An improved bound for the stepping-up lemma

David Conlon[*] Jacob Fox[†] Benny Sudakov[‡]

Abstract

The partition relation $N \to (n)_\ell^k$ means that whenever the k-tuples of an N-element set are ℓ-colored, there is a monochromatic set of size n, where a set is called monochromatic if all its k-tuples have the same color. The logical negation of $N \to (n)_\ell^k$ is written as $N \not\to (n)_\ell^k$. An ingenious construction of Erdős and Hajnal known as the stepping-up lemma gives a negative partition relation for higher uniformity from one of lower uniformity, effectively gaining an exponential in each application. Namely, if $\ell \geq 2$, $k \geq 3$, and $N \not\to (n)_\ell^k$, then $2^N \not\to (2n+k-4)_\ell^{k+1}$. In this note we give an improved construction for $k \geq 4$. We introduce a general class of colorings which extends the framework of Erdős and Hajnal and can be used to establish negative partition relations. We show that if $\ell \geq 2$, $k \geq 4$ and $N \not\to (n)_\ell^k$, then $2^N \not\to (n+3)_\ell^{k+1}$. If also k is odd or $\ell \geq 3$, then we get the better bound $2^N \not\to (n+2)_\ell^{k+1}$. This improved bound gives a coloring of the k-tuples whose largest monochromatic set is a factor $\Omega(2^k)$ smaller than given by the original version of the stepping-up lemma. We give several applications of our result to lower bounds on hypergraph Ramsey numbers. In particular, for fixed $\ell \geq 4$ we determine up to an absolute constant factor (which is independent of k) the size of the largest guaranteed monochromatic set in an ℓ-coloring of the k-tuples of an N-set.

1 Introduction

The partition relation $N \to (n)_\ell^k$ means that whenever the k-tuples of an N-element set are ℓ-colored, there is a monochromatic set of size n, where a set is called monochromatic if all its k-tuples have the same color. If $\ell = 2$ we simply write $N \to (n)^k$ instead of $N \to (n)_2^k$.

The *Ramsey number* $r(n)$ is the least integer N such that $N \to (n)^2$. That is, $r(n)$ is the least integer N such that every 2-coloring of the edges of the complete graph on N vertices contains a monochromatic clique of size n. Ramsey's theorem states that $r(n)$ exists for all n. Determining or estimating Ramsey numbers is one of the central problem in combinatorics (see the book Ramsey theory [11] for details). A classical result of Erdős and Szekeres [10], which is a quantitative version of Ramsey's theorem, implies that $r(n) \leq 2^{2n}$ for every positive integer n. Erdős [6] showed using probabilistic arguments that $r(n) > 2^{n/2}$ for $n > 2$. Over the last sixty years, there have been several improvements on these bounds (see, e.g., [4]). However, despite efforts by various researchers, the constant factors in the above exponents remain the same.

Although already for graph Ramsey numbers there are significant gaps between lower and upper bounds, our knowledge of hypergraph Ramsey numbers is even weaker. The Ramsey number $r_k(n)$

[*]St John's College, Cambridge, United Kingdom. E-mail: D.Conlon@dpmms.cam.ac.uk. Research supported by a research fellowship at St John's College.

[†]Department of Mathematics, Princeton, Princeton, NJ. Email: jacobfox@math.princeton.edu. Research supported by an NSF Graduate Research Fellowship and a Princeton Centennial Fellowship.

