WEAK QUASI-RANDOMNESS FOR UNIFORM HYPERGRAPHS

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Abstract. We study *quasi-random* properties of k-uniform hypergraphs. Our central notion is uniform edge distribution with respect to large vertex sets. We will find several equivalent characterisations of this property and our work can be viewed as an extension of the well known Chung-Graham-Wilson theorem for quasi-random graphs.

Moreover, let K_k be the complete graph on k vertices and $M(k)$ the line graph of the graph of the k-dimensional hypercube. We will show that the pair of graphs $(K_k, M(k))$ has the property that if the number of copies of both K_k and $M(k)$ in another graph G are as expected in the random graph of density d, then G is quasi-random (in the sense of the Chung-Graham-Wilson theorem) with density close to d.

1. Introduction

We study quasi-random properties of k-uniform hypergraphs, k-graphs for short. The systematic study of quasi-random or pseudo-random graphs was initiated by Thomason [28, 29]. Roughly speaking, Thomason studied deterministic graphs G_n of density p that "imitate" the binomial random graph $G(n,p)$, i.e., graphs G_n that share some important properties with $G(n,p)$. One of the key properties of $G(n,p)$ is its uniform edge distribution and Thomason chose a quantitative version of this property, so-called *jumbledness*, to define pseudo-random graphs. Subsequently Chung, Graham and Wilson [5] (building on the work of others) considered a variation of jumbledness (see property P_4 below) and showed that several other properties of $G(n,p)$ are equivalent to it in a deterministic sense. In particular, those authors proved the following beautiful result.

Theorem 1 (Chung, Graham, and Wilson)**.** *For any sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with $|V(G_n)| = n$ the following properties are equivalent:*

- P_1 : *for all graphs F we have $N^*_F(G_n) = (1/2)^{\binom{\ell}{2}} n^\ell + o(n^\ell)$, where $\ell = |V(F)|$ and $N^*_F(G_n)$ denotes the number of labeled, induced copies of F in G_n;*
- P_2 : *$e(G_n) \geq \frac{1}{2}\binom{n}{2} - o(n^2)$ and $N_{C_4}(G_n) \leq (n/2)^4 + o(n^4)$, where C_4 is the cycle on 4 vertices and $N_{C_4}(G)$ denotes the number of labeled (not necessarily induced) copies of C_4 in G_n;*
- P_3 : *$e(G_n) \geq \frac{1}{2}\binom{n}{2} - o(n^2)$, $\lambda_1(G_n) = n/2 + o(n)$, and $|\lambda_2(G_n)| = o(n)$, where $\lambda_i(G_n)$ is the i-th largest eigenvalue of the adjacency matrix of G_n in absolute value;*
- P_4 : *for every subset $U \subseteq V(G_n)$ we have $e(U) = \frac{1}{2}\binom{|U|}{2} + o(n^2)$;*

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P_5 : *for every subset* $U = \lfloor n/2 \rfloor$ *we have* $e(U) = n^2/16 + o(n^2)$*;*
P_6 : $\sum_{u,v} |s(u,v) - n/2| = o(n^3)$*, where for vertices* $u, v \in V(G_n)$ *we set* $s(u,v) = |\{x \in V(G_n)\colon\ ux \in E(G_n) \Leftrightarrow vx \in E(G_n)\}|$*;*
P_7 : $\sum_{u,v} |\operatorname{codeg}(u,v) - n/4| = o(n^3)$*, where for vertices* $u, v \in V(G_n)$ *we set* $\operatorname{codeg}(u,v) = |\{x \in V(G_n)\colon\ ux \in E(G_n)$ *and* $vx \in E(G_n)\}|$. □

Note that, e.g. due to property P_4, the density of G_n must tend to $1/2$. However, the properties $P_1, \dots, P_7$ can be altered in a straightforward way and the analogue of Theorem 1 holds for all fixed, positive densities. Moreover, graphs satisfying one (and hence all) of the properties $P_1, \dots, P_7$ are called *quasi-random* and $P_1, \dots, P_7$ are quasi-random properties. The list of quasi-random properties was extended by several authors (see, e.g., [19, 20, 22, 23, 24, 25, 30]). Another result related to our work here is the following due to Simonovits and Sós [23].

Theorem 2 (Simonovits and Sós)**.** *For every* $d > 0$*, every graph* F *on* ℓ *vertices containing at least one edge, and every* $\varepsilon > 0$ *exist* $\delta > 0$ *and* n_0 *such that the following is true. If* $G = (V, E)$ *is a graph with* $|V| = n \geq n_0$ *vertices such that* $N_F(U) = d^{e(F)} n^\ell \pm \delta n^\ell$ *for every subset* $U \subseteq V$*, where* $N_F(U)$ *denotes the number of labeled copies of* F *in the induced subgraph* $G[U]$*, then* $e(U) = d\binom{|U|}{2} \pm \varepsilon n^2$ *for every subset* $U \subseteq V$*.* □

We consider extensions of Theorem 1 and Theorem 2 to k-graphs. Chung and Graham [2, 4] and Kohayakawa, Rödl, and Skokan [15] studied extensions of some of the properties $P_1, \dots, P_7$ and showed their equivalences. Roughly speaking, a k-graph H_n of density d is quasi-random, in their sense, if the edges in H_n intersect a d-proportion of the cliques of order k of every $(k-1)$-graph on the same vertex set. In fact, this property can be viewed as a generalisation of P_4 and as it turned out, this notion of quasi-randomness implies the natural analogue of P_1 for k-graphs. On the other hand, for this notion of quasi-randomness there exist no appropriate extension of Szemerédi's regularity lemma [27], i.e., there exists no lemma, which guarantees a decomposition of any given k-graph into relatively "few" blocks, such that most of them satisfy this notion of quasi-randomness. However, a variation of this notion together with a corresponding regularity lemma for k-graphs was found by Gowers [11, 12] and Rödl et al. [9, 18] (see, e.g., [16] for more details).

We study a simpler notion of uniform edge distribution, which only enforces similar densities induced on vertex sets. More precisely, we consider the following straightforward extension of P_4.

$\mathrm{DISC}_d(\delta)$**:** We say a k-graph H_n on n vertices has $\mathrm{DISC}_d(\delta)$ for d, $\delta > 0$, if

$$e(U) = d\binom{|U|}{k} \pm \delta n^k \quad \text{for all} \quad U \subseteq V(H_n)\,,$$

where by $x = y \pm z$ we mean that x lies in the interval $[y - z, y + z]$.

Hypergraphs with property DISC_d were studied in [2, 14] and a straightforward generalisation of Szemerédi's regularity lemma for this concept was observed to hold in [3, 8, 26] (see Theorem 23 below).

We will suggest extensions of properties P_1, P_2, P_6, and P_7 to k-graphs which all turn out to be equivalent to DISC_d (the analogue of P_4 in this context). As a consequence we obtain a new generalisation of Theorem 1 to k-graphs, which we present in the next section, Section 1.1 (see Theorem 3). In Section 1.2 we will discuss a consequence of this generalisation for graphs. In particular, we will show that for every integer $k \geq 2$ the following is true: if the number of copies of the complete graph K_k and of the line graph of the k-dimensional hypercube $M(k)$ are "right" in a given graph G, then G is quasi-random (see Corollary 4). We will also verify the equivalence of another property for k-graphs, which is inspired by Theorem 2 and which we discuss in Section 1.3 (see Theorem 5). Finally, we show the equivalence of several partite variants of DISC_d (see Theorem 6 in Section 1.4).

1.1. Generalisation of Theorem 1.

We establish a generalisation of Theorem 1 for k-graphs which is based on DISC_d. Since DISC_d is the straightforward generalisation of P_4, we need to find generalisations of the other properties of Theorem 1, which are equivalent to DISC_d.

1.1.1. *Extension of P_1.* We start with property P_1. This property asserts that the number of induced copies of a fixed graph F in G_n is asymptotically the same as in the random graph $G(n, 1/2)$. It is well known that DISC_d does not imply such a property for $k \geq 3$ as the following example shows: let H_n be the 3-graph whose edges are formed by the triangles of the random graph $G(n, 1/2)$. Chernoff type estimates show that H_n satisfies $\mathrm{DISC}_{1/8}$ with high probability. On the other hand, the number of labeled (not necessarily induced) copies of $K^{(3)}_{1,1,2}$ (the 3-graph with two edges on four vertices) in H_n is $\sim n^4/32$, which is twice as much as the "right" number $(1/8)^2 n^4$. Moreover, the number of labeled, induced copies of $K^{(3)}_{1,1,2}$ in H_n is $\sim n^4/64$, while the "right" number would be $49n^4/64^2$.

However, it was shown in [14], that k-graphs having $\mathrm{DISC}_d(\delta)$ for sufficiently small δ must contain approximately the same number of copies of any fixed linear k-graph F as a genuine random k-graph of the same density. Here a *linear* k-graph F is defined as having no pair of edges which intersect in two or more vertices. In other words, the property DISC_d implies the following counting-lemma-type property,

$\mathrm{CL}_d(F, \varepsilon)$: We say a k-graph H_n on n vertices has $\mathrm{CL}_d(F, \varepsilon)$ for a given linear k-graph F on ℓ vertices and d, $\varepsilon > 0$, if

$$N_F(H_n) = d^{e(F)} n^\ell \pm \varepsilon n^\ell \,,$$

where $N_F(H)$ denotes the number of labeled copies of F in H.

