Hypergraph packing and sparse bipartite Ramsey numbers

David Conlon *

Abstract

We prove that there exists a constant c such that, for any integer Δ, the Ramsey number of a bipartite graph on n vertices with maximum degree Δ is less than $2^{c\Delta}n$. A probabilistic argument due to Graham, Rödl and Ruciński implies that this result is essentially sharp, up to the constant c in the exponent. Our proof hinges upon a quantitative form of a hypergraph packing result of Rödl, Ruciński and Taraz.

1 Introduction

For a graph G, the Ramsey number $r(G)$ is defined to be the smallest natural number n such that, in any two-colouring of the edges of K_n, there exists a monochromatic copy of G. That these numbers exist was proven by Ramsey [19] and rediscovered independently by Erdős and Szekeres [10].

Whereas the original focus was on finding the Ramsey numbers of complete graphs, in which case it is known that

$$\sqrt{2}^t \leq r(K_t) \leq 4^t,$$

the field has broadened considerably over the years. One of the most famous results in the area to date is the theorem, due to Chvatál, Rödl, Szemerédi and Trotter [7], that if a graph G, on n vertices, has maximum degree Δ, then

$$r(G) \leq c(\Delta)n,$$

where $c(\Delta)$ is just some appropriate constant depending only on Δ. Their proof makes use of the regularity lemma and because of this the bound it gives on the constant $c(\Delta)$ is (and is necessarily [11]) very bad.

The situation was improved somewhat by Eaton [8], who proved, by using a variant of the regularity lemma, that the function $c(\Delta)$ may be taken to be of the form $2^{2^{c\Delta}}$. Such a result follows for bipartite graphs from an earlier theorem of Komlós (Corollary 7.6, [15]).

This bound was not best for long, for Graham, Rödl and Ruciński proved [13], by a beautiful method which avoids any use of the regularity lemma, that

$$c(\Delta) \leq 2^{c\Delta(\log \Delta)^2}.$$

*St John's College, Cambridge, CB2 1TP, United Kingdom.

For bipartite graphs they were able to do even better [14], showing that

$$c(\Delta) \leq 2^{c\Delta \log \Delta}.$$

Moreover, they proved that these results were almost sharp in that there exists a bipartite graph G on n vertices, with maximum degree Δ, for which

$$r(G) \geq 2^{c\Delta} n.$$

Our main result in this paper is a proof that for bipartite graphs this latter bound is, up to the constant in the exponent, the correct one.

Theorem 1 *For all bipartite graphs G on n vertices with maximum degree Δ*

$$r(G) \leq 2^{(2+o_\Delta(1))\Delta} n.$$

The proof is related to a lemma whose use has become very common in Ramsey theory of late (see for example [12, 16, 22, 2, 18, 24, 23]). This lemma states that if we have a bipartite graph $G = (A, B; E)$ of density α, then, given a fixed constant $\beta < \alpha$, we can find a large induced subgraph (how large depends on the choice of β) on vertex sets A' and B such that every subset of A' of size r has at least $\beta^r |B|$ neighbours in B. We give a non-standard version of this lemma.

Lemma 1 *Let $G = (A, B; E)$ be a bipartite graph with $|A| = |B| = N$. Suppose that the graph has density α, that is that there are αN^2 edges. Then, for any $\beta < \alpha$ and any $r, s \in \mathbb{N}$, there exists a set $A' \subset A$ of size greater than $\frac{\alpha^s}{2} N$ which contains at most $\frac{4\beta^{rs}}{\alpha^s}\binom{N}{r}$ r-tuples which have less than $\beta^r N$ common neighbours.*

Proof: For each vertex v, let $d(v)$ be its degree. For any given vertices $b_1, \cdots, b_s \in B$, let I be the set of common neighbours. What is the expectation of $|I|$ over all possible random choices of $b_1, \cdots, b_s$ (allowing repetitions)? If we apply Jensen's inequality, we see that

$$\begin{aligned}
\mathbb{E}(|I|) &= \sum_{v \in A} \mathbb{P}(v \in I) = \sum_{v \in A} \left(\frac{d(v)}{N} \right)^s \\
&\geq \frac{N \left(\frac{\sum_{v \in A} d(v)}{N} \right)^s}{N^s} = \frac{N(\alpha N)^s}{N^s} = \alpha^s N.
\end{aligned}$$

Therefore we see that with probability at least $\alpha^s/2$ we have $|I| \geq \frac{\alpha^s}{2} N$.

