# On the existence of rainbow 4-term arithmetic progressions

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**Abstract**

For infinitely many natural numbers $n$, we construct 4-colorings of $[n] = \{1, 2, \ldots, n\}$, with equinumerous color classes, that contain no 4-term arithmetic progression whose elements are colored in distinct colors. This result solves an open problem of Jungić et al. [JL+03], Axenovich and Fon-der-Flaass [AF04].

## 1 Introduction

Throughout the paper, we will use $AP(k)$ to denote a $k$-term arithmetic progression. Moreover, a coloring of $[n]$ will be called *equinumerous* if all the color classes have the same cardinality. A famous result of van der Waerden [vW27] states that for every pair of positive integers $k$ and $r$, there exists a positive integer $W := W(k, r)$, such that every $r$-coloring of integers in $[W] = \{1, 2, \ldots, W\}$ contains a *monochromatic* $AP(k)$. This theorem was generalized in numerous ways [GRS90, LR03], one being the following "density"-type theorem of Szemerédi [Sz75]: for every $k \in \mathbb{N}$ and a real number $\delta > 0$, there exists a positive integer $N$, such that every $S \subseteq [N]$, with $|S| \geq \delta N$, contains an $AP(k)$.

In [JL+03], Jungić et al. initiated the search for a rainbow counterpart of van der Waerden's theorem. Namely, given positive integers $k$ and $r$, what conditions on the $r$-coloring of $[n]$ guarantee a *rainbow* $AP(k)$, that is, an arithmetic progression of length $k$ all of whose elements are colored in distinct colors? If every integer in $[n]$ is colored by the largest power of three that divides it, then one immediately obtains an $r$-coloring of $[n]$ with $r \leq \lfloor \log_3 n + 1 \rfloor$ and without rainbow $AP(3)$. So, while Szemerédi's theorem states that a large cardinality in only one color class ensures the existence of a monochromatic $AP(k)$, one needs *all* color classes to be "large" to force a rainbow $AP(k)$.

Jungić et al. [JL+03] proved that every 3-coloring of $\mathbb{N}$ with the upper density of each color class greater than $1/6$ yields a rainbow $AP(3)$. Using some tools from additive number theory, they obtained similar (and stronger) results for 3-colorings of $\mathbb{Z}_n$ and $\mathbb{Z}_p$, some of which were recently extended by Conlon [C05]. In [JR03] Jungić and Radoičić studied the more difficult *interval* case and showed that every equinumerous 3-coloring of $[n]$ contains a rainbow $AP(3)$. Finally, Axenovich and Fon-Der-Flaass cleverly combined the previous methods with some additional ideas to obtain the following theorem, conjectured in [JL+03].

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**Theorem 1** [AF04] *For every $n \geq 3$, every partition of $[n]$ into three color classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ with* $\min(|\mathcal{A}|, |\mathcal{B}|, |\mathcal{C}|) > r(n)$, *where*

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod 6 \\ (n+4)/6 & \text{if } n \equiv 2 \pmod 6 \end{cases} \tag{1}$$

*contains a rainbow $AP(3)$.*

For $n \not\equiv 2 \pmod 6$, the following coloring

$$c(i) := \begin{cases} A & \text{if } i \equiv 1 \pmod 6 \\ B & \text{if } i \equiv 4 \pmod 6 \\ C & \text{otherwise} \end{cases}$$

shows that Theorem 1 is the best possible. When $n = 6m + 2$, $m \in \mathbb{N}$, the coloring $\bar{c}$ shows the tightness of Theorem 1:

$$\bar{c}(i) := \begin{cases} A & \text{if } i \leq 2m+1 \text{ and } i \text{ is odd} \\ B & \text{if } i \geq 4m+2 \text{ and } i \text{ is even} \\ C & \text{otherwise} \end{cases}$$

Axenovich and Fon-Der-Flaass also demonstrated that for $k \geq 5$, no matter how large the smallest color class is, there is a $k$-coloring of $[n]$ with no rainbow $AP(k)$. Therefore, no statement similar to Theorem 1 holds for five or more colors. Their construction goes as follows: Let $n = 2mk$, $k \geq 5$. Subdivide $[n]$ into $k$ consecutive intervals of length $2m$ each, say $S_1, \ldots, S_k$, and let $t = \lfloor k/2 \rfloor$. Then, it is easy to see that the coloring

$$c(i) = \begin{cases} j & \text{if } i \in S_j \text{ and } j \neq t, j \neq t+2 \\ t & \text{if } i \in S_t \cup S_{t+2} \text{ and } i \text{ is even,} \\ t+2 & \text{if } i \in S_t \cup S_{t+2} \text{ and } i \text{ is odd.} \end{cases}$$

is equinumerous (the size of each color class is $n/k$) and does not contain a rainbow $AP(k)$. Notice that this coloring has large blocks of consecutive integers with the same color.

