ON THE ITERATES OF DIGIT MAPS

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Abstract. Given a base b, a "digit map" is a map $f : \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ of the form $f(\sum_{i=0}^{n} a_i b^i) = \sum_{i=0}^{n} f_*(a_i)$, $0 \leq a_i \leq b-1$ for each i, where $f_* : \{0, 1, \ldots, b-1\} \to \mathbb{Z}^{\geq 0}$ satisfies $f_*(0) = 0$ and $f_*(1) = 1$. It has been proven for $b = 10$ and $f_*(m) = m^2$, and various generalizations thereof, that there are arbitrarily long sequences of consecutive positive integers that end up at 1 under repeated application of f. In this paper, we significantly generalize these results, providing a complete classification of digit maps for which, given any periodic point n, there are arbitrarily long sequences of consecutive positive integers that end up n.

1. Introduction

In this paper, we look at functions that take in a positive integer and output the sum of its values on the digits of that integer. Precisely, for a fixed base b, we start with a function $f_* : \{0, 1, \ldots, b-1\} \to \mathbb{Z}^{\geq 0}$ and then obtain a map $f : \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ given by $f(\sum_{i=0}^{n} a_i b^i) = \sum_{i=0}^{n} f_*(a_i)$, where $0 \leq a_i \leq b-1$. We study long-term iterates of the map f; that is, we start with a positive integer n and repeatedly apply f, to obtain the sequence $n, f(n), f(f(n)), f(f(f(n))), \ldots$.

In Richard Guy's book "Unsolved Problems in Number Theory", Guy poses many questions regarding $(2, 10)$-happy numbers [3]. An (e, b)-*happy number* is a number that, under iterates of the digit map f induced by $f_*(m) = m^e$ in base b, eventually reaches 1. In [4], Pan proved that there exist arbitrarily long sequences of consecutive (e, b)-happy numbers assuming that if a prime p divides $b-1$, then the integer $p-1$ does not divide $e-1$.

A question appearing in Guy's book [3] is that of gaps in the happy number sequence. It is easy to see that, for any digit map, every positive integer eventually ends up in some finite cycle, i.e. a collection of positive integers $\{n_1, \ldots, n_k\}$ such that $f(n_i) = n_{i+1}$ for $1 \leq i \leq k-1$ and $f(n_k) = f(n_1)$. For example, the cycles generated by the $(2, 10)$-happy number digit map are $\{1\}$ and $\{4, 16, 37, 58, 89, 145, 42, 20\}$. A gap in the happy number sequence, therefore, corresponds to consecutive numbers that end up in the latter cycle. In this paper, a special case of what we prove is that indeed for any u in an (e, b)-happy number cycle, we can find arbitrarily long sequences of consecutive integers that end up in the same cycle as u. This answers the question of Guy.

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Significantly more broadly, we provide a complete classification of digit maps for which there are arbitrarily long sequences of consecutive integers ending up in any prespecified cycle. To state our main theorem, we say that a digit map f with base b has a *modular obstruction* if $\gcd(f_*(1)-1, \ldots, f_*(b-1)-(b-1), b-1) > 1$. We call a positive integer u in some cycle a *cycle number* and any positive integer ending up in that cycle a *u-integer.*

Theorem 1. *Let f be a digit map. If f has a modular obstruction, then for any cycle number u, there do not exist two consecutive u-integers. If f does not have a modular obstruction, then for any cycle number u and any positive integer n, there exist n consecutive u-integers.*

For example, working in base 10, if we construct the digit map $f_* : \{0, \ldots, 9\} \to \mathbb{Z}^{\geq 0}$ by setting $f_*(0) = 0, f_*(1) = 1$, $f_*(9) = 7$, and choosing *any* values for $2, 3, 4, 5, 6, 7$, and 8, then we are guaranteed that there will exist arbitrarily long sequences of consecutive positive integers that end up at 1 under repeated application of f. The result of Theorem 1 consumes the work of H. Pan [4], H. Grundman and E. A. Teeple in [2], and E. El-Sedy and S. Siksek in [1].