We also have that the expected number of bad r-tuples, that is r-tuples which have less than $\beta^r N$ common neighbours, is at most

$$\beta^{rs} \binom{N}{r}.$$

To see this note simply that any such bad r-tuple has at most $\beta^r n$ neighbours in B and therefore the probability that such an r-tuple be chosen is β^{rs}. Thus, by Markov's inequality, the probability that the number of bad r-tuples is larger than $\frac{4\beta^{rs}}{\alpha^s}\binom{N}{r}$ is at most $\alpha^s/4$.

We therefore see that with positive probability we may choose a set A' of size at least $\frac{\alpha^s}{2}N$ which contains at most $\frac{4\beta^{rs}}{\alpha^s}\binom{N}{r}$ bad r-tuples. □

This lemma is normally applied by choosing an appropriate constant s and showing that the number of bad r-tuples is smaller than the number of vertices. If we now delete one vertex for each bad r-tuple we get a graph which contains no bad r-tuples. This kind of graph is very well-behaved and we can easily find embedded graphs within it. For example, if we have such a graph for $r = \Delta$ and we want to find a graph of maximum degree Δ within it, then, since we know that every set of Δ points has at least $\beta^{\Delta}N$ neighbours, we can easily embed any graph $H = (U, V; F)$ with maximum degree Δ for which $|U| \leq |A'|$ and $|V| \leq \beta^{\Delta}N$. The problem is that, typically, to be able to delete the bad Δ-tuples, we need to take s to be of the order of $\log N$ and this implies that $|A'|$ will be smaller than N^{ϵ} for some $\epsilon < 1$.

The main concern of this paper is to avoid this annoying loss of size. What we will do is show that the set of bad Δ-tuples are not such a hassle as at first appears and that we may avoid them without deleting them outright. Indeed, suppose that we are given a bipartite graph $H = (U, V; F)$ with maximum degree Δ. Then, for each vertex $v \in V$, let $D(v)$ be the set of neighbours. Each such set may be considered to be a hyperedge in U. Since $|D(v)| \leq \Delta$ for all v, we may, by adding some dummy vertices, assume that the resulting hypergraph is Δ-uniform. Now, given a two-coloured graph $G = (A, B; E)$, one of the colours, say red, has density greater than $1/2$. Using our lemma, we see that there is a subset A' of A of size $|A|/2^s$ which contains at most $\beta^{s\Delta}\binom{N}{\Delta}$ bad Δ-tuples, that is, Δ-tuples with less than $\beta^{\Delta}N$ neighbours. The set of bad Δ-tuples again defines a Δ-uniform hypergraph. If now we could embed the hypergraph defined by H in the complement of that defined by the bad Δ-tuples, this would allow us to embed our graph H within the red subgraph of our complete graph, because, choosing $N \geq n/\beta^{\Delta}$, we see that every collection of Δ vertices in U which have a common neighbour in H will have n common neighbours in G. This naturally leads us to the following question: under what circumstances can we pack two hypergraphs? This will be the subject of the next section.

2 Packing hypergraphs

Let A and B be k-uniform hypergraphs, on vertex sets V and V' respectively, and suppose, without loss of generality, that $|V| \leq |V'|$. We say that an injection $\psi : V \to V'$ is a packing of A and B if the edge sets of B and $\psi(A)$ are disjoint.

Recall that, for a hypergraph H, $\Delta_j(H)$ denotes the maximum j-tuple degree in H, that is,

$$\Delta_j(H) = \max_{T \in [V]^j} \deg_H(T)$$

where

$$\deg_H(T) = |\{e \in H : T \subset e\}|.$$

For $j = 1$, we will simply write $\Delta(H) = \Delta_1(H)$.

The following result, which is the natural generalisation of the well-known graph packing result of Spencer and Sauer [21] to hypergraphs, was proven by Rödl, Ruciński and Taraz (Proposition 2.1, [20]):

Proposition 1 *Let A and B be k-uniform hypergraphs on no more than n vertices. Then, if*

$$\Delta(A)\Delta_{k-1}(B) + \Delta(B)\Delta_{k-1}(A) \leq n - k + 1,$$

there is a packing of A and B.