For a property $P_{x_1,\dots,x_p}(\alpha_1, \dots, \alpha_r)$ of k-graphs we say a sequence $(H_n)_{n\in\mathbb{N}}$ of k-graphs with $|V(H_n)| = n$ *has* or *satisfies* $P_{x_1,\dots,x_p}$, if for all choices of the parameters $\alpha_1, \dots, \alpha_r$ there exists an n_0 such that H_n satisfies $P_{x_1,\dots,x_p}(\alpha_1, \dots, \alpha_r)$ for all $n \geq n_0$. Note that the parameters $x_1, \dots, x_p$ are fixed for this definition and the fixed parameters always appear as subscripts on the name of the property. Moreover, the parameters $x_1, \dots, x_p$ and $\alpha_1, \dots, \alpha_r$ might be of different types, like k-graphs, integers, or real numbers. For example, in CL_d the parameter α_1 is an arbitrary linear k-graph, while x_1 and α_2 are positive reals. Furthermore, for two properties $P_{x_1,\dots,x_p}(\alpha_1, \dots, \alpha_r)$ and $Q_{y_1,\dots,y_q}(\beta_1, \dots, \beta_s)$ we say $P_{x_1,\dots,x_p}$ *implies* $Q_{y_1,\dots,y_q}$ ($P_{x_1,\dots,x_p} \Rightarrow Q_{y_1,\dots,y_q}$), if every sequence of k-graphs $(H_n)_{n\in\mathbb{N}}$ that satisfies property $P_{x_1,\dots,x_p}$ also satisfies property $Q_{y_1,\dots,y_q}$. Moreover, properties $P_{x_1,\dots,x_p}$ and $Q_{y_1,\dots,y_q}$ are called equivalent if $P_{x_1,\dots,x_p} \Rightarrow Q_{y_1,\dots,y_q}$ and $Q_{y_1,\dots,y_q} \Rightarrow P_{x_1,\dots,x_p}$. With this notation, the aforementioned result from [14] states that

$$\mathrm{DISC}_d \text{ implies } \mathrm{CL}_d. \tag{1}$$

The discussion above suggests that the "right" extension of P_1 in our context involves linear k-graphs, which leads to the following definition for the induced-counting-lemma-type property.

$\mathrm{ICL}_d(F', F, \varepsilon)$**:** We say a k-graph H_n on n vertices has $\mathrm{ICL}_d(F', F, \varepsilon)$ for given linear k-graphs $F' \subseteq F$ with $V(F') = V(F) = [\ell]$ and d, $\varepsilon > 0$, if

$$N^*_{F',F}(H_n) = d^{e(F')}(1-d)^{e(F)-e(F')}n^\ell \pm \varepsilon n^\ell \,,$$

where $N^*_{F',F}(H_n)$ denotes the number of labeled, induced copies of F' with respect to F in H_n, i.e., $N^*_{F',F}(H_n)$ is the number of injective mappings $\varphi\colon V(F) \to V(H_n)$ such that for all edges e of the supergraph F we have $\varphi(e) \in E(H_n)$ if and only if e is an edge of the subgraph F'.

The notion of induced copies with respect to a linear supergraph F may look a bit artificial. But it generalises the usual notion of induced graphs in the case of graphs, as may be seen by setting $F = K_\ell$ to be the complete graph on the same vertex set. We will show that ICL_d is equivalent to DISC_d for k-graphs (see Theorem 3 below).

1.1.2. *Extension of P_2.* Next we focus on a generalisation of P_2. For that we need to identify a k-graph which in some sense allows us to reverse the implication from (1). Note that there are k-graphs O known, which have the following property: if O appears asymptotically in the "right" frequency in H_n, then H_n must satisfy DISC_d. However, to our knowledge all known k-graphs O with this property are non-linear and, as shown for example in [14], $\mathrm{DISC}_d(\delta)$ never yields the "right" frequency for any non-linear k-graph O. Below we will define a linear k-graph M with the same property, i.e., M plays the role of C_4 for $k \geq 3$. (In fact, for $k = 2$ the graph M will be equal to C_4.)

For a k-partite k-graph A with vertex classes $X_1, \dots, X_k$ and $i \in [k]$ we define the *doubling* $\mathrm{db}_i(A)$ *of* A *around class* X_i to be the k-graph obtained from A by taking two disjoint copies of A and identifying the vertices of X_i. More formally, $\mathrm{db}_i(A)$ is the k-partite k-graph with vertex classes $Y_1, \dots, Y_k$, where $Y_i = X_i$ and for $j \neq i$ we have $Y_j = X_j \dot\cup \tilde{X}_j$ with $\tilde{X}_j = \{\tilde{x} \,|\, x \in X_j\}$. Thus $\tilde{x}$ denotes the copy of x. Moreover, the edge set of $\mathrm{db}_i(A)$ is given by

$$E(\mathrm{db}_i(A)) = E(A) \dot\cup \{\{\tilde{x}_1, \dots, \tilde{x}_{i-1}, x_i, \tilde{x}_{i+1}, \dots, \tilde{x}_k\} \colon \{x_1, x_2, \dots, x_k\} \in E(A)\}.$$

For the construction of the k-graph M we will start with a single hyperedge K_k, which can be seen as a k-partite k-graph with partition classes of size 1, and iteratively *double* this k-graph around the partition classes. More precisely,

$$M = \mathrm{db}_k(\mathrm{db}_{k-1}(\dots \mathrm{db}_1(K_k)\dots))\,.$$

More generally, set

$$M_0 = K_k \quad \text{and} \quad M_j = \mathrm{db}_j(M_{j-1}) \quad \text{for } j = 1, \dots, k,$$

so that $M = M_k$. We observe that for every $j = 0, \dots, k$ we have

$$|V(M_j)| = j2^{j-1} + (k-j)2^j \quad \text{and} \quad |E(M_j)| = 2^j\,.$$

Moreover, for the vertex partition $X_1 \dot\cup \dots \dot\cup X_k$ of M_j we have

$$|X_1| = \dots = |X_j| = 2^{j-1} \quad \text{and} \quad |X_{j+1}| = \dots = |X_k| = 2^j.$$

As already mentioned for graphs ($k = 2$) the corresponding graph M is C_4 and for $k \geq 3$ the k-graph M will turn out to be the "right" generalisation for our purposes. In fact, it follows from the Cauchy-Schwarz inequality that if H_n contains at least $\alpha n^{|V(A)|}$ labeled copies of some given k-partite k-graph A, then H_n contains at least $(\alpha^2 - o(1))n^{|V(\mathrm{db}_i(A))|}$ labeled copies of $\mathrm{db}_i(A)$. Consequently, every k-graph H_n

with at least $d\binom{n}{k} + o(n^k)$ edges contains at least $(d^{2^k} - o(1))n^{k2^{k-1}}$ labeled copies of M. Hence, the random k-graph of density d contains approximately the minimum number of copies of M and as we will see k-graphs H_n having $N_M(H_n)$ close to the minimum number will satisfy DISC_d. More precisely, we will show that MIN_d is another property equivalent to DISC_d (see Theorem 3 below), where MIN_d is defined as follows.

$\mathrm{MIN}_d(\varepsilon)$**:** We say a k-graph H_n on n vertices has $\mathrm{MIN}_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$e(H_n) \geq d\binom{n}{k} - \varepsilon n^k \quad \text{and} \quad N_M(H_n) \leq d^{2^k} n^{k2^{k-1}} + \varepsilon n^{k2^{k-1}} .$$

We did not find any interesting generalisation of property P_3 from Theorem 1 to k-graphs for $k \geq 3$. Moreover, the extension property P_4 in this work is DISC_d and the generalisation of P_5 is straightforward (and the implication $P_5 \Rightarrow P_4$ would follow similarly to Fact 18). Hence, we continue with the discussion of properties P_6 and P_7.

1.1.3. *Extension of* P_6. From our point of view property P_6 is closely related to the appearance of subgraphs of C_4. More precisely, for a graph G_n let $\mathrm{EVEN}_{C_4}(G_n)$ be the sum of the number of labeled induced copies of subgraphs of C_4 with an even number of edges, i.e.,

$$\mathrm{EVEN}_{C_4}(G_n) = N^*_{\emptyset,C_4}(G_n) + 4N^*_{P_2,C_4}(G_n) + 2N^*_{2K_2,C_4}(G_n) + N^*_{C_4,C_4}(G_n) ,$$

where $\emptyset$ is the subgraph of C_4 without any edges vertices, P_i is the path with i edges, and $2K_2$ is a matching consisting of two edges. Note, that there are four different ways to select a path of length two within a C_4 and there two different way to fix a matching of size two in any given C_4, while there is only one way to fix a C_4 or an "empty C_4" within a cycle of length four. Similarly, set

$$\mathrm{ODD}_{C_4}(G_n) = 4N^*_{P_1,C_4}(G_n) + 4N^*_{P_3,C_4}(G_n) .$$

We can rewrite $\mathrm{ODD}_{C_4}(G_n)$ and $\mathrm{EVEN}_{C_4}(G_n)$ in terms of $s(u,v)$ (cf. P_6 in Theorem 1) as follows

$$\mathrm{EVEN}_{C_4}(G_n) = \sum_{u,v\in V} \Big(s(u,v)^2 + (n - s(u,v))^2\Big) + o(n^4)$$

and

$$\mathrm{ODD}_{C_4}(G_n) = 2\sum_{u,v\in V} \Big(s(u,v)(n - s(u,v)\Big) + o(n^4) .$$

Hence, property P_6 is, due to the Cauchy-Schwarz inequality, equivalent to the following property.