[‡]Department of Mathematics, UCLA, Los Angeles, CA 90095. Email: bsudakov@math.ucla.edu. Research supported in part by NSF CAREER award DMS-0812005 and by USA-Israeli BSF grant.

is the minimum N such that $N \to (n)^k$. That is, $r_k(n)$ is the least N such that every 2-coloring of the k-tuples of an N-element set contains a monochromatic set of size n, where a set is called monochromatic if all its k-tuples have the same color. Erdős, Hajnal, and Rado [9] showed that there are positive constants c and c' such that

$$2^{cn^2} < r_3(n) < 2^{2^{c'n}}.$$

They also conjectured that $r_3(n) > 2^{2^{cn}}$ for some constant $c > 0$ and Erdős (see, e.g. [3]) offered a \$500 reward for a proof. Similarly, for $k \geq 4$, there is a difference of one exponential between known upper and lower bounds for $r_k(n)$, i.e.,

$$t_{k-1}(c2^{-k}n^2) \leq r_k(n) \leq t_k(c'n),$$

where the tower function $t_k(x)$ is defined by $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$.

The proof of the lower bound is a corollary of the following lemma of Erdős and Hajnal, known as the stepping-up lemma (see e.g. [11]).

Theorem 1 (Stepping-up Lemma) *If $k \geq 3$ and $N \not\to (n)^k_\ell$, then $2^N \not\to (2n + k - 4)^{k+1}_\ell$.*

By starting with the negative partition relation $2^{n^2/6} \not\to (n)^3$, which follows by considering a random 2-coloring of the triples of an N-set, after $k - 3$ iterations of the stepping-up lemma, we have $t_{k-1}(n^2/6) \not\to (2^{k-3}n - k + 3)^k$.

There is also a way to step-up from $k = 2$ to $k = 3$, but, in doing so, the number of colors jumps from 2 to 4. However, the benefit from stepping up from $k = 2$ is that already for four colors we know the correct height for the tower function. In particular, Erdős and Hajnal showed that there exist constants c and c' such that

$$2^{2^{cn}} \leq r_3(n; 4) \leq 2^{2^{c'n}},$$

where $r_k(n; \ell)$ is the minimum N such that every ℓ-coloring of the k-tuples of an N-element set contains a monochromatic subset of size n. The relevant variant of the stepping-up lemma is as follows.

Theorem 2 (Stepping-up Lemma for $k = 2$) *If $N \not\to (n)^2$, then $2^N \not\to (n + 1)^3_4$.*

Using the negative partition relation $2^{n/2} \not\to (n)^2$, this gives $t_3(n/2) \not\to (n + 1)^3_4$. Through $k - 3$ applications of Theorem 1, we have $t_k(n/2) \not\to (2^{k-3}(n + 1) - k + 3)^k_4$. Thus the four-color Ramsey number satisfies

$$t_k(c2^{-k}n) < r_k(n; 4) < t_k(c'n), \tag{1}$$

where c and c' are absolute constants. Satisfying as this result may be, one rather annoying aspect remains: the exponential dependence on k of the factor of n in the lower bound. We show here that this dependence on k can be removed. In particular, for fixed $\ell \geq 4$, up to an absolute constant factor, we know the size of the largest guaranteed monochromatic set an ℓ-coloring of the k-tuples of an N-set must have. We accomplish this by improving the bound in the stepping-up lemma.

Theorem 3 *Suppose $k \geq 4$ and $N \not\to (n)^k_\ell$. We have $2^N \not\to (n + 3)^{k+1}_\ell$. If k is odd or $\ell \geq 3$, we have the better bound $2^N \not\to (n + 2)^{k+1}_\ell$.*

This theorem implies that for all k and $n \geq 3k$ the four-color Ramsey number satisfies

$$r_k(n;4) > t_k(cn), \tag{2}$$

where $c > 0$ is an absolute constant. This bound improves on the lower bound in (1) and is tight apart from the absolute constant factor c. The lower bound in (1) for $k = 4$ shows that $N \not\rightarrow (n/3)^4_4$, where $N = t_4(cn)$ and $c > 0$ is an absolute constant. After $k-4$ iterations of Theorem 3, we get $t_{k-4}(N) \not\rightarrow (n/3+2(k-4))^k_4$. Substituting in $n \geq 3k$ and $N = t_4(cn)$, we obtain (2). The lower bound (2) also holds for any number $\ell \geq 4$ of colors instead of 4 as trivially $r_k(n;\ell) \geq r_k(n;\ell')$ if $\ell \geq \ell'$, and the upper bound in (1) holds with the same proof for any fixed number $\ell \geq 4$ of colors instead of 4. Theorem 3 also allows us to improve the lower bound for two colors for $n \geq 3k$ to

$$r_k(n) \geq t_{k-1}(cn^2),$$

where $c > 0$ is an absolute constant.