For hypergraphs, this isn't a particularly strong result, but it may be improved considerably by being more careful. We will need some more notation:

Given a k-uniform hypergraph H, let H^{k-1} be the shadow of H on the $(k-1)$st level, that is, the family of all $(k-1)$-element sets that are contained in any hyperedge of H. Now, given two k-uniform hypergraphs A and B, define

$$B_v = \{e \in [V]^{k-1} : e \cup \{v\} \in B\}$$

and

$$D_\psi(B) = \max_{v \in V} |B_v \cap \psi(A^{k-1})|.$$

We may now state a more useful packing result, due again to Rödl, Ruciński and Taraz (Lemma 2.1, [20]):

Proposition 2 *Let A and B be k-uniform hypergraphs on no more than n vertices. If there exists an attempted packing ψ such that*

$$\Delta(A)\Delta_{k-1}(B) + D_\psi(B)\Delta_{k-1}(A) + |B \cap \psi(A)|\Delta(A)\Delta_{k-1}(A) \leq n - k + 1$$

then there is a packing of A and B.

Rödl, Ruciński and Taraz then used this result and an application of the probabilistic method in order to prove the following beautiful packing result (Theorem 2.1, [20]):

Theorem 2 *For all integers $k \geq 2$ and $\Delta \geq 1$, there exist $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that, if A and B are k-uniform hypergraphs on no more than n vertices, where $n > n_0$, and*
(i) $\Delta(A) \leq \Delta$,
(ii) $\Delta(B) \leq \epsilon n^{k-1}$,
(iii) $\Delta_{k-1}(B) \leq \epsilon n$,
then there is a packing of A and B.

This is already very close to satisfying our needs. We would like to prove that we can pack a sparse hypergraph with bounded maximum degree Δ and a hypergraph with at most ϵn^k edges, for some appropriately chosen ϵ. Unfortunately we don't really have any local control over the maximum degree of the larger hypergraph so we can't apply this theorem directly. There is, however, one further packing theorem in [20] which proves more equal to the task. To state this theorem we will again need some notation:

For a k-uniform hypergraph H and a real number $\epsilon > 0$, let $H_{k-1}^{(\epsilon)}$ be the $(k-1)$-uniform hypergraph consisting of all $(k-1)$-element sets T with $\deg_H(T) > \epsilon n$. Moreover, for each j with $2 \leq j \leq k$, define $H_{j-1}^{(\epsilon)}$ to be the $(j-1)$-uniform hypergraph consisting of all $(j-1)$-element sets T with $\deg_{H_j^{(\epsilon)}}(T) > \epsilon n$. The theorem of Rödl, Ruciński and Taraz (Theorem 2.2, [20]) is now as follows:

Theorem 3 *For all integers $k \geq 2$ and $\Delta \geq 1$, there exist $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that, if A and B are k-uniform hypergraphs on no more than n vertices, where $n > n_0$, and*

(i) $\Delta(A) \leq \Delta$,

(ii) $B_1^{(\epsilon)} = \emptyset$,

then there is a packing of A and B. Moreover, ϵ and n_0 may be taken to be $\frac{1}{100k^c\Delta^d 2^{2k}}$ and $100k^c\Delta^d 2^{2k}$ respectively, where c and d are just some constants.

This theorem is exactly as stated by Rödl, Ruciński and Taraz except for the quantitative bounds on ϵ and n_0 that we have given. In order to verify that this quantitative bound does hold we will give a detailed sketch of the proof concentrating on those aspects where extra care is necessary.

Proof: For $k = 2$ the theorem follows from the result of Sauer and Spencer. For $k \geq 3$, we will prove the theorem by inductively packing $B_j^{(\epsilon)}$ and A^j for each $j = 2, \cdots, k$. We will study the properties of a randomly chosen embedding $\psi : V \leftarrow V'$.