However, the question about the existence of equinumerous 4-colorings of $[n]$ without rainbow $AP(4)$s remained unresolved. In [AF04] a 4-coloring of $[n]$, $n = 10m + 1$, with the smallest color class of size $2m = (n-1)/5$ and no rainbow $AP(4)$ was constructed.

In this note, we settle the question.

**Theorem 2** *For every positive integer $n$, $n \equiv 0 \pmod 8$, there exists an equinumerous 4-coloring of $[n]$ with no rainbow $AP(4)$.*

In the next section, we present our construction. It is important to note that in this coloring there is a color which appears on consecutive integers. An important step in establishing the existence of a rainbow $AP(3)$ in every equinumerous 3-coloring of $[n]$ is proving that at least one of the colors is *recessive*, i.e., it does not appear on consecutive integers. Therefore, a natural way to possibly force the existence of a rainbow $AP(4)$ is to assume that every color is recessive. This is our motivation for the second construction, presented in Section 3, where (to our surprise) for every $n \equiv 0 \pmod{24}$, we construct an equinumerous 4-coloring of $[n]$ with no rainbow $AP(4)$ and no two consecutive integers having the same color.

In fact, our example provides an equinumerous colouring, in four colours, of $\mathbb{Z}_{24}$ which does not contain an AP(4). This is easily seen to extend to $\mathbb{Z}_{24k}$ for any $k \in \mathbb{N}$, and thus also curtails any hope of a $\mathbb{Z}_n$ analogue of Theorem 1 for AP(4)s.

## 2 Proof of Theorem 2

Let $n = 8m$, $m \in \mathbb{N}$. Define the coloring $\lambda$ as follows: for every $i \in [n]$, let

$$\lambda(i) := \begin{cases} A & \text{if } i \equiv 1 \pmod 4 \text{ and } i < 4m; \text{ or if } i \equiv 3 \pmod 4 \text{ and } i > 4m \\ B & \text{if } i \equiv 2 \pmod 4 \text{ and } i < 4m; \text{ or if } i \equiv 0 \pmod 4 \text{ and } i > 4m \\ C & \text{if } i \equiv 3 \pmod 4 \text{ and } i < 4m; \text{ or if } i \equiv 0 \pmod 4 \text{ and } i \leq 4m \\ D & \text{if } i \equiv 1 \pmod 4 \text{ and } i > 4m; \text{ or if } i \equiv 2 \pmod 4 \text{ and } i > 4m \end{cases}$$

It is immediately clear that every color class has exactly $2m$ elements, so $\lambda$ is equinumerous. The proof that $\lambda$ does not contain a rainbow $AP(4)$ will be a straightforward case analysis. Suppose that $\{x, y, z, w\}$ form a rainbow $AP(4)$; more precisely, $\lambda(x) = A$, $\lambda(y) = B$, $\lambda(z) = C$, and $\lambda(w) = D$. Then, $z \leq 4m < w$. We will assume that $x \leq 4m$, the other case is symmetric. Therefore, we have the following three possibilites:

**Case 1.** $x < y < 4m$. Since $\lambda(x) = A$ and $\lambda(y) = B$, then $x \equiv 1 \pmod 4$ and $y \equiv 2 \pmod 4$. There are three subcases according to the order of $x$, $y$, and $z$.

**Subcase 1a.** $x < y < z \leq 4m < w$.

Since $x + z \equiv 2y \pmod 4$, we have $z \equiv 3 \pmod 4$. Then, $x + w \equiv y + z \pmod 4$ implies $w \equiv 0 \pmod 4$, which is a contradiction with $\lambda(w) = D$.

**Subcase 1b.** $x < z < y < 4m < w$.

Then, $x + y \equiv 2z \pmod 4$ yields $2z \equiv 3 \pmod 4$, which is impossible.

**Subcase 1c.** $z < x < y < 4m < w$.

Then, $x + w \equiv 2y \pmod 4$ yields $w \equiv 3 \pmod 4$, which is a contradiction with $\lambda(w) = D$.

**Case 2.** $y < x < 4m$. Again, $\lambda(x) = A$ and $\lambda(y) = B$ imply $x \equiv 1 \pmod 4$ and $y \equiv 2 \pmod 4$. There are three subcases according to the order of $x$, $y$, and $z$.