$P'_6:$ $|\mathrm{EVEN}_{C_4}(G_n) - \mathrm{ODD}_{C_4}(G_n)| = \sum_{u,v\in V}(2s(u,v) - n)^2 = o(n^4)$.

For the extension of P'_6 to k-graphs, we replace C_4 by M from property MIN_d and in order to deal with arbitrary densities $d > 0$ we need a different weight function for the subgraphs of M. For a k-graph H_n and $1 \geq d > 0$ we define a weight function $w\colon \binom{V(H_n)}{k} \to [-1,1]$ and set for $e \in \binom{V(H_n)}{k}$

$$w(e) = \begin{cases} 1-d & \text{if } e \in E(H_n) \\ -d & \text{if } e \notin E(H_n) . \end{cases}$$

For a labeled copy $\tilde{A}$ of a given k-graph A in the complete k-graph on $V(H_n)$ we set

$$w(\tilde{A}) = \prod_{e\in E(\tilde{A})} w(e)\,.$$

It is easy to check that for a graph G_n and $d = 1/2$ we have

$$|\mathrm{EVEN}_{C_4}(G_n) - \mathrm{ODD}_{C_4}(G_n)| = 16\left|\sum_{\tilde{C}_4} w(\tilde{C}_4)\right| + o(n^4)\,,$$

where the sum runs over all labeled copies $\tilde{C}_4$ of C_4 in the complete graph on $V(G_n)$.

With this in mind, we define the generalisation of P_6 as follows, which may be viewed as a weighted form of MIN_d.

$\mathrm{DEV}_d(\varepsilon)$**:** We say a k-graph H_n on n vertices has $\mathrm{DEV}_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$\left|\sum_{\tilde{M}} w(\tilde{M})\right| \le \varepsilon n^{k2^{k-1}}$$

where the sum runs over all labeled copies $\tilde{M}$ of M in the complete k-graph on $V(H_n)$.

Again Theorem 3 will show that DEV_d is equivalent to DISC_d.

1.1.4. *Extension of* P_7. The last property we consider here is P_7. Roughly speaking, P_7 asserts that most pairs of vertices of G_n have approximately $n/4$ neighbours and this implies, on the one hand, that the number of labeled C_4's in G_n is close to $n^4/16$, while, on the other hand, for most vertices v the number of labeled C_4's containing v satisfies $\sum_{w\in V}(\mathrm{codeg}(v,w))^2 \sim n \times (n/4)^2$ as well as $\sum_{u,u'\in N(v)} \mathrm{codeg}(u,u') \sim (\deg(v))^2(n/4)$, which yields $\deg(v) \sim n/2$. Consequently, P_7 implies P_2 and the reverse implication follows from the Cauchy-Schwarz inequality. From this point of view the obvious generalisation of P_7 concerns the number of labeled copies of M_{k-1} attached to a fixed, labeled set of 2^{k-1} vertices. We now make this precise.

Let H_n be a k-graph on n vertices. Let X_k be the (unique) largest vertex class of M_{k-1} and, for $q = 2^{k-1}$, let $x_1, \dots, x_q$ be an arbitrary labeling of the vertices of X_k. For an ordered set $\boldsymbol{u} = (u_1, \dots, u_q)$ of q vertices in $V(H_n)$, we denote by $\mathrm{ext}(M_{k-1}, H_n, \boldsymbol{u})$ the number of copies of M_{k-1} in H_n extending $\boldsymbol{u}$ in a canonical way, i.e., $\mathrm{ext}(M_{k-1}, H_n, \boldsymbol{u})$ is the number of injective, edge preserving mappings $\varphi\colon V(M_{k-1}) \to V(H_n)$ with $\varphi(x_i) = u_i$ for $i = 1, \dots, q$. The generalisation of P_7 then reads as follows.

$\mathrm{MDEG}_d(\varepsilon)$**:** We say a k-graph H_n on n vertices has $\mathrm{MDEG}_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$\sum_{\boldsymbol{u}} \left|\mathrm{ext}(M_{k-1}, H_n, \boldsymbol{u}) - d^{2^{k-1}} n^{(k-1)2^{k-2}}\right| \le \varepsilon n^{(k+1)2^{k-2}}$$

where the sum runs over all ordered 2^{k-1}-element subsets $\boldsymbol{u}$ in $V(H_n)$.

After this discussion of the extension of properties P_1, P_2, P_6, and P_7 we state our first result (for the proof see Section 2), which asserts that those generalisations are equivalent (recall the definition of equivalent properties in the paragraph above (1)).

Theorem 3. *For every integer $k \ge 2$ and every $d > 0$ the properties DISC_d, CL_d, ICL_d, MIN_d, DEV_d, and MDEG_d are equivalent.*

Note that, due to MIN_d, restricting ICL_d to all pairs of k-graphs $F' \subseteq F$ for $\ell \geq k2^{k-1}$ fixed is already equivalent to DISC_d and, in fact, P_1 was stated in [5] in that form.

In the proof of Theorem 3 we will use (1) which was proved in [14]. We will include a direct proof of the implication from DEV_d to CL_d in Section 2.5.

1.2. **Quasi-random pairs for graphs.** Theorem 3, although a result about k-graphs, has an interesting consequence for graphs. Recall property P_2 essentially says that if the density of a graph G is at least $d - o(1)$ and the density of 4-cycles is at most $d^4 + o(1)$, then G is a quasi-random graph with density d. In other words, lower and upper bounds on the number of K_2 and C_4 in G imply that G is quasi-random and the question arises which other pairs of graphs replacing K_2 and C_4 have the same effect. Such pairs are called *quasi-random pairs* (cf. notion of forcing pairs in [5]). For example, it was noted in [5] and [25] that C_4 may be replaced by any even cycle or any complete bipartite graph $K_{a,b}$ with $a, b \geq 2$ and, furthermore, every pair of even cycles of different length and every pair of distinct complete bipartite graphs with one vertex class of size 2 are quasi-random pairs. Moreover, it follows from the recent work of Hatami [13] that C_4 can be replaced by Q_k, the graph of the k-dimensional hypercube for $k \geq 2$.

However, all those quasi-random pairs consist of bipartite graphs and it would be interesting to find quasi-random pairs involving non-bipartite graphs (see, e.g., [20]). Below, we will use Theorem 3 combined with Theorem 2 to verify certain quasi-random pairs involving cliques.

For an integer k let $M(k)$ be the graph which we obtain if we replace every hyperedge of the k-graph M_k by a graph clique of order k. Since the k-graph M_k is linear, the graph $M(k)$ consists of 2^k graph cliques K_k, which intersect in at most one vertex. Hence, $M(k)$ consists of $k2^{k-1}$ vertices and $2^k\binom{k}{2}$ edges. (Alternatively, $M(k)$ is the graph we obtain from the k-dimensional hypercube, by letting $V(M(k))$ be the edges of the hypercube and letting edges of $M(k)$ connect two edges of the hypercube if they have a common end-vertex. In other words, $M(k)$ is the line graph of the graph of the k-dimensional hypercube Q_k.) The following corollary of Theorem 3 shows that for every $k \geq 2$ the pair of graphs K_k and $M(k)$ is a quasi-random pair.

Corollary 4. *For every integer $k \geq 2$, every $d > 0$, and every $\delta > 0$ exist $\varepsilon > 0$ and n_0 such that the following is true. If $G = (V, E)$ is a graph on $|V| = n \geq n_0$ vertices that satisfies*

$$N_{K_k}(G) \geq d^{\binom{k}{2}} n^k - \varepsilon n^k \quad \text{and} \quad N_{M(k)}(G) \leq d^{2^k\binom{k}{2}} n^{k2^{k-1}} + \varepsilon n^{k2^{k-1}},$$

then G satisfies $\mathrm{DISC}_d(\delta)$.