Note that

$$|B_j^{(\epsilon)}| \leq \frac{n|B_{j-1}^{(\epsilon)}| + \epsilon n\binom{n}{j-1}}{j},$$

and therefore, by induction,

$$|B_j^{(\epsilon)}| \leq \frac{\epsilon(j-1)n^j}{j!}.$$

Moreover, the size of the j-shadow A^j of A satisfies

$$|A^j| \leq \binom{k}{j}\Delta n \leq 2^k \Delta n.$$

Thus the expected number of conflicts between $B_j^{(\epsilon)}$ and $\psi(A^j)$ is at most

$$\frac{|A^j||B_j^{(\epsilon)}|}{\binom{n}{j}} \leq 2\epsilon k 2^k \Delta n,$$

provided $n \geq k^3$. Therefore, by Markov's inequality, we have, with probability $1/2k$, that

$$|B_j^{(\epsilon)} \cap \psi(A^j)| \leq 4\epsilon k^2 2^k \Delta n.$$

Now, for each vertex v of the hypergraph B, let B_v^j be the $(j-1)$-hypergraph

$$B_v^j = \{e \in [V]^{k-1} : e \cup \{v\} \in B_j^{(\epsilon)}\}$$

and

$$D_\psi^j(B) = \max_{v \in V} |B_v^j \cap \psi(A^{j-1})|.$$

For each j, we have that

$$|B_v^j| \leq \frac{\epsilon(j-1)n^{j-1}}{(j-1)!}.$$

This latter inequality, and more specifically the factorial factor in the denominator, is crucial for our purposes (and was ignored in [20]). The proof is similar to that bounding $|B_j^{(\epsilon)}|$ above, and follows from the simple inequality

$$|B_v^j| \leq \frac{n|B_v^{j-1}| + \epsilon n\binom{n}{j-2}}{j-1}.$$

Define X_v^j to be the random variable given by

$$X_v^j = |B_v^j \cap \psi(A^{j-1})|.$$

We have that

$$\mathbb{E}(X_v^j) = \frac{|B_v^j||A^{j-1}|}{\binom{n}{j-1}} \leq 2\epsilon k 2^k \Delta n,$$

provided again that $n \geq k^3$.

At this stage a rather involved application of Chernoff's inequality (which we omit, since it is almost exactly the same as that in [20]) implies that, for some appropriate c and d and $n \geq \frac{100k^c\Delta^d}{\epsilon}$, we have, with probability $1 - 1/4kn$,

$$X_v^j \leq 4\epsilon k 2^k \Delta n.$$

Therefore, with probability $1 - 1/4k$, we have

$$D_\psi(B_j^{(\epsilon)}) \leq 4\epsilon k 2^k \Delta n.$$

We see, therefore, that with non-zero probability, we may choose a mapping ψ_0 such that, for all $j = 2, \cdots, k$, the number of conflicts

$$|B_j^{(\epsilon)} \cap \psi_0(A^j)| \leq 4\epsilon k^2 2^k \Delta n,$$

and also

$$D_{\psi_0}(B_j^{(\epsilon)}) \leq 4\epsilon k 2^k \Delta n.$$

The remainder of the proof will be concerned with showing that we may remove conflicts of all orders (that is, all intersections of $B_j^{(\epsilon)}$ with $\psi_0(A^j)$). We will start by removing conflicts of small order, by switching appropriate 'bad' vertices with 'good' ones. To be more precise, for each conflict, of any possible order, choose a vertex involved in the conflict. There are at most $4\epsilon k^3 2^k \Delta n$ such vertices. Every time that we switch a vertex we will be switching one of these with some other vertex. We will do this so as to ensure that no extra conflicts of smaller order are created, but conflicts of higher order might well be. In any case, there will always be a set of size at most

$4\epsilon k^3 2^k \Delta n$ which represents all conflicts. By the time we remove all conflicts of order k we will have no conflicts remaining so we will be done.

We proceed by induction. Since there are no conflicts of order 1, there is nothing to do in this case. So suppose that we have eliminated all conflicts of order $i < j$ and that we are trying to eliminate conflicts of order j. Note that at this step our initial embedding ψ_0 may have been replaced by another embedding ψ_0, which, as we shall see, has related properties. Let $T \in B_j^{(\epsilon)}$ be a conflict of size j. We would like to swap one of the vertices x of T (which must be one of those that we have fixed from the start) with some other vertex y without producing any new conflicts of size j or less. If we cannot swap x with y then one of two possibilities must hold. For some $i \leq j$, either there exists a T' such that