**Subcase 2a.** $y < x < z \leq 4m < w$.

Since $y + z \equiv 2x \pmod 4$, we have $z \equiv 0 \pmod 4$. Then, $y + w \equiv x + z \pmod 4$ implies $w \equiv 3 \pmod 4$, which is a contradiction with $\lambda(w) = D$.

**Subcase 2b.** $y < z < x < 4m < w$.

Then, $y + x \equiv 2z \pmod 4$ yields $2z \equiv 3 \pmod 4$, which is impossible.

**Subcase 2c.** $z < y < x < 4m < w$.

Then, $y + w \equiv 2x \pmod 4$ yields $w \equiv 0 \pmod 4$, which is a contradiction with $\lambda(w) = D$.

**Case 3.** $x < 4m < y$. Since $\lambda(x) = A$ and $\lambda(y) = B$, then $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 4$. Now, there are four subcases according to the order of $x$ and $z$, and of $y$ and $w$.

**Subcase 3a.** $x < z \leq 4m < y < w$.

Then, $x + y \equiv 2z \pmod 4$ yields $2z \equiv 1 \pmod 4$, which is impossible.

**Subcase 3b.** $x < z \leq 4m < w < y$.

Since $x+w \equiv 2z \pmod 4$, we have $w \equiv 1 \pmod 4$, and, hence, $z \equiv 1 \pmod 4$. Then, $x+y \equiv z+w \pmod 4$ implies $y \equiv 1 \pmod 4$, which is a contradiction.

**Subcase 3c.** $z < x < 4m < y < w$.

This case is impossible, since $y + z \equiv 2x \equiv 2 \pmod 4$ and $y \equiv 0 \pmod 4$ yield $z \equiv 2 \pmod 4$, which contradicts $\lambda(z) = C$.

**Subcase 3d.** $z < x < 4m < w < y$.

Then, $x + y \equiv 2w \pmod 4$ yields $2w \equiv 1 \pmod 4$, which is impossible. □

## 3 A construction with recessive colors

First, we introduce a new notation: Given a 4-coloring $\nu = (\nu(0), \nu(1), \ldots, \nu(k-1)) \in \{A, B, C, D\}^k$ of $\mathbb{Z}_k$, let $\overline{\nu}$ denote the 4-coloring of $\mathbb{N}$ such that for every $i \in \mathbb{N}$, $\overline{\nu}(i) = \nu(i \pmod k)$.

Now, let $n = 24m$, $m \in \mathbb{N}$. Define the 4-coloring $\mu$ of $[n]$ as follows: for every $i \in [n]$, let

$$\mu(i) := \begin{cases} A & \text{if } i \equiv 3, 6, 9, 16, 18, 20 \pmod{24} \\ B & \text{if } i \equiv 1, 8, 10, 12, 19, 22 \pmod{24} \\ C & \text{if } i \equiv 5, 7, 13, 15, 21, 23 \pmod{24} \\ D & \text{if } i \equiv 0, 2, 4, 11, 14, 17 \pmod{24} \end{cases}$$

In other words, $\mu([n])$ is a prefix of $\overline{\nu}(\mathbb{N})$ of length $n$, where $\nu$ denotes the 4-coloring of $\mathbb{Z}_{24}$ given by:

$$\nu := (B, D, A, D, C, A, C, B, A, B, D, B, C, D, C, A, D, A, B, A, C, B, C, D).$$

It is immediately clear that every color class of $\mu$ has exactly $6m$ elements, so $\mu$ is equinumerous. Moreover, no two consecutive integers receive the same color. What remains to be checked is the non-existence of a rainbow $AP(4)$. Since $\mu([n])$ is a prefix of $\overline{\nu}(\mathbb{N})$, it suffices to show that there does not exist a rainbow $AP(4)$ in $\nu$, that is, a 4-tuple $(x, y, z, w)$, $x, y, z, w \in \mathbb{Z}_{24}$, and a *common difference* $d \in \mathbb{Z}_{24}$, such that

$$y \equiv x + d \pmod{24}, \;\; z \equiv x + 2d \pmod{24}, \text{ and } w \equiv x + 3d \pmod{24},$$

with $\nu(x)$, $\nu(y)$, $\nu(z)$, $\nu(w)$, being pairwise distinct. Notice that any such 4-tuple (with common difference $d$) yields $(w, z, y, x)$, another 4-tuple with the same property, whose (common) difference is $24 - d$. Hence, we can restrict our attention to 4-tuples with difference at most 12.