Proof. We briefly sketch the proof of Corollary 4. From the given graph G we construct a k-graph $H = H(G)$, where the hyperedges of H correspond to the cliques K_k of G. Therefore we have a one-to-one correspondence between the hyperedges of H and the K_k's of G, as well as, between the copies of M_k in H and the copies of $M(k)$ in G. Hence, the assumption on G implies that H satisfies $\mathrm{MIN}_{d'}$ for k-graphs for $d' = d^{\binom{k}{2}}$ and from Theorem 3 we infer that H satisfies $\mathrm{DISC}_{d'}(\varepsilon')$ for k-graphs for some $\varepsilon' = \varepsilon'(\varepsilon)$ with $\varepsilon' \to 0$ as $\varepsilon \to 0$. But $\mathrm{DISC}_{d'}(\varepsilon')$ for H implies that the assumption of Theorem 2 for the graphs $F = K_k$ and G are met and, hence, Theorem 2 yields that G satisfies $\mathrm{DISC}_d(\delta)$ for graphs for some $\delta = \delta(\varepsilon')$. □

1.3. **Hereditary subgraphs properties.** From Theorem 3 we know that k-graphs containing the "right" number of copies of M are quasi-random. However, note that for characterising quasi-randomness the linear k-graph M cannot be replaced by an arbitrary (linear) k-graph. For example, in the case of graphs, the C_4 in P_2 cannot be replaced by a triangle, as the following example from [5] shows: partition the vertex set $V(G_n)$ in four sets $X_1 \dot\cup X_2 \dot\cup X_3 \dot\cup X_4 = V(G_n)$ as equal as possible and add the edges of the complete graph on X_1, of the complete graph on X_2, of the complete bipartite graph with vertex classes X_3 and X_4, and of the random bipartite graph of density $1/2$ with vertex classes $X_1 \dot\cup X_2$ and $X_3 \dot\cup X_4$. Simple calculations show, that G_n defined this way has density $1/2 + o(1)$ and contains $n^3/8 + o(n^3)$ labeled triangles. On the other hand, G_n is not quasi-random, as it obviously violates P_4. Moreover, due to Theorem 1, a quasi-random graph must be *hereditarily* quasi-random, since if G_n satisfies P_4, then induced subgraphs $G_n[U]$ for large subsets also satisfy P_4 (with a bigger error). Consequently, any property equivalent to P_4 must directly apply to induced subgraphs of linear sized subsets. (It is not obvious that all the properties in Theorem 1 indeed have this quality, but e.g. due to Theorem 1 it follows.) Returning to the example of triangles, we note that the "counterexample" shows that there are graphs which have globally the "right" number of triangles, but there are large subsets on which the number of triangles is wrong, e.g. $G_n[X_1]$ contains too many (more than $(n/4)^3/8$) triangles. In order to rule out this phenomenon Simonovits and Sós suggested a notion of hereditary properties and in [23] they showed that a graph G with density d is quasi-random if and only if every induced subgraph of G contains the right number of copies of a fixed graph F (see Theorem 2). This result has been extended to the case of induced copies of F by Simonovits and Sós [24] and by Shapira and Yuster [20]. We will continue this line of research and introduce hereditary properties for k-graphs, which are equivalent to DISC_d.

Let H_n be a k-graph on n vertices and let F be a k-graph with vertex set $[\ell] = \{1, \dots, \ell\}$. For pairwise disjoint sets $U_1, \dots, U_\ell \subseteq V(H_n)$ let $N_F(U_1, \dots, U_\ell)$ denote the number of partite-isomorphic, copies of F in H_n, i.e., the number of ℓ-tuples $(h_1, \dots, h_\ell)$ with $h_1 \in U_1, \dots, h_\ell \in U_\ell$ such that $\{h_{i_1}, \dots, h_{i_k}\}$ is an edge in H_n if $\{i_1, \dots, i_k\}$ is an edge in F. We define the following properties and show that they are equivalent to DISC_d.

$\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\varepsilon)$**:** We say a k-graph H_n on n vertices has $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\varepsilon)$ for a linear k-graph F with $V(F) = [\ell]$, a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\ell) \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \alpha_i < 1$, and d, $\varepsilon > 0$, if for all choices of pairwise disjoint subsets $U_1, \dots, U_\ell \subset V(H_n)$ with $|U_i| = \lfloor \alpha_i n \rfloor$ for all $i \in [\ell]$ we have

$$N_F(U_1, \dots, U_\ell) = d^{e(F)} \prod_{i \in [\ell]} |U_i| \pm \varepsilon n^\ell \,.$$

$\mathrm{HCL}_{d,F}(\varepsilon)$**:** We say a k-graph H_n on n vertices has $\mathrm{HCL}_{d,F}(\varepsilon)$ for a linear k-graph F with $V(F) = [\ell]$ and d, $\varepsilon > 0$, if H_n satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\varepsilon)$ for every vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\ell) \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \alpha_i < 1$.

Theorem 5. *For every integer k, every linear k-graph F with at least one edge and $V(F) = [\ell]$, every $d > 0$, and every vector $\boldsymbol{\alpha} \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \alpha_i < 1$the properties DISC_d, $\mathrm{HCL}_{d,F}$, and $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$ are equivalent.*

We prove Theorem 5 in Section 3. We also like to mention that the property $\mathrm{HCL}_{d,F}$ can be weakened in the graph case. In fact, Theorem 2 shows that it suffices to ensure approximately the right number of copies of the fixed graph F in every subset $U \subseteq V(G_n)$ of the vertices of G_n to make G_n quasi-random. We, however, need the assumption for all partitions of U into ℓ classes. It seems quite plausible that this stronger looking assumption is not needed and, in fact, for 3-graphs this was proved recently by Dellamonica and Rödl [7].

1.4. **Partite versions of** DISC**.** Property P_4 of Theorem 1 has a very natural bipartite version, stating that the number of edges between two subsets is close to half of all possible edges between those sets. More precisely, we may consider the following property.

- **P_4':** $e(U,W) = |U||W|/2 + o(n^2)$ for all pairwise disjoint subsets U, $W \subseteq V(G_n)$, where $e(U,W)$ denotes the number of edges with one vertex in U and one vertex in W.

It is well known that in fact P_4 and P_4' are equivalent. For example P_4 implies P_4' due to the identity $e(U,W) = e(U \cup W) - e(U) - e(W)$, while P_4 follows from P_4' by considering $e(U',W')$ for a random partition $U = U' \dot\cup W'$ of a given set U into classes of size $|U|/2$.

Below we introduce several partite variants of DISC_d for k-graphs, which will turn out to be equivalent. We start with some definitions. For integers $1 \le \ell \le k$ we call $\tau\colon [\ell] \to [k]$ an *(ℓ,k)-function* if $\sum_{i\in[\ell]} \tau(i) = k$. The set of all (ℓ,k)-functions will be denoted by $T(\ell,k)$. For a fixed $\tau \in T(\ell,k)$ and ℓ pairwise disjoint sets $U_1,\dots,U_\ell \subset V$ of some vertex set V we say a k-set $K \in \binom{V}{k}$ has *type τ (with respect to $(U_1,\dots,U_\ell)$)*, if $|K \cap U_i| = \tau(i)$ for all $i \in [\ell]$. The family of all k-sets having type τ is denoted by

$$\mathrm{Vol}_\tau(U_1,\dots,U_\ell) = \left\{K \in \tbinom{V}{k}\colon\ K \text{ has type } \tau\right\}$$

and let $\mathrm{vol}_\tau(U_1,\dots,U_\ell) = |\mathrm{Vol}_\tau(U_1,\dots,U_\ell)| = \prod_{i\in[\ell]} \binom{|U_i|}{\tau(i)}$.

Alternatively $\mathrm{Vol}_\tau(U_1,\dots,U_\ell)$ can be considered the complete k-graph with respect to type τ. The actual edges of a k-graph H_n with vertex set V of type τ with respect to $(U_1,\dots,U_\ell)$ will be denoted by

$$E_\tau(U_1,\dots,U_\ell) = E(H_n) \cap \mathrm{Vol}_\tau(U_1,\dots,U_\ell)$$

and we set $e_\tau(U_1,\dots,U_\ell) = |E_\tau(U_1,\dots,U_\ell)|$. Note that for $k = 2$ and $\ell = 1,2$ there exists only one (ℓ,k)-function and edges of the corresponding type are considered in P_4 ($\ell = 1$) and in P_4' ($\ell = 2$). For general $k \ge 2$ we define the following property.

- **$\mathrm{DISC}_{d,\tau}(\varepsilon)$:** We say a k-graph H_n on n vertices has $\mathrm{DISC}_{d,\tau}(\varepsilon)$ for some (ℓ,k)-function τ, and d, $\varepsilon > 0$, if

$$e_\tau(U_1,\dots,U_\ell) = d\,\mathrm{vol}_\tau(U_1,\dots,U_\ell) \pm \varepsilon n^k$$

for all pairwise disjoint subsets $U_1,\dots,U_\ell \subseteq V(H_n)$.

Next, we define the notion of the ℓ-partite sub-k-graph with respect to the pairwise disjoint sets $U_1,\dots,U_\ell \subset V(H_n)$. The edge set of the complete ℓ-partite k-graph with respect to the classes $U_1,\dots,U_\ell$ is given by

$$\mathrm{Vol}(U_1,\dots,U_\ell) = \bigcup_{\tau\in T(\ell,k)} \mathrm{Vol}_\tau(U_1,\dots,U_\ell) \tag{2}$$

and the actual edge set of the ℓ-partite sub-k-graph on $U_1, \dots, U_\ell$ is

$$E(U_1, \dots, U_\ell) = E(H_n) \cap \mathrm{Vol}(U_1, \dots, U_\ell). \tag{3}$$

Finally, we consider the following notion of uniform edge distribution.