The off-diagonal partition relation $N \rightarrow (n_i)^k_{i<\ell}$ means that whenever the k-tuples of an N-element set are ℓ-colored with colors $0, 1, \ldots, \ell-1$, there is a color i and a monochromatic set in color i of size n_i. The off-diagonal version of the stepping-up lemma of Erdős and Hajnal asserts that if $k \geq 3$ and $N \not\rightarrow (n_i)^k_{i<\ell}$, then $2^N \not\rightarrow (2n_i + k - 4)^{k+1}_{i<\ell}$. Our main result is the following extension of Theorem 3.

Theorem 4 *Suppose $k \geq 4$ and $N \not\rightarrow (n_i)^k_{i<\ell}$.*

1. *If $\ell \geq 2$, letting $n'_i = n_i + 3$ for $i = 0,1$ and $n'_i = n_i + 1$ for $2 \leq i < \ell$, then $2^N \not\rightarrow (n'_i)^{k+1}_{i<\ell}$.*
2. *If $\ell \geq 2$ and k is odd, letting $n'_i = n_i + 2$ for $i = 0,1$ and $n'_i = n_i + 1$ for $2 \leq i < \ell$, then $2^N \not\rightarrow (n'_i)^{k+1}_{i<\ell}$.*
3. *If $\ell \geq 3$, letting $n'_i = n_i + 2$ for $i = 0,1,2$, and $n'_i = n_i + 1$ for $3 \leq i < \ell$, then $2^N \not\rightarrow (n'_i)^{k+1}_{i<\ell}$.*

Off-diagonal hypergraph Ramsey numbers have also received much interest. The Ramsey-number $r_k(s,n)$ is the minimum N such that every red-blue coloring of the k-tuples of an N-set contains a red s-set or a blue n-set. For $k = 2$, after several successive improvements, it is known (see [1], [2], [12], [13]) that there are constants c_1, c_2 such that for fixed $s \geq 3$,

$$c_1 \frac{n^{(s+1)/2}}{\log^t n} \leq r_2(s,n) \leq c_2 \frac{n^{s-1}}{\log^{s-2} n}, \tag{3}$$

where $t = (s^2 - s - 4)/(2s - 4)$.

Erdős and Hajnal [8] gave a simple construction showing that $r_3(4,n) \geq 2^{cn}$. They further conjectured that $\lim_{n\to\infty} \frac{\log r_3(4,n)}{n} = \infty$. The authors [5] recently settled this conjecture. They also improved on the lower and upper bounds for $r_3(s,n)$. For $s = 4$, they show that there are positive constants c_1 and c_2 such that

$$n^{c_1 n} \leq r_3(4,n) \leq n^{c_2 n^2}.$$

Using the off-diagonal stepping-up lemma of Erdős and Hajnal, it follows from the above result that for $s \geq 2^{k-1} - k + 3$,

$$r_k(s,n) \geq t_{k-1}(c2^{-k} n \log n).$$

In the other direction, it follows from the results in [5] that $r_k(s,n) \leq t_{k-1}(c'n^{s-k+1}\log n)$ for $s \geq k+1$ and n sufficiently large. This leads to the following interesting question: what is the minimum $s = s(k)$ for which $r_k(s,n)$ grows at least as a tower function of height $k-1$ in n? As in the cases $k = 2, 3$, it is natural to conjecture that $s(k) = k+1$. However, the original stepping-up lemma only gives $s(k) \leq 2^{k-1} - k + 3$. Our improved stepping-up lemma gives the linear upper bound $s(k) \leq \lceil \frac{5}{2}k \rceil - 3$ for $k \geq 4$. Indeed, starting from $s(4) \leq 7$, we have $s(k+1) \leq s(k) + 3$ from the first part of Theorem 4, and we have the better bound $s(k+1) \leq s(k) + 2$ for k odd from the second part of Theorem 4. This is a step toward the conjectured bound $s(k) = k+1$.

