$$

$$(i\mathbf{grad}\,\psi_0 + \mathbf{A}_0\psi_0)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\times[0,\tilde{S}]\,, \tag{6.14}$$

and

$$(i\frac{\partial}{\partial\tilde{z}} + \tilde{A}_{0z})\psi_0 = 0 \quad \text{on } (\Omega\times\{0\})\cup(\Omega\times\{\tilde{S}\})\,. \tag{6.15}$$

Clearly, (6.13)-(6.15) are isotropic equations.

Reversing the scaling process, the above reduction to the isotropic case may be interpreted as follows. Suppose one has in hand a solution

$$\psi(x,y,\tilde{z}) \quad \text{and} \quad \big(\mathbf{A},\tilde{A}_z\big)(x,y,\tilde{z})$$

of the homogeneous, isotropic Ginzburg-Landau equations for a domain $\Omega\times[0,\tilde{S}]$ and for an applied field $(\tilde{\mathbf{H}}, H_z)$ and suppose the value of κ is "large", e.g., $\kappa \geq 5$, and that the applied field is of $O(\kappa)$. (There are, for example, numerous codes in existence that yield such results.) Then, for any mass ratio ρ,

$$\psi(x,y,z) \quad \text{and} \quad \big(\mathbf{A},A_z\big)(x,y,z)\,,$$

where $z = \sqrt{\rho}\tilde{z}$ and $A_z = \tilde{A}_z/\sqrt{\rho}$, is an accurate approximation to the solution of the anisotropic Ginzburg-Landau equations for the domain $\Omega\times[0,S]$ and for the applied field $(\mathbf{H}, H_z)$, where $S = \sqrt{\rho}\tilde{S}$ and $\mathbf{H} = \tilde{\mathbf{H}}/\sqrt{\rho}$. Thus, a single isotropic calculation can be rescaled to yield, at negligible cost, solutions for a one parameter family of solutions of the anisotropic Ginzburg-Landau equations.

One refinement of these observations is that if the applied field is parallel to the z-axis, then a change in the mass ratio ρ by a factor of K is equivalent to a change in the thickness of the superconducting sample by a factor of $\sqrt{K}$. If the applied field is not parallel to the z-axis, then the the tangent of the angle that the field makes with that axis also changes by a factor of $\sqrt{1/K}$.

6.2. The simplified Lawrence-Doniach model

We again change to the in-layer coherence length for the basic length scale for nondimensionalization; see §6.1. Then, the Lawrence-Doniach equations (4.3)-(4.11) are transformed to

$$\begin{aligned}&-\psi_n+|\psi_n|^2\psi_n+(i\mathbf{grad}+\frac{1}{\kappa}\mathbf{A_n})^2\psi_n\\&\quad+\sigma\Big[2\psi_n-\psi_{n+1}\exp(-i\phi_n^{n+1})-\psi_{n-1}\exp(i\phi_{n-1}^{n})\Big]=0\\&\qquad\qquad\text{in }\Omega\text{ and for }n=0,1,\dots,N\,,\end{aligned}\tag{6.16}$$

where

$$\phi_n^{n+1}=\frac{1}{\kappa}\int_{ns}^{(n+1)s}A_z\,dz\quad\text{for }n=0,\dots,N-1\,,\tag{6.17}$$

and

$$\vec{\mathrm{curl}}\,\vec{\mathrm{curl}}\,\vec{A}=\begin{pmatrix}\mathbf{j}\\ j_z\end{pmatrix}\quad\text{in }\mathbb{R}^3\ ,\tag{6.18}$$

where

$$\mathbf{j}=s\sum_{n=0}^{N}\left(\frac{i}{\kappa}(\psi_n\mathbf{grad}\,\psi_n{}^*-\psi_n{}^*\mathbf{grad}\,\psi_n)-\frac{i}{\kappa^2}|\psi_n|^2\mathbf{A_n}\right)\delta(z-ns)\quad\text{in }\mathbb{R}^3\ ,\tag{6.19}$$