$\mathrm{DISC}_{d,\ell}(\varepsilon)$**:** We say a k-graph H_n on n vertices has $\mathrm{DISC}_{d,\ell}(\varepsilon)$ for some positive integer $\ell \le k$, and d, $\varepsilon > 0$, if

$$e(U_1, \dots, U_\ell) = d\mathrm{vol}(U_1, \dots, U_\ell) \pm \varepsilon n^k$$

for all pairwise disjoint subsets $U_1, \dots, U_\ell \subseteq V(H_n)$.

Note that for arbitrary k the properties DISC_d, $\mathrm{DISC}_{d,1}$, and $\mathrm{DISC}_{d,(1)}$ are the same and $\mathrm{DISC}_{d,k}$ and $\mathrm{DISC}_{d,(1,\dots,1)}$ are the same. Moreover, for $k = 2$ these two properties are equivalent. The following result states that in fact any version of DISC defined above is equivalent to any other.

Theorem 6. *For all integer ℓ and k with $1 \le \ell \le k$, every fixed (ℓ, k)-function τ, and every $d > 0$ the properties DISC_d, $\mathrm{DISC}_{d,\ell}$, and $\mathrm{DISC}_{d,\tau}$ are equivalent.*

2. Proof of Theorem 3

In this section we present the proof of Theorem 3. We have to show that for every $k \ge 2$ and every $d > 0$ the properties DISC_d, CL_d, ICL_d, MIN_d, DEV_d, and MDEG_d are equivalent. As already noted in (1) it was shown in [14] that DISC_d implies CL_d. In Section 2.1 we will show the following obvious implications

$$\begin{array}{ccc} \mathrm{CL}_d & \xRightarrow{\text{Fact } 8} & \mathrm{ICL}_d \\ \text{Fact } 7 \Big\Downarrow & & \Big\Downarrow \text{Fact } 9 \\ \mathrm{MIN}_d & & \mathrm{DEV}_d \end{array} \tag{4}$$

and the proofs of the main implications

$$\mathrm{MIN}_d \xRightarrow{\text{Lemma } 10} \mathrm{DISC}_d \quad \text{and} \quad \mathrm{DEV}_d \xRightarrow{\text{Lemma } 13} \mathrm{DISC}_d$$

will be given in Sections 2.2 and 2.3. Finally, we prove the equivalence of MDEG_d and MIN_d in Section 2.4 (see Lemma 14), which concludes the proof of Theorem 3.

In addition in Section 2.5 we verify a more direct proof of the implication from DEV_d to CL_d.

2.1. Simple facts.

In this section we verify the simple implications from (4). The first implication, $\mathrm{CL}_d \Rightarrow \mathrm{MIN}_d$, follows from the definition that a sequence $(H_n)_{n\in\mathbb{N}}$ satisfies CL_d if for every linear k-graph F and every $\varepsilon > 0$ all but finitely many k-graphs H_n of the sequence satisfy $\mathrm{CL}_d(F, \varepsilon)$.

Fact 7. *For every integer $k \ge 2$, every $d > 0$, and every $\varepsilon > 0$ there exists n_0 such that the following is true. If H is a k-graph that satisfies $\mathrm{CL}_d(K_k, \varepsilon/2)$ and $\mathrm{CL}_d(M, \varepsilon)$, then H satisfies $\mathrm{MIN}_d(\varepsilon)$.*

Proof. Clearly, satisfying $\mathrm{CL}_d(K_k, \varepsilon/2)$ implies $e(H_n) \ge d\binom{n}{k} - \varepsilon n^k$ for sufficiently large n and satisfying $\mathrm{CL}_d(M, \varepsilon)$ yields $N_M(H) \le d^{|E(M)|} n^{|V(M)|} + \varepsilon n^{|V(M)|}$, which gives $\mathrm{MIN}_d(\varepsilon)$. □

A standard argument using the principle of inclusion and exclusion yields the implication from CL_d to ICL_d.

Fact 8. *For every integer $k \geq 2$, every $d > 0$, all linear k-graphs $F' \subseteq F$ with $V(F') = V(F) = [\ell]$ for some integer ℓ, and every $\varepsilon > 0$, there exists $\delta > 0$ such that the following is true. If H is a k-graph that satisfies $\mathrm{CL}_d(\hat{F}, \delta)$ for every k-graph $\hat{F}$ with $F' \subseteq \hat{F} \subseteq F$, then H satisfies $\mathrm{ICL}_d(F', F, \varepsilon)$.*

Proof. Let $\delta = \varepsilon / 2^{e(F)-e(F')}$ and H be a k-graph on n vertices. Note that by the principle of inclusion and exclusion we have

$$N^*_{F',F}(H) = \sum_{F' \subseteq \hat{F} \subseteq F} (-1)^{e(\hat{F})-e(F')} N_{\hat{F}}(H)\,.$$

Since H satisfies $\mathrm{CL}_d(\hat{F}, \delta)$ for every k-graph $\hat{F}$ with $F' \subseteq \hat{F} \subseteq F$ we obtain

$$N^*_{F',F}(H) = d^{e(F')}(1-d)^{e(F)-e(F')} n^\ell \pm 2^{e(F)-e(F')} \delta n^\ell\,,$$

which shows that H satisfies $\mathrm{ICL}_d(F', F, \varepsilon)$. □

We close this section by observing that ICL_d implies DEV_d.

Fact 9. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$, there exists $\delta > 0$ such that the following is true. If H is a k-graph that satisfies $\mathrm{ICL}_d(M', M, \delta)$ for every k-graph $M' \subseteq M$, then H satisfies $\mathrm{DEV}_d(\varepsilon)$.*

Proof. Set $\delta = \varepsilon / 2^{2^k}$. Let H be a k-graph on n vertices with vertex set $V = V(H)$ satisfying $\mathrm{ICL}_d(M', M, \delta)$ for every $M' \subseteq M$. Recall that the edge weights w of the complete k-graph K_V on V are $1-d$ for edges of H and $-d$ for edges of the complement of H. Moreover, $w(\tilde{A})$ for subgraph $\tilde{A} \subseteq K_V$ is $\prod_{e \in E(\tilde{A})} w(e)$. Summing over all copies $\tilde{M}$ of M in K_V we obtain

$$\sum_{\tilde{M}} w(\tilde{M}) = \sum_{M' \subseteq M} (1-d)^{e(M')} (-d)^{2^k - e(M')} N^*_{M',M}(H)\,.$$

Applying the assumption that H satisfies $\mathrm{ICL}_d(M', M, \delta)$ for all k-graphs $M' \subseteq M$ we get

$$\begin{aligned}\sum_{\tilde{M}} w(\tilde{M}) &= \sum_{M' \subseteq M} (1-d)^{e(M')} (-d)^{2^k - e(M')} \Big(d^{e(M')} (1-d)^{2^k - e(M')} \pm \delta \Big) n^{|V(M)|} \\ &= \sum_{j=0}^{2^k} \binom{2^k}{j} \Big(d(1-d) \Big)^j \Big((-d)(1-d) \Big)^{2^k - j} n^{|V(M)|} \pm 2^{2^k} \delta n^{|V(M)|}\,.\end{aligned}$$

Consequently, the binomial theorem and the choice of δ yields DEV_d,

$$\Big| \textstyle\sum_{\tilde{M}} w(\tilde{M}) \Big| \leq \varepsilon n^{|V(M)|}\,.$$

□

2.2. MIN **implies** DISC**.** In this section we focus on one of the central implication of Theorem 3 and prove the following lemma, which asserts that MIN_d implies DISC_d.

Lemma 10. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$, there exists $\delta > 0$ and n_0 such that the following is true. If H is a k-graph on $n \geq n_0$ vertices that satisfies $\mathrm{MIN}_d(\delta)$, then H satisfies $\mathrm{DISC}_d(\varepsilon)$.*

Before we prove Lemma 10 we introduce a bit of notation, which will be also useful for the proof of Lemma 13. It will be convenient to consider the number of homomorphisms from certain k-graphs A to some k-graph H, instead of the number of labeled copies of A in H. Recall that a homomorphism from A to H is a (not necessarily injective) mapping from $V(A)$ to $V(H)$ that preserves edges. Note that the difference of the number of homomorphisms and the number of labeled copies of A in H is $o(|V(H)|^{|V(A)|})$, which is inessential for the properties considered in Theorem 3.

Let A be a k-partite k-graph given with its partition classes $X_1,\dots,X_k$ and let $U_1,\dots,U_k$ be (not necessarily pairwise disjoint) subsets of $V(H)$ and set $\mathcal{U}=(U_1,\dots,U_k)$. We denote by $\mathrm{Hom}(A,H,\mathcal{U})$ those homomorphisms φ from A to H that map every X_i into U_i, i.e. $\varphi(X_i)\subseteq U_i$ for all $i\in[k]$. Furthermore, let $\hom(A,H,\mathcal{U})=|\mathrm{Hom}(A,H,\mathcal{U})|$.