$$T' \cup \{x\} \in \psi(A^i), T' \cup \{y\} \in B_i^{(\epsilon)},$$

or there exists T'' for which

$$T'' \cup \{x\} \in B_i^{(\epsilon)}, T'' \cup \{y\} \in \psi(A^i).$$

The question now arises: how many such vertices y can be ruled out by these conditions? For any i there are at most $2^k\Delta$ choices of T' for which $T' \cup \{x\} \in A^i$ and because $T' \notin B_{i-1}^{(\epsilon)}$ (by the induction hypothesis), at most $\epsilon k 2^k \Delta n$ possibilities for y are ruled out by the first case (the extra k comes from counting over the i). In the second case there are at most $|B_x^i \cap \psi(A^{i-1})|$ choices for T'' and any such choice gives rise to at most $k\Delta$ choices of y (because T'' is contained in at most Δ edges of A). It is tempting now to use our bound on $D_\psi(B)$, but we must be careful. Any switch that we have done may potentially increase the maximum value of $|B_v^i \cap \psi(A^{i-1})|$ by $\Delta(A^{i-1})$ (any vertex may belong to this many sets in A^{i-1}) and we may make as many as $4\epsilon k^4 2^k \Delta n$ switchings overall (changing each vertex in each turn). Therefore the value of $|B_x^i \cap \psi(A^{i-1})|$ may be as large as

$$D_{\psi_0}(B_i^{(\epsilon)}) + 4\epsilon k^4 \Delta(A^{i-1}) 2^k \Delta n \leq 4\epsilon k 2^k \Delta n + 4\epsilon k^4 2^{2k} \Delta^2 n,$$

since $\Delta(A^i) \leq 2^k\Delta$. Therefore, in the second case, there are at most

$$4\epsilon k^3 2^k \Delta^2 n + 4\epsilon k^6 2^{2k} \Delta^3 n$$

bad choices for y (we get an extra k from counting over the i). If we now choose

$$\epsilon = \frac{1}{15k^6\Delta^3 2^{2k}}$$

we find that the total number of bad choices for y is smaller than $n - k + 1$ and therefore we can swap in such a way as to remove our conflict of size j. The result therefore follows. □

The theorem that we will apply to prove our main result is the following corollary of Theorem 2.4:

Corollary 1 *For all integers $k \geq 2$ and $\Delta \geq 1$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that, if A and B are k-uniform hypergraphs on $n/2$ and n vertices respectively, and*
(i) $\Delta(A) \leq \Delta$,
(ii) $|B| \leq \delta\binom{n}{k}$,
then there is a packing of A and B. Moreover, δ and n_0 may be taken to be $\frac{1}{100^k k^{ck} \Delta^{dk} 2^{2k^2}}$ and $200k^c\Delta^d 2^{2k}$ respectively, where c and d are just some constants.

Proof: Note that

$$|B_j^{(\epsilon)}| \geq \frac{\epsilon n |B_{j-1}^{(\epsilon)}|}{j},$$

and therefore

$$|B_1^{(\epsilon)}| \leq \frac{\delta n}{\epsilon^{k-1}}.$$

If we choose $\delta \leq \epsilon^{k-1}/4$, we then see that $|B_1^{(\epsilon)}| \leq n/2$. Removing all of these points we have $B_1^{(\epsilon)} = \emptyset$.

Applying Theorem 2.4 to the graph that remains, we see that there exists a packing, as required. □

3 Finding sparse bipartite graphs

We are now ready to give the proof of Theorem 1.1. More precisely, we prove the following result:

Theorem 4 *There exists a constant C such that, for a bipartite graph H with maximum degree Δ on $n \geq 2^{2\Delta + C\log\Delta}$ vertices,*

$$r(H) \leq 2^{2\Delta + C\log\Delta} n.$$

Proof: Suppose that we have a bipartite graph $G = (A, B; E)$ with $|A| = |B| = N$ the edges of which have been coloured red and blue. One of the colours, say red, has density greater than 1/2. If we apply Lemma 1.2 to this graph, with $r = \Delta$, $s = 2\Delta$ and

$$\beta = \frac{1}{2^{2 + C\log\Delta/\Delta}},$$

for some C to be chosen large, we find a set A' of size at least $N/2^{2\Delta+1}$ which contains at most

$$\frac{1}{2^{4\Delta^2 + C'\Delta\log\Delta}} \binom{N}{\Delta} \leq \frac{1}{2^{2\Delta^2 + C''\Delta\log\Delta}} \binom{|A'|}{\Delta}$$

bad Δ-tuples, for some C', C'' depending only on C.

