Lines in Euclidean Ramsey theory

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Abstract

Let ℓ_m be a sequence of m points on a line with consecutive points of distance one. For every natural number n, we construct a red/blue-coloring of $\mathbb{E}^n$ containing no red copy of ℓ_2 and no blue copy of ℓ_m for any $m \geq 2^{cn}$. This is best possible up to the constant c in the exponent. It also answers a question of Erdős, Graham, Montgomery, Rothschild, Spencer and Straus from 1973. They asked if, for every natural number n, there is a set $K \subset \mathbb{E}^1$ and a red/blue-coloring of $\mathbb{E}^n$ containing no red copy of ℓ_2 and no blue copy of K.

1 Introduction

Let $\mathbb{E}^n$ denote n-dimensional Euclidean space, that is, $\mathbb{R}^n$ equipped with the Euclidean distance. Following Erdős, Graham, Montgomery, Rothschild, Spencer and Straus [4], we study the following question.

Question 1.1 *For which subsets $K \subset \mathbb{E}^n$ does every red/blue-coloring of $\mathbb{E}^n$ contain a red pair of points of distance one or a blue isometric copy of K?*

In what follows, we will write ℓ_m for a sequence of m points on a line with consecutive points of distance one and $\mathbb{E}^n \longrightarrow (\ell_2, K)$ if every red/blue-coloring of $\mathbb{E}^n$ contains either a red copy of ℓ_2 or a blue copy of K, where a copy of a set will always mean an isometric copy. Conversely, $\mathbb{E}^n \not\longrightarrow (\ell_2, K)$ expresses the fact that there is some red/blue-coloring of $\mathbb{E}^n$ which contains neither a red copy of ℓ_2 nor a blue copy of K.

The problem of determining which n and K satisfy the relation $\mathbb{E}^n \longrightarrow (\ell_2, K)$ has received considerable attention, with a particular focus on small values of n. For example, Erdős et al. [4] showed that $\mathbb{E}^2 \longrightarrow (\ell_2, \ell_4)$ and $\mathbb{E}^2 \longrightarrow (\ell_2, K)$ for any three-point set K. Juhász [8] later improved the latter result to cover all four-point planar sets, while just recently Tsaturian [14] improved the former result by showing that $\mathbb{E}^2 \longrightarrow (\ell_2, \ell_5)$. In three dimensions, Iván [7] showed that $\mathbb{E}^3 \longrightarrow (\ell_2, K)$ for any five-point set $K \subset \mathbb{E}^3$. The particular case where $K = \ell_5$ was recently improved by Arman and Tsaturian [1], who showed that $\mathbb{E}^3 \longrightarrow (\ell_2, \ell_6)$.

On the other hand, Csizmadia and Tóth [2] identified a set K of 8 points in the plane, namely, a regular heptagon with its center, such that $\mathbb{E}^2 \not\longrightarrow (\ell_2, K)$. This improved a result of Juhász [8], who had previously identified a set K of 12 points with the same property. Our chief concern in this paper will be with extending these results to higher dimensions by studying the smallest possible size of a set $K \subset \mathbb{E}^n$ such that $\mathbb{E}^n \not\longrightarrow (\ell_2, K)$.

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In general, $|K|$ can be unbounded in terms of n and still satisfy $\mathbb{E}^n \longrightarrow (\ell_2, K)$. For example, any subset K of the unit sphere in $\mathbb{E}^n$ satisfies $\mathbb{E}^n \longrightarrow (\ell_2, K)$. Indeed, in a red/blue-coloring of $\mathbb{E}^n$, if there is no red point, then we clearly get a copy of K, while if there is a red point, then the sphere of radius one around that point must be blue, so we again get a blue copy of K.

However, our main result shows that under some mild conditions, a set $K \subset \mathbb{E}^n$ such that $\mathbb{E}^n \longrightarrow (\ell_2, K)$ can have size at most exponential in n. To state the result, we say that a point set $S \subset \mathbb{E}^n$ is *t-separated* if any two points in S have distance at least t.