Partition relations with infinite cardinals have an important role in modern set theory and there is an analogous stepping-up lemma for partition relations with infinite cardinals. The still open problem of improving the bound given by the stepping-up lemma for infinite cardinals was raised by Erdős and Hajnal (see [7]). However, we do not investigate this problem here.

2 Step-Up Colorings

Here we construct a large family of colorings from which we derive negative partition relations in the next section. The coloring given by Erdős and Hajnal to prove the stepping-up lemma is a particularly simple coloring in this family.

Suppose $\phi : [N]^k \to \{0, \ldots, \ell-1\}$ is an ℓ-coloring of the k-tuples of an N-set such that for $0 \leq i \leq \ell-1$, there is no monochromatic n_i-set in color i. Let

$$T = \{(\gamma_1, \ldots, \gamma_N) : \gamma_i = 0 \text{ or } 1\}.$$

Let $\mathcal{P}$ denote the family of nonempty subsets of $\{2, \ldots, k-1\}$ and α a function $\alpha : \mathcal{P} \times \{0,1\} \to \{0, \ldots, \ell-1\}$. The main goal of this section is to define a coloring $C = C_{\phi,\alpha} : T \to \{0, \ldots, \ell-1\}$. We show in the next section that, for certain choices of α, if ϕ has no large monochromatic set, then C also has no large monochromatic set.

If $\epsilon = (\gamma_1, \cdots, \gamma_N)$, $\epsilon' = (\gamma'_1, \cdots, \gamma'_N)$ and $\epsilon \neq \epsilon'$, define

$$\delta(\epsilon, \epsilon') = \max\{i : \gamma_i \neq \gamma'_i\},$$

that is, $\delta(\epsilon, \epsilon')$ is the largest coordinate at which they differ. Given this, we can define an ordering on T, saying that

$$\epsilon < \epsilon' \text{ if } \gamma_i = 0, \gamma'_i = 1,$$
$$\epsilon' < \epsilon \text{ if } \gamma_i = 1, \gamma'_i = 0,$$

where $i = \delta(\epsilon, \epsilon')$. Equivalently, associate to any ϵ the number $b(\epsilon) = \sum_{i=1}^{N} \gamma_i 2^{i-1}$. The ordering then says simply that $\epsilon < \epsilon'$ if and only if $b(\epsilon) < b(\epsilon')$.

We will further need the following two properties of the function δ which one can easily prove.

(a) If $\epsilon_1 < \epsilon_2 < \epsilon_3$, then $\delta(\epsilon_1, \epsilon_2) \neq \delta(\epsilon_2, \epsilon_3)$ and

(b) if $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_p$, then $\delta(\epsilon_1, \epsilon_p) = \max_{1 \leq i \leq p-1} \delta(\epsilon_i, \epsilon_{i+1})$.

In particular, these properties imply that there is a unique index i which achieves the maximum of $\delta(\epsilon_i, \epsilon_{i+1})$. Indeed, suppose that there are indices $i < i'$ such that

$$\ell = \delta(\epsilon_i, \epsilon_{i+1}) = \delta(\epsilon_{i'}, \epsilon_{i'+1}) = \max_{1 \leq j \leq p-1} \delta(\epsilon_j, \epsilon_{j+1}).$$

Then, by property (b) we also have that $\ell = \delta(\epsilon_i, \epsilon_{i'}) = \delta(\epsilon_{i'}, \epsilon_{i'+1})$. This contradicts property (a) since $\epsilon_i < \epsilon_{i'} < \epsilon_{i'+1}$.