Now, associate to the graph $H = (U, V; F)$, with $|U|, |V| \leq n$, the Δ-uniform hypergraph whose edge set is the set of neighbours $D(v)$ of any vertex $v \in V$. If we add dummy vertices to ensure that every vertex has a corresponding hyperedge the resulting hypergraph has at most Δn vertices. Now, by Corollary 2.5, applied with $k = \Delta$, we see that if we choose C'' large enough (slightly larger than $c + d$ will do), and take

$$\frac{N}{2^{2\Delta+1}} \geq 2\Delta n,$$

we may pack the set of bad Δ-tuples and the hypergraph formed by H.

Fix such a packing. Then, for each hyperedge corresponding to a vertex $v \in V$, there are at least $N/2^{2\Delta + C\log\Delta}$ vertices to which they are connected. Taking

$$N \geq 2^{2\Delta + C\log\Delta} n,$$

we see that there are always n vertices to choose from, so even if we have already chosen some of these vertices there will always be some left. We therefore see that we can find a copy of H in the red component of G, as required. □

Some extra consideration now allows us to tidy up this result to get our main result:

Corollary 2 *There exists a constant D such that, for a bipartite graph H with maximum degree Δ on n vertices,*

$$r(H) \leq 2^{2\Delta + D\log\Delta} n.$$

Proof: For $n \geq 2^{2\Delta + C\log\Delta}$, the result follows immediately from Theorem 3.1, which tells us that

$$r(H) \leq 2^{2\Delta + C\log\Delta} n.$$

For $n < 2^{2\Delta + C\log\Delta}$, we use Lemma 1.2 in the more usual way, by simply removing all the bad Δ-tuples. More precisely, suppose we are given a two-colouring of a bipartite graph $G = (A, B; E)$ with $|A| = |B| = N$, and suppose also that the colour red has density greater than 1/2. Applying Lemma 1 with $r = \Delta$, $s = 2\Delta + C'\log\Delta$ (for some C' to be chosen later) and $\beta = 1/4$ tells us that we have a set $A' \subset A$ with size at least $N/2^{2\Delta + C'\log\Delta + 1}$ containing at most

$$\frac{1}{2^{4\Delta^2 + 2C'\Delta\log\Delta - O(\Delta)}} \binom{N}{\Delta}$$

bad Δ-tuples. But for

$$N = 2^{2\Delta + C'\log\Delta + 1} n \leq 2^{4\Delta + (C + C')\log\Delta + 1},$$

this is necessarily less than 1 (provided C' is chosen appropriately), so we have no bad Δ-tuples. Therefore, noting that every Δ-tuple in A has a set of common neighbours of size $N/2^{2\Delta}$, and taking $N = 2^{2\Delta + C'\log\Delta} n$, we're done. □

4 Concluding remarks

It seems unlikely that our method can be extended to improve upon the general upper bound for sparse Ramsey numbers. It would, however, be very interesting to obtain the correct bound for this problem as it could potentially allow a solution to the following problem of Erdős (see [6], [9]):

Problem 1 *Does there exist a constant c such that*

$$r(G) \leq 2^{c\sqrt{e(G)}}?$$

The reason underlying this belief is that the best known result on this problem,

$$r(G) \leq 2^{c\sqrt{e(G)}\log e(G)},$$

due to Alon, Krivelevich and Sudakov [2], is proved by applying the methods of Graham, Rödl and Ruciński [13]. There is, however, no guarantee that the determination of the correct asymptotic for sparse Ramsey numbers would yield the full conjecture. Accordingly, the interesting question of determining the correct upper bound for sparse Ramsey numbers must be stated as another independent open problem:

Problem 2 *Does there exist a constant c such that, for all graphs G on n vertices with maximum degree Δ,*

$$r(G) \leq 2^{c\Delta} n?$$

Note added in proof. It recently came to the author's attention that the main theorem of this paper, that for any bipartite graph G on n vertices with maximum degree Δ the Ramsey number $r(G)$ is smaller than $2^{c\Delta}n$, has been proved simultaneously and independently by Jacob Fox and Benny Sudakov. They give the bound $r(G) \leq \Delta 2^{\Delta+5} n$.