Moreover, let $X_i=\{x_{i,1},\dots,x_{i,|X_i|}\}$ be a labeling of the vertices of the partition class X_i. Then, for an $|X_i|$-tuple $\boldsymbol{u}_i=(u_1,\dots,u_{|X_i|})\in U_i^{|X_i|}$ denote by $\mathrm{Hom}(M,H,\mathcal{U},i,\boldsymbol{u}_i)$ those homomorphisms φ from $\mathrm{Hom}(M,H,\mathcal{U})$, that map the j-th vertex in the ordering of X_i to u_j, i.e., $\varphi(x_{i,j})=u_j$. Similarly, let $\hom(A,H,\mathcal{U},i,\boldsymbol{u}_i)=|\mathrm{Hom}(A,H,\mathcal{U},i,\boldsymbol{u}_i)|$.

The following well known fact (see, e.g. [25]) will be useful for the proof of Lemma 10.

Fact 11. *For every $\gamma>0$ there exists $\eta>0$ such that for all non-negative reals $a_1,\dots,a_N$ and a satisfying $\sum_{i=1}^N a_i\geq(1-\eta)aN$ and $\sum_{i=1}^N a_i^2\leq(1+\eta)a^2N$, we have $|\{i\in[N]\colon\ |a-a_i|<\gamma a\}|>(1-\gamma)N$.* □

Proof of Lemma 10. We first make a few observations (see Claim 12 below). For that let H be a k-graph with vertex set $V=V(H)$ and let $U_1,\dots,U_k$ be arbitrary, not necessarily disjoint, subsets of V. Set $\mathcal{U}=(U_1,\dots,U_k)$. For every $j\in[k]$ the Cauchy-Schwarz inequality yields

$$\sum_{\boldsymbol{u}_j\in U_j^{2^{j-1}}}\big(\hom(M_{j-1},H,\mathcal{U},j,\boldsymbol{u}_j)\big)^2 \geq\frac{1}{|U_j|^{2^{j-1}}}\Bigg(\sum_{\boldsymbol{u}_j\in U_j^{2^{j-1}}}\hom(M_{j-1},H,\mathcal{U},j,\boldsymbol{u}_j)\Bigg)^2. \quad (5)$$

Furthermore note, that $M_j=\mathrm{db}_j(M_{j-1})$, i.e., M_j arises from M_{j-1} by "fixing" the vertices from the j-th partition class of M_{j-1}, denoted by $X_j(M_{j-1})$, and "doubling" all other vertices of M_{j-1} and the corresponding edges. Thus, this definition yields the following identity for every $j\in[k]$.

$$\hom(M_j,H,\mathcal{U})=\sum_{\boldsymbol{u}_j\in U_j^{2^{j-1}}}\hom(M_j,H,\mathcal{U},j,\boldsymbol{u}_j) =\sum_{\boldsymbol{u}_j\in U_j^{2^{j-1}}}\big(\hom(M_{j-1},H,\mathcal{U},j,\boldsymbol{u}_j)\big)^2. \quad (6)$$

Combining (5) and (6), we get

$$\begin{aligned}\hom(M_j,H,\mathcal{U}) &\overset{(6)}{=} \sum_{\boldsymbol{u}_j\in U_j^{2^{j-1}}} \big(\hom(M_{j-1},H,\mathcal{U},j,\boldsymbol{u}_j)\big)^2\\ &\overset{(5)}{\geq} \frac{1}{|U_j|^{2^{j-1}}}\Bigg(\sum_{\boldsymbol{u}_j\in U_j^{2^{j-1}}} \hom(M_{j-1},H,\mathcal{U},j,\boldsymbol{u}_j)\Bigg)^2\\ &= \frac{1}{|U_j|^{2^{j-1}}}\big(\hom(M_{j-1},H,\mathcal{U})\big)^2 .\end{aligned}$$

Iterating the last estimate $j-\ell+1$ times for some $1\leq \ell\leq j$ we get the following line of inequalities for every integer r between ℓ and j

$$\hom(M_j,H,\mathcal{U}) = \sum_{\boldsymbol{u}_j\in U_j^{2^{j-1}}} \big(\hom(M_{j-1},H,\mathcal{U},j,\boldsymbol{u}_j)\big)^2 \tag{7}$$

$$\geq \left(\frac{1}{|U_j|}\right)^{2^{j-1}} \Bigg(\sum_{\boldsymbol{u}_j\in U_j^{2^{j-1}}} \hom(M_{j-1},H,\mathcal{U},j,\boldsymbol{u}_j)\Bigg)^2$$

$$\dots$$

$$\geq \left(\prod_{i=r+1}^{j}\frac{1}{|U_i|}\right)^{2^{j-1}} \Bigg(\sum_{\boldsymbol{u}_r\in U_r^{2^{r-1}}} \big(\hom(M_{r-1},H,\mathcal{U},r,\boldsymbol{u}_r)\big)^2\Bigg)^{2^{j-r}} \tag{8}$$

$$\geq \left(\prod_{i=r}^{j}\frac{1}{|U_i|}\right)^{2^{j-1}} \Bigg(\sum_{\boldsymbol{u}_r\in U_r^{2^{r-1}}} \hom(M_{r-1},H,\mathcal{U},r,\boldsymbol{u}_r)\Bigg)^{2^{j-r+1}} \tag{9}$$

$$\dots$$

$$= \left(\prod_{i=\ell}^{j}\frac{1}{|U_i|}\right)^{2^{j-1}} \Big(\hom(M_{\ell-1},H,\mathcal{U})\Big)^{2^{j-\ell+1}} . \tag{10}$$

Combining the last line of inequalities with Fact 11 yields the following claim.

Claim 12. *For all integers $k\geq j\geq \ell\geq 1$ and every $\gamma_{j,\ell}>0$ there exists $\eta_{j,\ell}>0$ such that for all $\mathcal{U}=(U_1,\dots,U_k)$ with $U_i\subseteq V$ the following is true. If*

(*a*) $\hom(M_{\ell-1},H,\mathcal{U})\geq (1-\eta_{j,\ell})d^{2^{\ell-1}}\prod_{i=1}^{\ell-1}|U_i|^{2^{\ell-2}}\prod_{i=\ell}^{k}|U_i|^{2^{\ell-1}}$ *and*
(*b*) $\hom(M_j,H,\mathcal{U})\leq (1+\eta_{j,\ell})d^{2^j}\prod_{i=1}^{j}|U_i|^{2^{j-1}}\prod_{i=j+1}^{k}|U_i|^{2^j}$

hold, then for every r with $\ell\leq r\leq j$ the following holds. For all but at most $\gamma_{j,\ell}|U_r|^{2^{r-1}}$ tuples $\boldsymbol{u}_r=(u_1,\dots,u_{2^{r-1}})$ from $U_r^{2^{r-1}}$ we have

$$\hom(M_{r-1},H,\mathcal{U},r,\boldsymbol{u}_r)=(1\pm\gamma_{j,\ell})d^{2^{r-1}}\prod_{i=1}^{r-1}|U_i|^{2^{r-2}}\prod_{i=r+1}^{k}|U_i|^{2^{r-1}} .$$

Proof of Claim 12. Note that the assumptions (*a*) and (*b*) of the claim yield a lower bound for the right-hand side of (10) and an upper bound for the left-hand side in (7). Consequently, for every r between ℓ and j we obtain from (8) and (9)

$$\sum_{\boldsymbol{u}_r\in U_r^{2^{r-1}}}\big(\hom(M_{r-1},H,\mathcal{U},r,\boldsymbol{u}_r)\big)^2\le(1+\eta_{j,\ell})^{1/2^{j-r}}d^{2^r}\prod_{i=1}^{r}|U_i|^{2^{r-1}}\prod_{i=r+1}^{k}|U_i|^{2^r}$$

and

$$\sum_{\boldsymbol{u}_r\in U_r^{2^{r-1}}}\hom(M_{r-1},H,\mathcal{U},r,\boldsymbol{u}_r)\ge(1-\eta_{j,\ell})^{2^{r-\ell}}d^{2^{r-1}}\prod_{i=1}^{r-1}|U_i|^{2^{r-2}}\prod_{i=r}^{k}|U_i|^{2^{r-1}}\,.$$

Hence, a sufficiently small choice of $\eta_{j,\ell}>0$ yields the conclusion of Claim 12 due to Fact 11 applied with $N=|U_r|^{2^{r-1}}$ and $a=d^{2^{r-1}}\prod_{i=1}^{r-1}|U_i|^{2^{r-2}}\prod_{i=r+1}^{k}|U_i|^{2^{r-1}}$. □

After those preparations we finally prove Lemma 10. Let k, d, and ε be given. We determine $\delta>0$ as follows: Set $\gamma_{1,1}=\varepsilon/4$ and for $j=2,\dots,k$ let

$$\gamma_{j,1}=\tfrac{1}{2}(d\varepsilon)^{2^{j-1}}\eta_{j-1,1}\,,$$

where $\eta_{j-1,1}$ is given by Claim 12 applied for $j-1$, $\ell=1$ with $\gamma_{j-1,1}$. We then set $\delta=\eta_{k,1}/2$ and let n_0 be sufficiently large.

