A new upper bound for the bipartite Ramsey problem

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Abstract

We consider the following question: how large does n have to be to guarantee that in any two-colouring of the edges of the complete graph $K_{n,n}$ there is a monochromatic $K_{k,k}$? In the late seventies, Irving [5] showed that it was sufficient, for k large, that $n \geq 2^{k-1}(k-1)-1$. Here we improve upon this bound, showing that it is sufficient to take

$$n \geq (1+o(1))2^{k+1} \log k,$$

where the log is taken to the base 2.

1 Introduction

One of the classic questions of graph theory is to determine accurate bounds on the Ramsey number $r(k)$, the smallest number n such that, in any two-colouring of the complete graph on n vertices, there is guaranteed to be a clique of size k all of whose edges are the same colour. It is only known ([7], [2]) that

$$(1+o(1))\frac{\sqrt{2}}{e}k\sqrt{2}^k \leq r(k) \leq k^{-c\frac{\log k}{\log \log k}}4^k,$$

and it seems unlikely at present that either of these bounds can be significantly improved (an improvement of the exponentiated constant in either bound would be a major result).

The most natural bipartite analogue of this problem, introduced by Beineke and Schwenk [1], is to determine $b(k)$, which we define to be the smallest number n such that, in any two-colouring of the bipartite graph $K_{n,n}$, there is guaranteed to be a monochromatic biclique $K_{k,k}$. The known bounds in this case ([4], [5]) are, for $k \geq 21$,

$$(1+o(1))\frac{\sqrt{2}}{e}k\sqrt{2}^k \leq b(k) < 2^{k-1}(k-1).$$

While the proof of the lower bound is, like Spencer's lower bound [7] for ordinary Ramsey numbers, an application of the Lovász Local Lemma, the upper bound, due to Irving [5], is proved in a very different fashion to its counterpart. Indeed, the only Ramsey property that is necessary for the proof is the fact that one or other of the colour classes has density greater than $\frac{1}{2}$, and from there on the proof is that of a Turán-type theorem (or, as such theorems are usually called in the bipartite setting, a Zarankiewicz-type theorem).

In this note we make the following improvement upon Irving's bound:

Theorem 1

$$b(k) \leq (1+o(1))2^{k+1} \log k,$$

where the log *is taken to the base* 2.

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The proof splits into two pieces, the first of which is to observe that in order for a two-coloured bipartite graph $K_{m,n}$ to necessarily contain a monochromatic $K_{k,k}$, it is only necessary that one of m and n be very large. The other may be as small as k^2. A Turán version of this observation is the subject of the next section. Then, in section 3, we will show how this result may be applied to improve upon Irving's bound.

2 The Turán argument

The following lemma, which we will need to apply twice to prove our theorem, has many predecessors in the literature, perhaps the earliest being [6]. Note that when we say a function is $o_{a\to\infty;b,c,\cdots}(1)$, we mean that as a tends to infinity the function tends to 0, but in a way which is perhaps dependent upon the parameters $b, c, \cdots$.

Lemma 1 *Let $\omega(r)$ be a function of r that tends to infinity with r and suppose that G is a subgraph of the graph $K_{m,n}$ with density p, where $\epsilon \leq p \leq 1$, for some fixed ϵ. Then, provided that*

$$m \geq r^2\omega(r) \text{ and } n \geq (1 + o_{r\to\infty;\omega,\epsilon}(1))p^{-r}(s-1),$$

the graph G must contain a complete subgraph $K_{r,s}$.

Proof: Suppose that we have a bipartite graph $K_{m,n}$, based on the two vertex sets M and N, which does not contain a complete $K_{r,s}$. We shall show that this cannot be the case.

To begin, note that there can be at most

$$\binom{m}{r}(s-1)$$

$K_{r,1}$s (with the set of r vertices always chosen to be a subset of M), since there at most $\binom{m}{r}$ choices for the r vertices, and each such choice can be contained in at most $s-1$ subgraphs (otherwise we would have a $K_{r,s}$).