2. Proof of Theorem 1

We first quickly prove the first part of Theorem 1. Suppose that f has a modular obstruction: there is some $g > 1$ with $g \mid b-1$ and $f_*(m) \equiv m \pmod{g}$ for each $1 \leq m \leq b-1$. Then, for any $n \in \mathbb{N}$, it holds that $f(n) \equiv n \pmod{g}$; indeed, if $n = \sum_{j=0}^{k} a_j b^j$, then, since $b \equiv 1 \pmod{g}$,

$$f(n) \equiv \sum_{j=0}^{k} f_*(a_j) \equiv \sum_{j=0}^{k} a_j \equiv \sum_{j=0}^{k} a_j b^j \pmod{g}.$$

Therefore, for any $n \in \mathbb{N}$ and $r \geq 1$, the iterate $f^r(n)$ is congruent to n mod g. Consequently, if there were a cycle number u and corresponding $n, r_1, r_2 \geq 1$ with $f^{r_1}(n) = f^{r_2}(n+1) = u$, we'd have $n \equiv n+1 \pmod{g}$, absurd.

We now move on to the second part of Theorem 1. We first use a few short results of Pan and introduce new techniques and results in Lemma 3 and Corollary 2.2. Specifically, the proofs of Lemma 1, Corollary 2.1, and Lemma 2 are basically identical to the proofs given by Pan; we just fit them to our notation.

Lemma 1. *Let x and m be arbitrary positive integers. Then for each $r \geq 1$, there exists a positive integer l such that*

$$f^r(l+y) = f^r(l) + f^r(y) = x + f^r(y)$$

for each $1 \leq y \leq m$.

Proof. We use induction on r. When $r = 1$, choose a positive integer s such that $b^s > m$ and let

$$l_1 = \sum_{j=0}^{x-1} b^{s+j}.$$

Clearly for any $1 \leq y \leq m$,

$$f(l_1 + y) = f(l_1) + f(y) = x + f(y).$$

Now assume $r > 1$ and the assertion of Lemma 1 holds for the smaller values of r. Note there exists an m' such that $f(y) \leq m'$ for $1 \leq y \leq m$. Therefore, by induction hypothesis, there exists an l_{r-1} such that

$$f^{r-1}(l_{r-1} + f(y)) = f^{r-1}(l_{r-1}) + f^{r-1}(f(y)) = x + f^r(y)$$

for $1 \leq y \leq m$. Let

$$l_r = \sum_{j=0}^{l_{r-1}-1} b^{s+j}$$

where s satisfies $b^s > m$. Then,

$$f^r(l_r) = f^{r-1}(f(l_r)) = f^{r-1}(l_{r-1}) = x$$

and for each $1 \leq y \leq m$,

$$\begin{aligned} f^r(l_r + y) &= f^{r-1}(f(l_r + y)) = f^{r-1}(f(l_r) + f(y)) \\ &= f^{r-1}(l_{r-1} + f(y)) = f^{r-1}(l_{r-1}) + f^r(y) = f^r(l_r) + f^r(y). \end{aligned}$$

□

Definition 2.1. Let $D = D(f_*, b)$ be the set of all positive integers that are in some cycle, that is $u \in D$ if and only if $f^r(u) = u$ for some $r \geq 1$. It is easy to see that D is finite.

Definition 2.2. Take some $u \in D$. We say a positive integer n is a *u-integer* if $f^r(n) = u$ for some $r \geq 1$. We say two positive integers m, n are *concurrently u-integers* if for some $r \geq 1$, $f^r(m) = f^r(n) = u$.

Note that two u-integers m, n are not concurrently u-integers only if u belongs to a cycle of length greater than 1 in D and m, n are at different places in the cycle at a certain time. Note "concurrently u-integers" is a transitive relation. Now fix u and we will prove that there are arbitrarily long sequences of consecutive u-integers. First, we make a reduction.