Suppose the k-graph H with vertex set V satisfies $\mathrm{MIN}_d(\delta)$. We have to show that H satisfies $\mathrm{DISC}_d(\varepsilon)$. For that fix an arbitrary set $U\subseteq V$. We have to show that

$$e(U)=d\tbinom{|U|}{k}\pm\varepsilon n^k\,. \tag{11}$$

This claim is trivial for sets U of size at most εn, so we assume $|U|\ge\varepsilon n$.

We are going to apply Claim 12 k times. We start with $j=k$, $\ell=1$, and $\mathcal{U}_k=(U_{k,1},\dots,U_{k,k})$, where all sets $U_{k,i}$ are equal to V for $i=1,\dots,k$. Note that the property $\mathrm{MIN}_d(\delta)$ shows that for sufficiently large n the assumptions (a) and (b) of Claim 12 are satisfied by H. Recall, that $M_0=K_k$ consists of one edge and

$$\hom(M_0,H,(V,\dots,V))=k!e(H)$$

here. Now the conclusion of Claim 12 for $r=k$ shows that, due to the choice of $\gamma_{k,1}$ and $|U|\ge\varepsilon n$, the assumption (b) of Claim 12 for $j=k-1$, $\ell=1$, and $\mathcal{U}_{k-1}=(U_{k-1,1},\dots,U_{k-1,k})$ with $U_{k-1,i}=V$ for $i=1,\dots,k-1$ and $U_{k-1,k}=U$ is met.

Moreover, noting that in general if $U_1=U_i$, then $\hom(M_0,H,\mathcal{U},1,(u))=\hom(M_0,H,\mathcal{U},i,(u))$ for every $u\in U_1=U_i$, we see that conclusion of Claim 12 for $r=1$ applied for $j=k$, $\ell=1$, and $\mathcal{U}_k$, yields the assumption (a) of Claim 12 for $j=k-1$, $\ell=1$, and $\mathcal{U}_{k-1}$.

In general we apply Claim 12 for $j=k,\dots,1$, always with $\ell=1$, and $\mathcal{U}_j=(U_{j,1},\dots,U_{j,k})$, where $U_{j,1}=\dots=U_{j,j}=V$ and $U_{j,j+1}=\dots=U_{j,k}=U$ and observe, as above, that the conclusion of Claim 12 for j yield the assumptions for $j-1$.

This way the conclusion of the last application of Claim 12 for $j=\ell=1$ and $r=1$ gives a lower and an upper bound for $\hom(M_0,H,(V,U,\dots,U),1,(u))$ for all but at most $\gamma_{1,1}|V|$ vertices of $u\in V$. Consequently,

$$\begin{aligned}k!e(U)&=\sum_{u\in U}\hom(M_0,H,(V,U,\dots,U),1,(u))\\&=|U|(1\pm\gamma_{1,1})d|U|^{k-1}\pm\gamma_{1,1}|V||U|^{k-1}=d|U|^k\pm\tfrac{\varepsilon}{2}n^k\,,\end{aligned}$$

which yields (11) for sufficiently large n. □

2.3. DEV **implies** DISC. In this section we verify another of the key implication of Theorem 3, by showing that DEV_d implies DISC_d.

Lemma 13. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$, there exists $\delta > 0$ and n_0 such that the following is true. If H is a k-graph on $n \geq n_0$ vertices that satisfies $\mathrm{DEV}_d(\delta)$, then H satisfies $\mathrm{DISC}_d(\varepsilon)$.*

Proof. For given k, d and ε we set $\delta = (\varepsilon/4)^{2^k}$ and n_0 sufficiently large. Let H be a k-graph with vertex set $V = V(H)$ and $|V| = n \geq n_0$, which satisfies $\mathrm{DEV}_d(\delta)$. We want to verify $\mathrm{DISC}_d(\varepsilon)$ and for that let $U \subseteq V$ be a subset of vertices. Again we may assume without loss of generality that $|U| \geq \varepsilon n$.

Again, as in Section 2.2, we consider homomorphisms of M (and its subhypergraphs) instead of labeled copies. Additionally to the notation from Section 2.2, we denote by $\mathcal{V} = (V, \dots, V)$ the vector which contains the vertex set V k times. Moreover, we denote by K_V the complete k-graph with vertex set V. Recall that $w\colon E(K_V) \to [-1, 1]$, where $w(e) = 1 - d$ if $e \in E(H)$ and $w(e) = -d$ otherwise. We introduce $f(M_j, H, U)$, which is a short hand notation for the total weight of all homomorphisms of M_j into K_V with the property that the "last" $k - j$ vertex classes $X_{j+1}(M_j), \dots, X_k(M_j)$ of M_j are mapped into U. More precisely, for $j = 0, \dots, k$ we set

$$f(M_j, H, U) = \sum_{\varphi \in \mathrm{Hom}(M_j, K_V, \mathcal{V})} \prod_{e \in E(M_j)} w(\varphi(e)) \prod_{i=j+1}^{k} \prod_{x \in X_i(M_j)} \mathbb{1}_U(\varphi(x)), \tag{12}$$

where $\mathbb{1}_U$ denotes the indicator function of U. Fixing first the image of $X_{j+1}(M_j)$ and summing over all homomorphisms φ which extend this choice to a full homomorphism of M_j, we can rewrite $f(M_j, H, U)$ as follows

$$\sum_{\boldsymbol{v} \in V^{2^j}} \prod_{i=1}^{2^j} \mathbb{1}_U(v_i) \sum_{\varphi \in \mathrm{Hom}(M_j, K_V, \mathcal{V}, j+1, \boldsymbol{v})} \prod_{e \in E(M_j)} w(\varphi(e)) \prod_{i=j+2}^{k} \prod_{x \in X_i(M_j)} \mathbb{1}_U(\varphi(x)) \,.$$

Recalling, that $M_{j+1} = \mathrm{db}_{j+1}(M_j)$, i.e., M_{j+1} arises from M_j by fixing the $(j+1)$-st vertex class $X_{j+1}(M_j)$ of M_j and "doubling" all the edges together with the remaining vertices, and applying the Cauchy-Schwarz inequality to $f(M_j, H, U)$ (to the form stated above), we obtain

$$\big(f(M_j, H, U)\big)^2 \leq |U|^{2^j} f(M_{j+1}, H, U)$$

for every $j \in \{0, \dots, k-1\}$ and, consequently,

$$\big(f(M_j, H, U)\big)^{2^{k-j}} \leq |U|^{2^{k-1}} \big(f(M_{j+1}, H, U)\big)^{2^{k-j-1}} \,.$$

Applying the last inequality inductively for $j = 0, \dots, k-1$ we obtain

$$\big|f(M_0, H, U)\big|^{2^k} \leq |U|^{k2^{k-1}} \big|f(M_k, H, U)\big| \,. \tag{13}$$

Since M_0 consists of a single edge we have

$$f(M_0, H, U) = k!e(U) - dk!\tbinom{|U|}{k} = k!e(U) - d|U|^k \pm \delta n^k \,,$$

since $|U| \geq \varepsilon n$ and n is sufficiently large. On the other hand, since $M_k = M$ we have for sufficiently large n

$$f(M_k, H, U) = \sum_{\varphi \in \mathrm{Hom}(M, K_V, \mathcal{V})} \prod_{e \in E(M)} w(\varphi(e)) = \sum_{\tilde{M}} \prod_{e \in E(\tilde{M})} w(\varphi(e)) \pm \delta n^{|V(M)|},$$

where the sum runs over all copies $\tilde{M}$ of M in K_V. Since H satisfies $\mathrm{DEV}_d(\delta)$ we obtain for sufficiently large n

$$|f(M_k, H, U)| \leq 2\delta n^{|V(M)|}$$

and consequently (13) yields

$$|k!e(U) - d|U|^k| \leq (\delta + (2\delta)^{1/2^k})n^k$$

which implies

$$e_H(U) = d\binom{|U|}{k} \pm \varepsilon n^k,$$

for sufficiently large n by our choice of δ. □

2.4. Equivalence of MIN and MDEG.

In this section we verify the equivalence of MIN_d and MDEG_d. As we will see the implication from MIN_d to MDEG_d is quite straightforward. Moreover, the reverse implication would be trivial, if MDEG_d would comprise the assumption that $e(H) \geq d\binom{n}{k} - o(n^k)$. In fact, in the main part of the proof we will deduce that k-graphs having MDEG_d must have the right density.

Lemma 14. *For every integer $k \geq 2$, every $d > 0$, and every ε, $\varepsilon' > 0$, there exists δ, $\delta' > 0$ and n_0 such that the following is true.*

- (*i*) *If H is a k-graph on $n \geq n_0$ vertices that satisfies $\mathrm{MIN}_d(\delta)$, then H satisfies $\mathrm{MDEG}_d(\varepsilon)$.*
- (*ii*) *If H is a k-graph on $n \geq n_0$ vertices that satisfies $\mathrm{MDEG}_d(\delta')$, then H satisfies $\mathrm{MIN}_d(\varepsilon')$.*

Proof. We start with the proof of (i). Let k, d and ε be given. We set $\gamma_{k,1} = \varepsilon/4$ and we let $\eta_{k,1}$ be given by Claim 12 applied with $j = k$ and $\gamma_{k,1}$. Then set $\delta = \eta_{k,1}/2$ and let n_0 be sufficiently large.