We are now ready to color the complete $(k+1)$-uniform hypergraph on the set T. If $\epsilon_1 < \ldots < \epsilon_{k+1}$, for $1 \le i \le k$, let $\delta_i = \delta(\epsilon_i, \epsilon_{i+1})$. Notice that δ_i is different from δ_{i+1} by property (a). If $\delta_1, \ldots, \delta_k$ form a monotone sequence (increasing or decreasing), then let $C(\{\epsilon_1, \ldots, \epsilon_{k+1}\}) = \phi(\{\delta_1, \ldots, \delta_k\})$. That is, we color this $(k+1)$-tuple of ϵ's by the color of the k-tuple of the δ's. So suppose $\delta_1, \ldots, \delta_k$ is not monotone.

For $2 \le i \le k-1$, we say that i is a *local maximum* if $\delta_{i-1} < \delta_i > \delta_{i+1}$, and i is a *local minimum* if $\delta_{i-1} > \delta_i < \delta_{i+1}$. We say that i is a local *extremum* if it is a local minimum or a local maximum. If $\delta_1, \ldots, \delta_k$ is not monotone, then, by property (a), this sequence has a local extremum. Let $S = \{i_1, \ldots, i_d\}_<$ denote the local extrema labeled in increasing order. The number i_j is the *location* of the jth local extremum. Note that the *type of local extremum* (maximum or minimum) of i_j and i_{j+1} is different for $1 \le j \le d-1$. That is, the type of local extremum alternates. We now can define the desired coloring:

$$C(\{\epsilon_1, \ldots, \epsilon_{k+1}\}) = \begin{cases} \alpha(S, 0) & \text{if } i_1 \text{ is a local minimum} \\ \alpha(S, 1) & \text{if } i_1 \text{ is a local maximum.} \end{cases} \tag{4}$$

Note that the domain of α has size $2^{k-1} - 2$, so there are $\ell^{2^{k-1}-2}$ different choices for α. That is quite a lot of possible α to choose from! However, some choices are clearly more useful then others. For example, they may give better negative partition relations, or may be simpler to define or analyze. The coloring used by Erdős and Hajnal in their proof of the stepping-up lemma is given by taking α to be the projection map onto the second coordinate. That is, they take $\alpha(S, i) = i$ for all $S \in \mathcal{P}$ and $i \in \{0, 1\}$.

We next define two more particular functions α which we use to establish negative partition relations. In the first coloring, the color of a $(k+1)$-tuple is determined by the parity of the location and the type of the first local extremum. Define the coloring $\alpha_1 : \mathcal{P} \times \{0, 1\} \to \{0, 1, \ldots, \ell - 1\}$ as follows:

$$\alpha_1(S, i) = i_1 + i \pmod 2,$$

where i_1 is the least element of S. Note that α_1 takes only the values 0 and 1 in its image, but the size of its range is ℓ. If $(\delta_1, \ldots, \delta_k)$ is not monotone, the coloring C of $\{\epsilon_1, \ldots, \epsilon_{k+1}\}$ we get from α_1 is given by

$$C(\{\epsilon_1, \ldots, \epsilon_{k+1}\}) = \begin{cases} 0 & \text{if } i_1 \text{ is even and a local min. or is odd and a local max.} \\ 1 & \text{if } i_1 \text{ is even and a local max. or is odd and a local min.} \end{cases} \tag{5}$$