Corollary 2.1. *Assume that there exists $h \in \mathbb{N}$ such that $h + x$ is a u-integer for all $x \in D$. Then for arbitrary $m \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that $l + 1, l + 2, \ldots, l + m$ are u-integers.*

Proof. By the definition of D, there exists $r \in \mathbb{N}$ such that $f^r(y) \in D$ for all $1 \le y \le m$. By Lemma 1, there exists $l \in \mathbb{N}$ so that

$$f^r(l+y) = h + f^r(y)$$

for $1 \le y \le m$. Since $f^r(l+y)$ is then a u-integer, $l+y$ is as well, for $1 \le y \le m$. □

Lemma 2. *Assume that for each $x \in D$ there exists $h_x \in \mathbb{N}$ such that $h_x + u$ and $h_x + x$ are concurrently u-integers. Then there exists $h \in \mathbb{N}$ such that $h + x$ is a u-integer for each $x \in D$.*

Proof. We shall prove that, under the assumption of Lemma 2, for each subset X of D containing u, there exists $h_X \in \mathbb{N}$ such that $h_X + x$ is a u-integer for each $x \in X$.

The cases $|X| = 1$ and $|X| = 2$ are clear. Assume $|X| > 2$ and that the assertion holds for every smaller value of $|X|$. Take some $x \in X$, with $x \ne u$. Then $h_x + u$ and $h_x + x$ are concurrently u-integers, so take $r \in \mathbb{N}$ large enough so that $f^r(h_x + u) = f^r(h_x + x) = u$ and $f^r(h_x + y) \in D$ for all $y \in X$. Let $X^* = \{f^r(h_x + y) | y \in X\}$. Then, X^* is clearly a subset of D containing u with $|X^*| < |X|$. Therefore, by induction, there exists $h_{X^*} \in \mathbb{N}$ such that $h_{X^*} + f^r(h_x + y)$ is a u-integer for each $y \in X$. By Lemma 1, there exists $l \in \mathbb{N}$ satisfying

$$f^r(l + h_x + y) = h_{X^*} + f^r(h_x + y)$$

for every $y \in X$. Thus, $h_X := l + h_x$ works. The induction is complete. □

We now proceed to prove the hypothesis of Lemma 2. Note it suffices to show that for any fixed difference d, we can find two concurrent u-integers with difference d. This is the statement of Corollary 2.2. We first need one more lemma.

Lemma 3. *Let h be a u-integer. Then for every integer a, there exists a u-integer l such that $l \equiv a \pmod{f(b-1)}$, and such that l and h are concurrently u-integers.*

Proof. Let l_1 be a u-integer such that

$$l_1 > f(a) + (b-1)f(b-1) \max_{1 \le m \le b-1} f(m).$$

We now find some l_2 with $f(l_2) = l_1$ and $l_2 \equiv a \pmod{f(b-1)}$. Since

$$\gcd(f(1) - 1, \ldots, f(b-1) - (b-1), f(b-1)) = 1,$$

we may take $r_1, \ldots, r_{b-1} \in \{0, \ldots, f(b-1)\}$ so that

$$r_1(1 - f(1)) + \cdots + r_{b-1}(b - 1 - f(b-1)) \equiv f(a) - l_1 \pmod{f(b-1)}.$$

Note that

$$L := l_1 - f(a) - r_1 f(1) - \cdots - r_{b-1} f(b-1)$$

satisfies $L \ge 1$. By the pigeonhole principle, there is some $b' \in \{0, \ldots, b-1\}$ such that $b^j \equiv b' \pmod{f(b-1)}$ for infinitely many j. Let $j_1 < j_2 < \cdots < j_L < t_1^{(1)} <$

$\cdots < t_{r_1}^{(1)} < \cdots < t_1^{(b-1)} < \cdots < t_{r_{b-1}}^{(b-1)}$ satisfy $b^{j_i} \equiv b^{t_s^{(k)}} \equiv b' \pmod{f(b-1)}$ for each i, s, and k, and satisfy $b^{j_1} > a$.