On the other hand, the number of $K_{r,1}$s is given by

$$\sum_{v\in N}\binom{d_v}{r}.$$

Applying Jensen's inequality tells us that

$$\begin{aligned}\sum_{v\in N}\binom{d_v}{r} &\geq n\binom{\frac{1}{n}\sum_{v\in N}d_v}{r}\\ &\geq n\binom{pm}{r}.\end{aligned}$$

Therefore we have that

$$n\binom{pm}{r} \leq \binom{m}{r}(s-1),$$

which implies that

$$n \leq \frac{\binom{m}{r}}{\binom{pm}{r}}(s-1).$$

Simplifying, and using the fact that $m \geq r^2\omega(r)$, we see that

$$n \leq (1 + o_{r\to\infty;\omega,\epsilon}(1))p^{-r}(s-1).$$

But this contradicts our assumption about n, so we're done. □

It is worth noting that a simple probabilistic argument gives us a subgraph of $K_{m,n}$, where

$$m = r^2 \text{ and } n = (1 + o_{r\to\infty}(1))\frac{p^{-r}s}{e(er)^{r/s}},$$

with density p or more which contains no $K_{r,s}$. We therefore see that Lemma 1 gives quite a good bound, especially when s is large compared with r.

3 The Ramsey argument

Note that all logs taken in this section we will be assumed to be to the base 2.

Proof: [Proof of Theorem 1] Suppose that we have a two-colouring of the edges of the complete graph $K_{n,n}$, based on the two vertex sets M and N. Then, for each vertex in M, we may choose an associated colour, red or blue, such that our vertex is connected by edges in this colour to at least half the vertices of N. Let the set of vertices whose associated colours are red or blue be M_R and M_B respectively. Then one of these sets, say M_R, has size greater than $\frac{1}{2}|M|$.

Now consider the red bipartite graph (which necessarily has density greater than $\frac{1}{2}$) lying between M_R and N. An application of Lemma 1, with $\omega(k) = \log k$, tells us that we can find a red biclique $K_{k^2 \log k, k-2\log k}$ provided that

$$|M_R| \geq (1 + o(1))2^{k-2\log k}k^2 \log k \text{ and } |N| \geq k^2 \log k,$$

both of which are easily seen to hold true provided that $n \geq (1 + o(1))2^{k+1} \log k$.

Now, let M' be the set of $k^2 \log k$ vertices in M_R which are contained in our red biclique, and let N' be the set of vertices in N which are not contained in this biclique. Consider the induced red bipartite subgraph on the vertex sets M' and N'. The degree of each vertex in M' is (since each such vertex is also in M_R) at least

$$\frac{|N|}{2} - (k - 2\log k),$$

so that the density of the induced subgraph is at least

$$\frac{1}{2} - \frac{k}{2^k}.$$

Therefore, applying Lemma 1 to this bipartite graph, we see that since

$$|M'| \geq k^2 \log k \text{ and } |N'| \geq (1 + o(1))2^{k+1} \log k,$$

we can find a red $K_{k,2\log k}$. Adding the vertices in $N \backslash N'$ to this graph then produces a complete red $K_{k,k}$, as required. □

4 Conclusion

A more general version of the bipartite Ramsey problem is to determine $b(k,l)$, the smallest number n such that in any two-colouring of $K_{n,n}$ there is guaranteed to be a monochromatic $K_{k,l}$. Applying Lemma 1 in the obvious way allows us to show that, for $k \leq l$,

$$b(k,l) \leq (1 + o_{k\to\infty}(1))2^k(l-1).$$

Thomason, however, has found a different approach [8], which allows one to show that

$$b(k,l) \leq 2^k(l-1)+1.$$

We cannot do better than this bound in general, though we do have the following natural analogue of Theorem 1 which does better than Thomason's result for a fairly wide range of values of k and l:

Theorem 2 *Suppose that $l \geq k$. Then*

$$b(k,l) \leq (1+o_{k\to\infty}(1))2^k\left(l-k+2\log k\right),$$

where the log *is taken to the base* 2.

The form of this result suggests a connection with the result of Füredi [3] on upper bounds for Zarankiewicz numbers. Recall that the Zarankiewicz number $z(n,n,k,l)$ is the maximum number of edges that one can have in a bipartite graph, both of whose vertex sets are of size n, without containing a $K_{k,l}$. What Füredi's result states is that

$$\lim_{n\to\infty}\frac{z(n,n,k,l)}{n^{2-1/k}} \leq (l-k+1)^{1/k},$$

improving upon the classical result, due to Kővári, Sós and Turán [6], that

$$\lim_{n\to\infty}\frac{z(n,n,k,l)}{n^{2-1/k}} \leq l^{1/k}.$$

Our result bears the same relation to Füredi's theorem as Irving's bound does to that of Kővári, Sós and Turán. However, whereas Irving's bound (other than the extra factor of 2) can be derived from a direct application of the latter, our result can only be derived from Füredi's by an appropriate reworking of his method (the error terms in the theorem itself being too large to make it directly applicable). Although the connection only emerged after the fact, the proof of Theorem 1 is such a reworking.

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