There are many more colorings similar to α_1 which we can define that are useful in establishing negative partition relations. To do so, we will consider proper colorings of a graph G_k, which we define next. Let G_k be the graph with vertex set $\{2, \ldots, k-1\} \times \{0, 1\}$, where $(2, 0)$ is adjacent to $(2, 1)$, (j, i) is adjacent to $(j+1, i)$ for $2 \le j \le k-2$, $(2, 0)$ is adjacent to $(k-1, 1)$, and $(2, 1)$ is adjacent to $(k-1, 0)$. Graph G_k is bipartite if and only if k is odd, and is always three-colorable. If k is odd, the coloring $\chi(j, i) = j + i \pmod 2$ is a proper coloring, i.e., adjacent vertices get different colors. Let χ be a proper coloring of the vertices of G_k with colors $0, 1, \ldots, \ell - 1$. Define the coloring $\alpha_\chi : \mathcal{P} \times \{0, 1\} \to \{0, 1, \ldots, \ell - 1\}$ as follows:

$$\alpha_\chi(S, i) = \chi(i_1, i),$$

where i_1 is the smallest element of S. Note that if k is odd, α_1 is of the form α_χ, with $\chi(j,i) = j + i \pmod 2$.

In the next section, we show that coloring C with $\alpha = \alpha_1$ demonstrates the first part of Theorem 4. We then use coloring C with α of the form α_χ to establish the second and third parts of Theorem 4.

3 Proof of Theorem 4

Here we prove Theorem 4. We first prove part 1 of the theorem, which states that if $k \geq 4$, $\ell \geq 2$, and $N \not\to (n_i)^k_{i<\ell}$, letting $n'_i = n_i + 3$ for $i = 0, 1$ and $n'_i = n_i + 1$ for $2 \leq i < \ell$, then $2^N \not\to (n'_i)^{k+1}_{i<\ell}$. We then prove parts 2 and 3 of Theorem 4.

Proof of Theorem 4, Part 1: Suppose $\phi : [N]^k \to \{0, \ldots, \ell - 1\}$ is an ℓ-coloring of the k-tuples of an N-set such that for $0 \leq i \leq \ell - 1$, there is no monochromatic n_i-set in color i. Let

$$T = \{(\gamma_1, \ldots, \gamma_N) : \gamma_i = 0 \text{ or } 1\}.$$

We show that the coloring $C : T \to \{0, \ldots, \ell - 1\}$ defined as $C = C_{\phi,\alpha_1}$ from the previous section demonstrates the negative partition relation $2^N \not\to (n'_i)^{k+1}_{i<\ell}$.

We use the following simple observation several times which holds since we color each $(k+1)$-tuple of ϵ's whose δ's are monotone by the color of the k-tuple of the δ's. If we have a monochromatic clique with n vertices $\{\epsilon_1, \ldots, \epsilon_n\}_<$ labeled in increasing order whose δ's are monotone, then in coloring ϕ the δ's are the vertices of a monochromatic clique of size $n - 1$ in the same color.

Suppose for contradiction we have a monochromatic clique with vertices $\{\epsilon_1, \ldots, \epsilon_{n'_j}\}_<$ of size n'_j in color j labeled in increasing order. From the above observation, there can be no monotone consecutive sequence of δ's of length n_j. In particular, if $j > 1$, since the only $(k+1)$-tuples with color j are those whose string of δ's are monotone, there is no monochromatic clique of size $n_j + 1 = n'_j$ in color j. So we may suppose $j = 0$ or 1. By symmetry, we may suppose $j = 0$. So $n'_0 = n_0 + 3$. As the sequence $\delta_1, \ldots, \delta_{n_0}$ can not be monotone, there is a first local extremum $e_1 \leq n_0 - 1$.

Case 1: The first local extremum e_1 satisfies $e_1 > 2$. Already in this case, we use the assumption $k \geq 4$. We claim that, if $e_1 < k$, then the $(k+1)$-tuples $(\epsilon_1, \ldots, \epsilon_{k+1})$ and $(\epsilon_2, \ldots, \epsilon_{k+2})$ have different colors. On the other hand, if $e_1 \geq k$, then the $(k+1)$-tuples $(\epsilon_{e_1-k+2}, \ldots, \epsilon_{e_1+2})$ and $(\epsilon_{e_1-k+3}, \ldots, \epsilon_{e_1+3})$ have different colors. Indeed, in both cases, the first local extremum for the pair of $(k+1)$-tuples are of the same type (minimum or maximum) but their locations differ by one and hence have different parity, which by (5) implies that these $(k+1)$-tuples have different colors.