Theorem 1.1 *If $R > 1$ and K is a 1-separated set of points in $\mathbb{E}^n$ with diameter at most R and $|K| > 10^{4n} \log R$, then $\mathbb{E}^n \not\longrightarrow (\ell_2, K)$.*

In particular, for $m = 10^{5n}$, we see that $\mathbb{E}^n \not\longrightarrow (\ell_2, \ell_m)$. This simple corollary is already enough to answer a problem raised by Erdős et al. [4], namely, whether, for every natural number d, there is a natural number n depending only on d such that $\mathbb{E}^n \to (\ell_2, K)$ for every $K \subset \mathbb{E}^d$. Erdős et al. state that they expect the answer to this question to be negative and our result confirms this already for $d = 1$, a special case stressed in [4], showing that n must grow logarithmically in the size of $|K|$.

The exponential dependence in Theorem 1.1, and hence the logarithmic dependence above, is also necessary. To see this, we combine a proof idea originating in the work of Erdős et al. [4] with a result from the seminal paper of Frankl and Wilson [5], saying that there exist positive constants C and ϵ such that, for each natural number n, there is a point set $P \subset \mathbb{E}^n$ of size at most C^n such that any subset of P of size at least $|P|/(1+\epsilon)^n$ contains two points of distance one.

Indeed, suppose that $K \subset \mathbb{E}^n$ is a set of size at most $(1+\epsilon)^n$ and there is a red/blue-coloring of $\mathbb{E}^n$ with no blue copy of K. Then, for each $p \in P$, where $P \subset \mathbb{E}^n$ is the set given by the result of Frankl and Wilson described above, the set $p + K$ is a copy of K, so at least one of the points in this set, say $p + k_p$, is colored red. By the pigeonhole principle, since K has size at most $(1+\epsilon)^n$, there must be a set $P' \subset P$ of size at least $|P|/(1+\epsilon)^n$ such that $k_{p'}$ is the same for all $p' \in P'$. If we write k_0 for this common value, we have that $p' + k_0$ is red for all $p' \in P'$. Moreover, by the choice of P, there are two points p'_1 and p'_2 in P' of distance one, so $p'_1 + k_0$ and $p'_2 + k_0$ are red points of distance one. We therefore have the following result.

Theorem 1.2 *There exists a positive constant c' such that $\mathbb{E}^n \longrightarrow (\ell_2, K)$ for any set $K \subset \mathbb{E}^n$ of size at most $2^{c'n}$.*

In fact, our proof yields the stronger result that every red/blue-coloring of $\mathbb{E}^n$ contains either a red copy of ℓ_2 or a blue *translate* of any set K of size at most $2^{c'n}$.

2 Proof of Theorem 1.1

We will prove the existence of a periodic red/blue-coloring (with period R in the standard coordinates) of $\mathbb{E}^n$ such that no two red points have distance one and there is no blue copy of K.

Let $\mathbb{T}_R^n = (\mathbb{E}/R\mathbb{Z})^n$ be the n-dimensional torus with period R in each direction. Let P be any maximal $1/3$-separated subset of $\mathbb{T}_R^n$. One can simply construct such a set P greedily. Consider the Voronoi decomposition of $\mathbb{T}_R^n$ with respect to P. This partitions $\mathbb{T}_R^n$ into cells V_p, one for each point $p \in P$, where V_p consists of the set of points closer to p than any other point in P. From the maximality of P, every point in V_p has distance at most $1/3$ from p. In particular, each V_p has diameter at most $2/3$.

Lemma 2.1 $|P| \leq (4n^{1/2}R)^n$.

Proof: Since each pair of points in P have distance at least $1/3$, the balls of radius $r = 1/6$ around each point are disjoint. A ball in n-space of radius r has volume $r^n\pi^{n/2}/\Gamma(n/2+1)$, where the gamma function satisfies $\Gamma(n/2+1) = (n/2)!$ if n is even and $\Gamma(n/2+1) = \sqrt{\pi} \cdot n!!/2^{(n+1)/2}$ if n is odd. In either case, we have $\Gamma(n/2+1) \leq n^{n/2}$, so the volume of the n-dimensional ball is at least $(r^2\pi/n)^{n/2}$. The balls of radius $1/6$ around the points of P are disjoint and the volume of the torus $\mathbb{T}_R^n$ is R^n, so the number of points in P is at most $R^n/((1/6)^2\pi/n)^{n/2} = (36nR^2/\pi)^{n/2} < (4n^{1/2}R)^n$. $\square$