Let

$$l_2 = a + \sum_{n=1}^{L} b^{j_n} + \sum_{m=1}^{b-1} \sum_{j=1}^{r_m} m b^{t_j^{(m)}}.$$

Due to the inequality $b^{j_1} > a$, we have

$$f(l_2) = f(a) + L + r_1 f(1) + \cdots + r_{b-1} f(b-1) = l_1,$$

and due to the choice of r_i's, we have

$$l_2 \equiv a + b' \left[L + r_1 + 2r_2 + \cdots + (b-1) r_{b-1}\right] \equiv a \pmod{f(b-1)}.$$

Now we generate $l_3, l_4, \ldots$ inductively by choosing l_{n+1} so that $l_{n+1} \equiv a \pmod{f(b-1)}$ and $f(l_{n+1}) = l_n$. Note that since the cycle that u is in is finite, it must be that one of the l_n's is concurrently a u-integer with h. □

Corollary 2.2. *For each $x \in \mathbb{N}$, there is a u-integer l such that l and $l + x$ are concurrently u-integers.*

Proof. Fix $x \in \mathbb{N}$. Take $s \in \mathbb{N}$ such that $b^s > x$. Let $x_1 = b^s - x$. Take a u-integer h' such that

$$h' \equiv f(x_1) \pmod{f(b-1)}.$$

Let V be the cycle set that u is in. By Lemma 3, for each $v' \in V$, there exists $l_{v'}$ such that $l_{v'} \equiv 1 \pmod{f(b-1)}$, and $l_{v'}$ and v' are concurrently u-integers. Fixing an $l_{v'}$ for each $v' \in V$, let $M = \max_{v' \in V} l_{v'}$.

Since the proof of Lemma 3 guarantees infinitely many u-integers in a given residue, we may (and do) fix $h > f(x_1) + M$ to be a u-integer with $h \equiv f(x_1) \pmod{f(b-1)}$. Let v be in the cycle of u so that h and v are concurrently u-integers. Now take the u-integer $N = l_v$ so that $N \equiv 1 \pmod{f(b-1)}$, and N and v are concurrently u-integers. Take a positive integer t so that $b^t > b^{s + \lfloor \frac{h}{f(b-1)} \rfloor + 1}$. Let $x_2 = x_1 + b^t \sum_{j=1}^{N-1} b^j$. Note $f(x_2) = f(x_1) + (N-1)$ since $b^t > b^s > x_1$. Thus,

$$f(x_2) \equiv f(x_1) \equiv h \pmod{f(b-1)}.$$

Also, $f(x_2) = f(x_1) + (N-1) \leq f(x_1) + M - 1 < h$. Write $h = f(b-1)k + f(x_2)$ and note that we have $k > 0$. Also note $k \leq \lfloor \frac{h}{f(b-1)} \rfloor + 1 < t - s$. Let

$$l = x_2 + \sum_{j=0}^{k-1} (b-1) b^{s+j}.$$

Then,

$$
\begin{aligned}
f(l) &= f\left(x_1 + b^t\sum_{j=1}^{N-1} b^j + b^s\sum_{j=0}^{k-1}(b-1)b^j\right)\\
&= f\left(x_1 + b^s[b^{t-s}\sum_{j=1}^{N-1} b^j + \sum_{j=0}^{k-1}(b-1)b^j]\right)\\
&= f(x_1) + f(b^{t-s}\sum_{j=1}^{N-1} b^j + \sum_{j=0}^{k-1}(b-1)b^j),
\end{aligned}
$$

and since $\sum_{j=0}^{k-1}(b-1)b^j = b^k - 1 < b^{t-s}$, we have

$$f(l) = f(x_1) + (N-1) + kf(b-1) = f(x_2) + kf(b-1) = h.$$

Further,

$$f(l+x) = f\left(b^s + \sum_{j=0}^{k-1}(b-1)b^{s+j} + b^t\sum_{j=1}^{N-1} b^j\right) = f\left(b^{s+k} + b^t\sum_{j=1}^{N-1} b^j\right),$$

which is equal to N. Since h and N are concurrently u-integers, it follows that l and $l+x$ are concurrently u-integers, as desired. □

Theorem 1.1 now follows from Corollary 2.1, Lemma 2, and Corollary 2.2.

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