Next, it is easy to resolve the cases when the difference is an even number. Indeed, suppose there exists a rainbow $AP(4)$ $(x, y, z, w)$ in $\mathbb{Z}_{24}$, whose difference $d$ is an even number. Observe that $\nu(i) = C$ if and only if $i \equiv 5 \pmod 8$, or $i \equiv 7 \pmod 8$. So, if $x$ is an even number, then every element of the 4-tuple $(x, y, z, w)$ is even and, thus, is not colored with $C$. This contradicts $(x, y, z, w)$ being rainbow. If $x$ is an odd number, then $\{x \pmod 8, y \pmod 8, z \pmod 8, w \pmod 8\}$ is one of the following: $\{1, 1, 1, 1\}$, $\{3, 3, 3, 3\}$, $\{5, 5, 5, 5\}$, $\{7, 7, 7, 7\}$ (when $d \equiv 0 \pmod 8$, $x \equiv 1, 3, 5, 7 \pmod 8$); $\{1, 3, 5, 7\}$ (when $d \equiv 2, 6 \pmod 8$, $x \equiv 1, 3, 5, 7 \pmod 8$); $\{1, 5, 1, 5\}$ (when $d \equiv 4 \pmod 8$, $x \equiv 1, 5 \pmod 8$); or $\{3, 7, 3, 7\}$ (when $d \equiv 4 \pmod 8$, $x \equiv 3, 7 \pmod 8$). Therefore, either two or none of the elements of the rainbow $(x, y, z, w)$ receive color $C$ in $\nu$, which is a contradiction.

Finally, assume that the common difference $d$ is odd. Our coloring $\nu$ of $\mathbb{Z}_{24}$ does not contain a rainbow $AP(4)$ if and only if none of the sequences $\{\overline{\nu}(i+jd)\}_{j=0}^{\infty}$, $i \in [24]$, contains all the colors ($A$, $B$, $C$, and $D$) in four consecutive positions. We partition our analysis into three cases.

**Case 1.** $d \in \{1, 5\}$.

The case $d = 1$ is trivial, as every sequence $\{\overline{\nu}(i+j)\}_{j=0}^{\infty}$, $i \in [24]$, is just a suffix of $\overline{\nu}(\mathbb{N})$, which does not contain all the colors in four consecutive positions. If $d = 5$, then every sequence $\{\overline{\nu}(i+5j)\}_{j=0}^{\infty}$, $i \in [24]$, is a suffix of $\overline{\nu'}(\mathbb{N})$, where $\nu'$ is the same coloring as $\nu$, except that the colors $A$ and $D$ are interchanged.

**Case 2.** $d \in \{7, 11\}$.

If $d = 7$, then every sequence $\{\overline{\nu}(i+7j)\}_{j=0}^{\infty}$, $i \in [24]$, is a suffix of $\overline{\gamma}(\mathbb{N})$, where

$$\gamma := (B, B, C, B, C, B, B, D, A, A, C, A, C, A, A, B, D, D, C, D, C, D, D, A).$$

Clearly, no four consecutive positions receive pairwise distinct colors.

The case $d = 11$ is similar: every sequence $\{\overline{\nu}(i+11j)\}_{j=0}^{\infty}$, $i \in [24]$, is a suffix of $\overline{\gamma'}(\mathbb{N})$, where $\gamma'$ is the same coloring as $\gamma$, except that the colors $A$ and $D$ are interchanged.

**Case 3.** $d \in \{3, 9\}$. Unlike in the previous cases, each sequence $\{\overline{\nu}(i+dj)\}_{j=0}^{\infty}$ is periodic modulo 8, rather than 24, since the greatest common divisor of $d$ and 24 is 3.

If $d = 3$, then every sequence $\{\overline{\nu}(i+3j)\}_{j=0}^{\infty}$, $i \in [24]$, is a suffix of one of the following three sequences: $\overline{\beta^{(1)}}(\mathbb{N})$, $\overline{\beta^{(2)}}(\mathbb{N})$, $\overline{\beta^{(3)}}(\mathbb{N})$, where

$$\beta^{(1)} := (B, D, C, B, C, A, B, B),$$

$$\beta^{(2)} := (D, C, B, D, D, D, A, C),$$

$$\beta^{(3)} := (A, A, A, B, C, A, C, D).$$

Clearly, no four consecutive positions receive pairwise distinct colors.

The case $d = 9$ is similar; the reader can easily check that each sequence $\{\overline{\nu}(i+9j)\}_{j=0}^{\infty}$, $i \in [24]$, is a suffix of one of the above three sequences. □

# 4 Concluding remarks

We are still puzzled by the contrast discovered in [JL+03, AF04] and further sharpened in this note; namely, every equinumerous $k$-coloring of $[kn]$ contains a rainbow $AP(k)$ if and only if $k = 3$. We are not anywhere close to understanding this phenomenon.

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