$$\begin{aligned}j_z=-i\frac{\sigma s}{\kappa}&\Big[\psi_n\exp(i\phi_n^{n+1})\psi_{n+1}^*-\psi_n^*\exp(-i\phi_n^{n+1})\psi_{n+1}\Big]\\&\qquad\text{in }\Omega\times[ns,(n+1)s],\ n=0,\dots,N-1\,,\end{aligned}\tag{6.20}$$

and

$$j_z=0\quad\text{in }\mathcal{D}_e\ .\tag{6.21}$$

One also obtains the boundary conditions

$$(\mathbf{grad}\,\psi_n+\frac{1}{\kappa}\mathbf{A_n}\psi_n)\cdot\mathbf{n}=0\quad\text{on }\Gamma\text{ and for }n=0,1,\dots,N\,,\tag{6.22}$$

$$[\vec{A}\times\vec{n}]=\mathbf{0}\quad\text{and}\quad[\vec{\mathrm{curl}}\,\vec{A}\times\vec{n}]=\mathbf{0}\quad\text{on the boundary }\partial\mathcal{D}\text{ of }\Omega\times[0,S]\,,\tag{6.23}$$

and

$$\vec{\text{curl}}\,\vec{A} \to \vec{H} \quad \text{as } |\vec{x}| \to \infty\,. \tag{6.24}$$

We again assume that the applied field $\vec{H} = \kappa\vec{H}_0$ and again let $\vec{A}_0$ be a magnetic potential such that $\vec{\text{curl}} \times \vec{A}_0 = \vec{H}_0$. Then, using the techniques of [5], it can be shown that there exists a solution ψ_n, $n = 0, 1, \ldots, N$, and $\vec{A}$ of the Lawerence-Doniach equations (6.16)-(6.24) satisfying

$$\vec{A} = \kappa\left(\vec{A}_0 + 0\,(\kappa^{-2})\right) \quad \text{and} \quad \psi_n = \psi_{0n} + 0\,(\kappa^{-2})\;,\; n = 0, 1, \ldots, N, \tag{6.25}$$

where ψ_{0n}, $n = 0, 1, \ldots, N$, satisfies

$$\begin{aligned} &-\psi_{0n} + |\psi_{0n}|^2\psi_{0n} + (\mathbf{grad} + \mathbf{A}_{0n})^2\psi_{0n} \\ &\quad + \sigma\Big[2\psi_{0n} - \psi_{0(n+1)}\exp(-i\phi_{0n}^{n+1}) - \psi_{0(n-1)}\exp(i\phi_{0(n-1)}^{n})\Big] = 0 \\ &\qquad\qquad \text{in } \Omega \text{ and for } n = 0, 1, \ldots, N\,, \end{aligned} \tag{6.26}$$

where

$$\phi_{0n}^{n+1} = \int_{ns}^{(n+1)s} A_{0z}\,dz \quad \text{for } n = 0, \ldots, N-1\,, \tag{6.27}$$

and

$$(i\mathbf{grad}\,\psi_{0n} + \mathbf{A}_{0n}\psi_{0n}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \text{ and for } n = 0, 1, \ldots, N\,. \tag{6.28}$$

In (6.25)-(6.28), we have that $\vec{A}_0 = (\mathbf{A}_0, A_{0z})$ and $\mathbf{A}_{0n}$ denotes the restriction of $\mathbf{A}_0$ to the n-th superconducting plane.

Thus, for large κ and large applied fields, the magnetic potential is trivially determined, and subsequently, the order parameter in the superconducting planes can be determined from (6.26)-(6.28). Clearly, this is a great simplification over the full Lawrence-Doniach equations (6.16)-(6.24). It is worth noting that, except for the gauge invariant form of the z-difference in (6.26), the original model proposed by Lawrence and Doniach in [14] is given by exactly (6.26) and (6.28).

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