References

[1] Alon, N. (1994) Subdivided graphs have linear Ramsey numbers. *J. Graph Theory* **18** 343–347.

[2] Alon, N., Krivelevich, M. and Sudakov, B. (2003) Turán numbers of bipartite graphs and related Ramsey-type questions. *Combinatorics, Probability and Computing* **12** 477–494.

[3] Beck, J. (1983) An upper bound for diagonal Ramsey numbers. *Studia Sci. Math. Hungar.* **18** 401–406.

[4] Burr, S. A. and Erdős, P. (1975) On the magnitude of generalized Ramsey numbers for graphs. In Infinite and Finite Combinatorics, vol. 1 *Colloq. Math. Soc. János Bolyai* **10** 214–240.

[5] Chen, G. and Schelp, R. (1993) Graphs with linearly bounded Ramsey numbers. *J. Combin. Theory Ser. B* **57** 138–149.

[6] Chung, F. and Graham, R. L. (1998) *Erdős on graphs: His legacy of unsolved problems.* A.K. Peters Ltd., Wellesley, MA.

[7] Chvatál, V., Rödl, V., Szemerédi, E. and Trotter Jr., W. T. (1983) The Ramsey number of a graph with bounded maximum degree. *J. Combin. Theory Ser. B* **34** 239–243.

[8] Eaton, N. (1998) Ramsey numbers for sparse graphs. *Discrete Math.* **185** 63–75.

[9] Erdős, P. (1984) On some problems in graph theory, combinatorial analysis and combinatorial number theory. *Graph theory and combinatorics (Cambridge, 1983)* Academic Press, London, 1–17.

[10] Erdős, P. and Szekeres, G. (1935) A combinatorial problem in geometry. *Compositio Mathematica* **2** 463–470.

[11] Gowers, W. T. (1997) Lower bounds of tower type for Szemerédi's uniformity lemma. *Geom. Funct. Anal.* **7** 322–337.

[12] Gowers, W.T́. (1998) A new proof of Szemerédi's theorem for arithmetic progressions of length 4. *Geom. Funct. Anal.* **8** 529–551.

[13] Graham, R. L., Rödl, V. and Ruciński, A. (2000) On graphs with linear Ramsey numbers. *J. Graph Theory* **35** 176–192.

[14] Graham, R. L., Rödl, V. and Ruciński, A. (2001) On bipartite graphs with linear Ramsey numbers. *Combinatorica* **21** 199–209.

[15] Komlós, J. and Simonovits, M. (1996) Szemerédi's regularity lemma and its applications in graph theory. *Combinatorics, Paul Erdős is Eighty (Volume 2), Keszthely (Hungary), 1993,* Budapest, 295–352.

[16] Kostochka, A. and Rödl, V. (2001) On graphs with small Ramsey numbers. *J. Graph Theory* **37** 198–204.

[17] Kostochka, A. and Rödl, V. (2004) On graphs with small Ramsey numbers, II. *Combinatorica* **24** 389–401.

[18] Kostochka, A. and Sudakov, B. (2003) On Ramsey numbers of sparse graphs. *Combinatorics, Probability and Computing* **12** 627–641.

[19] Ramsey, F. P. (1930) On a problem of formal logic. *Proc. London Math. Soc. Ser. 2* **30** 264–286.

[20] Rödl, V., Ruciński, A. and Taraz, A. (1999) Hypergraph Packing and Graph Embedding. *Combinatorics, Probability and Computing* **8** 363–376.

[21] Sauer, N. and Spencer, J. (1978) Edge disjoint placement of graphs. *J. Combin. Theory Ser. B* **25** 295–302.

[22] Shi, L. (2001) Cube Ramsey numbers are polynomial. *Random Struct. Algorithms* **19** 99–101.

[23] Shi, L. (2007) The tail is cut for Ramsey numbers of cubes. *Discrete Math.* **307** 290–292.

[24] Sudakov, B. (2003) A few remarks on Ramsey-Turán-type problems. *J. Combin. Theory Ser. B* **88** 99–106.