Lemma 2.2 *If $S \subset \mathbb{E}^n$ is t-separated, then, for any point $p \in \mathbb{E}^n$ and any $s \geq 0$, the number of points of S within distance s of p is at most $(2s/t+1)^n$.*

Proof: The balls of radius $t/2$ around each point of S are disjoint and, for each point $p' \in S$ with distance at most s from p, the ball of radius $s + t/2$ around p contains the ball of radius $t/2$ around p'. Hence, by a volume argument, there are at most $(\frac{s+t/2}{t/2})^n = (2s/t+1)^n$ points of distance at most s from p. $\square$

Lemma 2.3 *Each Voronoi cell V_p is a convex body defined by the intersection of at most 5^n half-spaces.*

Proof: A point q on the boundary of V_p is on the hyperplane equidistant to p and some other point $p' \in P$, where this distance is at most $1/3$. This implies that p' has distance at most $2/3$ from p. Since the points in P are $1/3$-separated, Lemma 2.2 implies that there are at most 5^n points of P of distance at most $2/3$ from p. Therefore, since the Voronoi cell V_p is the intersection of half-spaces that are defined by hyperplanes which are equidistant from p and p' for some p' of distance at most $2/3$ from p, the result follows. $\square$

Lemma 2.4 *If K is a 1-separated point set in $\mathbb{E}^n$ and $s \geq 1$, then there is a set $K' \subset K$ that is s-separated and has size at least $|K|/(2s+1)^n$.*

Proof: By Lemma 2.2 with $t = 1$, for each point p, there are at most $(2s+1)^n$ points of K within distance s of p (including p itself). We can then greedily construct the set K', getting one point in K' at the expense of at most $(2s+1)^n$ points from K, giving the desired bound. $\square$

Let Q be a random subset of P formed by picking each point in P with probability $x = 20^{-n}$ independently of the other points. Let S be the subset of Q where $s \in S$ if and only if there is no other point $s' \in Q$ of distance at most $5/3$ from s. By Lemma 2.2, there are at most $(2(5/3)/(1/3)+1)^n = 11^n$ points of P of distance at most $5/3$ from any point. For a given point $p \in P$, the probability that $p \in S$ is therefore at least $x(1-x)^{11^n} > x/2$ as $x = 20^{-n}$.

Let $V_1, \ldots, V_m$ be the Voronoi cells of points in S. We will color the points in these Voronoi cells red, including the boundaries, and everything else blue. We consider the periodic coloring of $\mathbb{E}^n$ given by the coloring of $\mathbb{T}_R^n$. Observe that there is a pair of red points of distance one in $\mathbb{T}_R^n$ if and only if there is a pair of red points of distance one in the periodic coloring of $\mathbb{E}^n$ and there is a blue copy of K in $\mathbb{T}^n_R$ if and only if there is a blue copy of K in $\mathbb{E}^n$.

We first claim that there are no two red points q and q' at distance one. Indeed, if q and q' are in the same Voronoi cell, then, as the diameter of each Voronoi cell is at most $2/3$, we have a contradiction. If q and q' are in different cells, with $q \in V_p$ and $q' \in V_{p'}$, then, since q has distance at most $1/3$ from p and q' has distance at most $1/3$ from p', p and p' have distance at most $5/3$. However, by construction, if $p \in S$, then p' is not in S, so these Voronoi cells are not both red and q and q' cannot both be red.

To finish the proof, we need to show that with positive probability, there is no blue copy of K. Observe that since the points of K have distance at most R from each other, if there is a blue copy of K in the coloring of $\mathbb{E}^n$, then we already have a blue copy in the axis-aligned box with one corner at the origin and side length $3R$. This box contains $3^n|P|$ Voronoi cells, which we label $U_1, \ldots, U_{3^n|P|}$.