Case 2: The first local extremum is $e_1 = 2$.

Case 2(a): 3 is a local extremum. Recall that consecutive extrema have different type and thus types of 2 and 3 are distinct. This implies that the $(k+1)$-tuples $(\epsilon_1, \ldots, \epsilon_{k+1})$ and $(\epsilon_2, \ldots, \epsilon_{k+2})$ have different colors. To see this, note that for each of these $(k+1)$-tuples, the first local extremum is the second δ, but these extrema are of different type, and hence by (5) these $(k+1)$-tuples have different colors.

Case 2(b): 3 is not a local extremum. As the sequence $\delta_2, \delta_3, \ldots, \delta_{n_0+1}$ of length n_0 cannot be monotone, the sequence of δ's has a second extremum $e_2 \leq n_0$. If $e_2 < k + 1$, then the $(k+1)$-tuples $(\epsilon_2, \ldots, \epsilon_{k+2})$ and $(\epsilon_3, \ldots, \epsilon_{k+3})$ have different colors. If $e_2 \geq k + 1$, then the $(k+1)$-tuples $(\epsilon_{e_2-k+2}, \ldots, \epsilon_{e_2+2})$ and $(\epsilon_{e_2-k+3}, \ldots, \epsilon_{e_2+3})$ have different colors. Indeed, in either case, the first local extremum for the pair of $(k+1)$-tuples are of the same type but their locations differ by one and hence have different parity, which by (5) implies that they have different colors. This completes the proof of part 1 of the theorem. $\square$

The last two parts of Theorem 4 follow from the discussion at the end of the previous section together with the following lemma. The proof is similar to the previous proof.

Lemma 1 *Suppose $\phi : [N]^k \to \{0, \ldots, \ell-1\}$ is an ℓ-coloring of the k-tuples of an N-set such that for $0 \leq j \leq \ell - 1$, there is no monochromatic n_j-set in color j. Let $\chi : \{2, \ldots, k-1\} \times \{0,1\} \longrightarrow \{0, 1, \ldots, \ell-1\}$ be a proper coloring of the vertices of the graph G_k defined at the end of the previous section. Then the coloring C_{ϕ,α_χ} defined at the end of the previous section has no monochromatic (n_j+2)-set in color j if j is a color used by χ, and χ has no monochromatic (n_j+1)-set in color j if j is a color not used by χ.*

Proof: For each color j not used by χ, the only $(k+1)$-tuples $\{\epsilon_1, \ldots, \epsilon_{k+1}\}_<$ of color j are those whose corresponding sequence of δ's is monotonic. Hence, as we already explained in the proof of the first part of Theorem 4, for each color j not used by χ, there is no monochromatic clique of size n_j+1 in color j in coloring C_{ϕ,α_χ}.

So suppose for contradiction that there is a color j used by χ such that the coloring C_{ϕ,α_χ} has a monochromatic clique $\{\epsilon_1, \ldots, \epsilon_{n_j+2}\}_<$ in color j. As we color the $(k+1)$-tuples of ϵ's whose δ's are monotone by the color of the k-tuples of the δ's, there is no monotone consecutive sequence of δ's of length n_j. As the sequence $\delta_1, \ldots, \delta_{n_j}$ cannot be monotone, there is a first local extremum $e_1 \leq n_j - 1$.