Let K' be a maximum subset of K which is 5-separated, so $|K'| \geq 11^{-n}|K|$ by Lemma 2.4. Denote the points of K' by $K' = \{k_1, \ldots, k_{|K'|}\}$, where $k_1, \ldots, k_d$ with $d \leq n$ form a basis for the vector space spanned by K', so each element of K' is a linear combination of $k_1, \ldots, k_d$. It suffices to show that with positive probability there is no blue copy of K'. For a map $f : |K'| \to 3^n|P|$, consider the bad event B_f that there is a blue copy of K' with the copy of k_i in $U_{f(i)}$. As each pair of points from K' have distance at least 5, the Voronoi cells V_p and $V_{p'}$ that their images map to have centers of distance at least $5 - 2/3 = 13/3 > 2 \cdot 5/3$ apart and, hence, the probability that p and p' are in S are independent. Therefore, the probability that B_f happens is at most $(1 - x/2)^{|K'|} < e^{-x|K'|/2}$.

We next estimate the number of bad events B_f that are realizable. That is, the number of f for which there is a copy of K' with the copy of k_i in $U_{f(i)}$. Given a copy of K' in $\mathbb{R}^n$ where k_i maps to $g(i) \in \mathbb{E}^n$ for each i, we map the copy of K' to the point $(g(1), \ldots, g(d)) \in \mathbb{R}^{nd}$. This is an injective map from the copies of K' in $\mathbb{E}^n$ to $\mathbb{R}^{nd}$ since the copy of K' is determined by which points $k_1, \ldots, k_d$ map to.

Let U be one of the Voronoi cells, with center p. The Voronoi cell U is given as the intersection of half-spaces $H_{pp'}$ which contain p and whose boundary is the hyperplane equidistant from p and p'. By Lemma 2.3, there are at most 5^n such half-spaces. The linear inequality defining whether a point $(x_1, \ldots, x_n)$ is in a half-space H is of the form $a_1x_1 + \cdots + a_nx_n \leq b$ for some $a_1, \ldots, a_n, b \in \mathbb{R}^n$. As k_i is a linear combination of $k_1, \ldots, k_d$, these observations show that, for any copy of K' in $\mathbb{E}^n$, we can determine whether k_i is mapped into U_j by considering a system of at most 5^n linear inequalities in the nd coordinates of the point $(g(1), \ldots, g(d)) \in \mathbb{R}^{nd}$ that K' is mapped to. Since the number of pairs (i, j) is $|K'| \cdot 3^n|P|$, we can therefore tell which B_f are feasible (i.e., which mappings of the points of K' to Voronoi cells are actually realizable by a copy of K') by the sign patterns of a sequence of $5^n|K'| \cdot 3^n|P|$ linear equations.

We can now bound the number of feasible B_f by using an appropriate version of the Milnor–Thom theorem [11, 12, 13]. For a discussion of this theorem and its history, as well as the statement we present below, we refer the interested reader to Section 6.2 of Matoušek's book [10].

Theorem 2.1 *For $m \geq n \geq 2$, the number of sign patterns of m polynomials in n variables, each of degree at most D, is at most $\left(\frac{50Dm}{n}\right)^n$.*

Taking $D = 1$ and $m = 5^n|K'| \cdot 3^n|P| \leq |K'|(60n^{1/2}R)^n$, we see that the number of feasible bad events B_f is at most

$$\left(\frac{50Dm}{nd}\right)^{nd} \leq \left(|K'|(3000n^{1/2}R)^n\right)^{n^2} \leq e^{8n^4 \ln(|R|+|K'|)}.$$

Therefore, since each event B_f holds with probability at most $e^{-x|K'|/2}$, we see that as long as $x|K'|/2 > 8n^4 \ln(|R| + |K'|)$, then, with positive probability, the desired coloring exists. But this holds for $x = 20^{-n}$, $|K'| \geq 11^{-n}|K|$ and $|K| \geq 10^{4n} \log |R|$, completing the proof.

3 Concluding remarks

Following Frankl and Rödl [6], we say that a set $X \subset \mathbb{E}^d$ is *super-Ramsey* if there exist positive constants C and ϵ such that, for each natural number $n \geq d$, there is a set $P \subset \mathbb{E}^n$ with $|P| \leq C^n$ with the property that any subset of P of size at least $|P|/(1+\epsilon)^n$ contains a copy of X. The result of Frankl and Wilson [5] used to prove Theorem 1.2 was precisely the statement that ℓ_2 is super-Ramsey. By substituting any super-Ramsey set X for ℓ_2 in the proof of that theorem, we can easily deduce the following result.

Theorem 3.1 *For any super-Ramsey set X, there exists a positive constant c' such that $\mathbb{E}^n \longrightarrow (X, K)$ for any set $K \subset \mathbb{E}^n$ of size at most $2^{c'n}$.*

For example, this result applies when X is a rectangular parallelepiped or a non-degenerate simplex, by results of Frankl and Rödl [6]. For all such X, Theorem 1.1 easily implies that the estimate on the size of K in Theorem 3.1 is best possible up to the constant c' in the exponent.

Of course, it is not true in general that for any fixed X and K in $\mathbb{E}^d$ there exists an n such that $\mathbb{E}^n \longrightarrow (X, K)$. For example, Erdős et al. [3] showed that $\mathbb{E}^n \not\longrightarrow (\ell_6, \ell_6)$ for all n. To say more, we recall that a set X is *Ramsey* if for every natural number r there exists a natural number n such that every r-coloring of $\mathbb{E}^n$ contains a monochromatic copy of X, a property we denote by $\mathbb{E}^n \xrightarrow{r} X$.

Theorem 3.2 *Assuming the axiom of choice, if $X \subset \mathbb{E}^d$ is not Ramsey, there exists a natural number m and a finite set $K \subset \mathbb{E}^m$ such that $\mathbb{E}^n \not\longrightarrow (X, K)$ for all n.*

Proof: Since X is not Ramsey, there exists a least natural number r such that $\mathbb{E}^n \overset{r}{\not\longrightarrow} X$ for all n. By the minimality of r, there is an m such that every $(r-1)$-coloring of $\mathbb{E}^m$ contains a monochromatic copy of X. But then, by compactness (and it is here that we invoke the axiom of choice), there must be a finite set $K \subset \mathbb{E}^m$ such that every $(r-1)$-coloring of K contains a monochromatic copy of K.

Suppose now that $\chi : \mathbb{E}^n \to \{1, 2, \ldots, r\}$ is an r-coloring of $\mathbb{E}^n$ containing no monochromatic copy of X. We claim that the red/blue-coloring of $\mathbb{E}^n$ where a point is colored red if it received color 1 under χ and blue otherwise contains no red copy of X and no blue copy of K. Indeed, a red copy of X would yield a copy of X in color 1 under χ, while a blue copy of K would yield an $(r-1)$-colored copy of K under χ, which, by the choice of K, would contain a monochromatic copy of X. In either case, this would contradict the definition of χ. □

We do not know if the converse also holds, but an interesting test case might be to show that if X is a regular polygon, then, for any $K \subset \mathbb{E}^d$, there exists an n such that $\mathbb{E}^n \longrightarrow (X, K)$. This would be an asymmetric counterpart to Kříž's beautiful result [9] that all regular polygons are Ramsey. We were also unable to decide the seemingly simpler question of whether, for every natural number m, there exists a natural number n such that $\mathbb{E}^n \longrightarrow (\ell_3, \ell_m)$. It seems unlikely that this holds for large m, but we were at a loss to exhibit a coloring which confirms our suspicion.

As a final note, we mention another problem of Erdős et al. [4]: for any natural number n, does there exist a natural number m depending only on n such that for every set $K \subset \mathbb{E}^n$ of size m there is a two-coloring of $\mathbb{E}^n$ with no monochromatic copy of K? By rescaling, we may assume that the smallest distance between any two points in K is equal to one and, therefore, that $\ell_2 \subset K$ and K is 1-separated. Theorem 1.1 then implies that if the diameter of K is at most R and $m \geq 10^{4n} \log R$, there is indeed a two-coloring of $\mathbb{E}^n$ with no monochromatic copy of K. This partially answers the question of Erdős et al. and a complete answer would follow if we could remove the dependence on R in Theorem 1.1 (a problem which is also interesting in its own right).

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