Case 1: The first local extremum e_1 satisfies $e_1 > 2$. If $e_1 < k$, then the $(k+1)$-tuples $(\epsilon_1, \ldots, \epsilon_{k+1})$ and $(\epsilon_2, \ldots, \epsilon_{k+2})$ have different colors. If $e_1 \geq k$, then the $(k+1)$-tuples $(\epsilon_{e_1-k+2}, \ldots, \epsilon_{e_1+2})$ and $(\epsilon_{e_1-k+3}, \ldots, \epsilon_{e_1+3})$ are different colors. Indeed, in either case, the first local extremum for the pair of $(k+1)$-tuples are of the same type but their locations differ by one and hence, since, for all $2 \leq j \leq k-2$ and $i = 0$ or 1, $\chi(j,i) \neq \chi(j+1,i)$, these $(k+1)$-tuples have different colors.

Case 2: The first local extremum is $e_1 = 2$.

Case 2(a): 3 is a local extremum. The $(k+1)$-tuples $(\epsilon_1, \ldots, \epsilon_{k+1})$ and $(\epsilon_2, \ldots, \epsilon_{k+2})$ have different colors. Indeed, for each of these $(k+1)$-tuples, the first local extremum is the second δ, but they are of different type, and hence, since $\chi(2,0) \neq \chi(2,1)$, these $(k+1)$-tuples have different colors.

Case 2(b): 3 is not a local extremum. As the sequence $\delta_2, \delta_3, \ldots, \delta_{n_j+1}$ of length n_j cannot be monotone, the sequence of δ's has a second extremum $e_2 \leq n_j$. If $e_2 = n_j$, then the $(k+1)$-tuples $(\epsilon_1, \ldots, \epsilon_{k+1})$ and $(\epsilon_{n_j-k+2}, \ldots, \epsilon_{n_j+2})$ have different colors. To see this, note that their only local extrema are the second δ and the $(k-1)$th δ, respectively, and are of different type. Therefore, since $\chi(2,0) \neq \chi(k-1,1)$ and $\chi(2,1) \neq \chi(k-1,0)$, these $(k+1)$-tuples have different colors. Hence $e_2 < n_j$.

If $e_2 < k+1$, then the $(k+1)$-tuples $(\epsilon_2, \ldots, \epsilon_{k+2})$ and $(\epsilon_3, \ldots, \epsilon_{k+3})$ have different colors. If $e_2 \geq k+1$, then the $(k+1)$-tuples $(\epsilon_{e_2-k+2}, \ldots, \epsilon_{e_2+2})$ and $(\epsilon_{e_2-k+3}, \ldots, \epsilon_{e_2+3})$ are different colors. Indeed, in either case, the first local extremum for the pair of $(k+1)$-tuples are of the same type but their locations differ by one, which, as in case 1, implies that they have different colors. This completes the proof of the lemma. $\square$

References

[1] M. Ajtai, J. Komlós, and E. Szemerédi, A note on Ramsey numbers, *J. Combinatorial Theory, Ser. A* **29** (1980), 354–360.

[2] T. Bohman and P. Keevash, in preparation.

[3] F. Chung and R.L. Graham, **Erdős on Graphs. His Legacy of Unsolved Problems**, A K Peters, Ltd., Wellesley, MA, 1998.

[4] D. Conlon, A new upper bound for diagonal Ramsey numbers, *Annals of Mathematics, to appear.*

[5] D. Conlon, J. Fox, and B. Sudakov, Hypergraph Ramsey numbers, submitted.

[6] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[7] P. Erdős, A. Hajnal, Unsolved problems in set theory, Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif,. 1967), pp. 17–48, Amer. Math. Soc., Providence, R.I., 1971.

[8] P. Erdős, A. Hajnal, On Ramsey like theorems, problems and results, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972) , pp. 123–140, Inst. Math. Appl., Southend-on-Sea, 1972.

[9] P. Erdős, A. Hajnal, R. Rado, Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 93–196.

[10] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.

[11] R.L. Graham, B.L. Rothschild, J.H. Spencer, **Ramsey theory**, 2nd edition, John Wiley & Sons (1980).

[12] J. H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Structures and Algorithms* **7** (1995), 173–207.

[13] J. Spencer, Asymptotic lower bounds for Ramsey functions, *Discrete Math.* **20** (1977/78), 69–76.