Let H be a k-graph on n vertices satisfying $\mathrm{MIN}_d(\delta)$, i.e., $e(H) \geq d\binom{n}{k} - \delta n^k$ and $N_M(H) \leq d^{e(M)}n^{|V(M)|} + \delta n^{|V(M)|}$ and, consequently, for sufficiently large n we have

$$\hom(M_0, H, \mathcal{V}) \geq dn^k - 2\delta n^k$$

and

$$\hom(M_k, H, \mathcal{V}) \leq d^{e(M_k)}n^{|V(M_k)|} + 2\delta n^{|V(M_k)|}.$$

Hence, the conclusion of Claim 12 implies that

$$\mathrm{ext}(M_{k-1}, H, \boldsymbol{u}) = \hom(M_{k-1}, H, \mathcal{V}, k, \boldsymbol{u}) \pm \tfrac{\varepsilon}{4}n^{(k-1)2^{k-2}} = (d^{2^{k-1}} \pm \tfrac{\varepsilon}{2})n^{(k-1)2^{k-2}}$$

for all but at most $\gamma_{k,1}n^{2^{k-1}}$ labeled subsets $\boldsymbol{u}_k = (u_1, \ldots, u_{2^{k-1}})$ of 2^{k-1} vertices in V. Therefore, from our choice of $\gamma_{k,1} \leq \varepsilon/4$ we obtain

$$\sum_{\boldsymbol{u}} \left| \mathrm{ext}(M_{k-1}, H, \boldsymbol{u}) - d^{2^{k-1}}n^{(k-1)2^{k-2}} \right| \leq \varepsilon n^{(k+1)2^{k-2}},$$

where the sum runs over all labeled 2^{k-1}-element subsets $\boldsymbol{u}$ of V. This shows that H satisfies $\mathrm{MDEG}_d(\varepsilon)$ and concludes the proof of (i) from the lemma.

For the second implication of the lemma, we first note that, due to

$$N_M(H) \leq \sum_{\boldsymbol{u}} \big(\operatorname{ext}(M_{k-1}, H, \boldsymbol{u})\big)^2$$

property $\mathrm{MDEG}_d(\delta')$, for sufficiently small choice of δ', immediately implies

$$N_M(H) \leq d^{2^k} n^{k2^{k-1}} + \varepsilon' n^{k2^{k-1}}\,.$$

Consequently, we have to show that $\mathrm{MDEG}_d(\delta')$ also implies $e(H) \geq d\binom{n}{k} - \varepsilon' n^k$. For that we will verify the following claim.

Claim 15. *For all integers $k-1 \geq j \geq 1$, every $d > 0$ and every $\gamma_j > 0$, there exists $\eta_j \geq 0$ such that the following is true. If*

$$\sum_{\boldsymbol{u}_{j+1} \in V^{2^j}} \Big|\hom(M_j, H, \mathcal{V}, j+1, \boldsymbol{u}_{j+1}) - d^{2^j} n^{|V(M_j)|-2^j}\Big| \leq \eta_j n^{|V(M_j)|}$$

for $\mathcal{V} = (V, \dots, V)$, then

$$\sum_{\boldsymbol{u}_j \in V^{2^{j-1}}} \Big|\hom(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j) - d^{2^{j-1}} n^{|V(M_{j-1})|-2^{j-1}}\Big| \leq \gamma_j n^{|V(M_{j-1})|}\,.$$

Before we verify Claim 15, we deduce part (ii) of Lemma 14 from the claim. For given $\varepsilon' > 0$ let $\gamma_1 = \varepsilon'/2$ and for $j = 1, \dots, k-1$ let η_j be given by Claim 15 applied with γ_j and set $\gamma_{j+1} = \eta_j$. Finally, set $\delta' = \eta_{k-1}/2$ and let n_0 be sufficiently large. From the assumption $\mathrm{MDEG}_d(\delta')$ standard calculations show that the assumption of Claim 15 for $j = k-1$ is satisfied and the conclusion yields the assumption for the claim with $j = k-2$. Repeating this argument for $j = k-2, \dots, 1$ we infer

$$\sum_{u \in V} \Big|\hom(M_0, H, \mathcal{V}, 1, (v)) - dn^{k-1}\Big| \leq \gamma_1 n^k = \frac{\varepsilon'}{2} n^k\,,$$

which yields $e(H) = d\binom{n}{k} \pm \varepsilon' n^k$ for sufficiently large n. □

Proof of Claim 15. For given γ_j let η_j be sufficiently small, determined later. For $\boldsymbol{u}_j \in V^{2^{j-1}}$ set

$$\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j) = \sum_{\boldsymbol{u}'_j \in V^{2^{j-1}}} \hom(M_{j+1}, H, \mathcal{V}, j+1, (\boldsymbol{u}_j, \boldsymbol{u}'_j))\,,$$

i.e., $\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j)$ denotes the number of homomorphisms φ from M_{j+1} to H, where the "first" 2^{j-1} vertices of $X_{j+1}(M_{j+1})$ are mapped to $\boldsymbol{u}_j$. Here we have to clarify what mean "first" 2^{j-1} vertices. By that we mean those vertices in $X_{j+1}(M_{j+1})$ which form $X_{j+1}(M_{j-1})$, i.e., the originals before the j-th "doubling" step. First we observe

$$\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j) = \sum_{\boldsymbol{u}'_j \in V^{2^{j-1}}} \Big(\hom(M_j, H, \mathcal{V}, j+1, (\boldsymbol{u}_j, \boldsymbol{u}'_j))\Big)^2 \tag{14}$$

and the assumption of the claim enables us to control the right-hand side of (14). Indeed, due to the assumption of the claim we know that all but at most $\sqrt[4]{\eta_j} n^{2^{j-1}}$ vectors $\boldsymbol{u}_j \in V^{2^{j-1}}$ there exist at most $\sqrt[4]{\eta_j} n^{2^{j-1}}$ vectors $\boldsymbol{u}'_j \in V^{2^{j-1}}$ such that

$$\big|\hom(M_j, H, \mathcal{V}, j+1, (\boldsymbol{u}_j, \boldsymbol{u}'_j)) - d^{2^j} n^{|V(M_j)|-2^j}\big| \geq \sqrt{\eta_j} n^{|V(M_j)|-2^j}$$

and we call such vectors $\boldsymbol{u}_j \in V^{2^{j-1}}$ *deviant*. For a non-deviant vector $\boldsymbol{u}_j \in V^{2^{j-1}}$ we infer from (14)

$$\begin{aligned}\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j) \\ &= n^{2^{j-1}} d^{2^{j+1}} n^{2|V(M_j)|-2^{j+1}} \pm (3\sqrt{\eta_j} + \sqrt[4]{\eta_j}) n^{2^{j-1}} n^{2|V(M_j)|-2^{j+1}} \\ &= (d^{2^{j+1}} \pm 4\sqrt[4]{\eta_j}) n^{2|V(M_j)|-2^{j+1}+2^{j-1}}. \end{aligned} \tag{15}$$

On the other hand, for all $\boldsymbol{u}_j \in V^{2^{j-1}}$, we have

$$\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j) = \hom(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j), \tag{16}$$

where $\hom(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j)$ denotes the number of homomorphisms φ from M_{j+1} to H, where the "first" 2^{j-1} vertices of $X_j(M_{j+1})$ are mapped to $\boldsymbol{u}_j$. Again, by "first" 2^{j-1} vertices we mean those vertices in $X_j(M_{j+1})$ which form $X_j(M_{j-1}) = X_j(M_j)$, i.e., those vertices which are fixed in the j-th "doubling" step. Now, we further rewrite $\hom(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j)$ and observe that it equals

$$\begin{aligned}\hom(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j) \\ &= \sum_{(\varphi,\varphi')} \hom(M_j, H, \mathcal{V}, j+1, (\varphi(X_{j+1}(M_{j-1})), \varphi'(X_{j+1}(M_{j-1}))), \end{aligned} \tag{17}$$

where the sum is indexed by all pairs of homomorphisms

$$(\varphi, \varphi') \in (\mathrm{Hom}(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j))^2,$$

i.e., over all those pairs of homomorphism each of which extends $\boldsymbol{u}_j$ to a homomorphic image of M_{j-1}. The identity simply says that we obtain all homomorphic images of M_{j+1} which extend $\boldsymbol{u}_j$ as the first 2^{j-1} vertices in $X_j(M_{j+1})$ by taking two homomorphic extensions of $\boldsymbol{u}_j$ to M_{j-1} (to obtain a homomorphic image of M_j) and attaching another homomorphic image of M_j to the image to the thereby fixed images of $X_{j+1}(M_j)$. From (15) we obtain another possibility to apply the assumption of the claim and more importantly to connect it with the conclusion. Note that, given the fixed choice of $\boldsymbol{u}_j$ and $X_{j+1}(M_j)$, there are at most $n^{|V(M_j)|-2^{j-1}-2^j}$ ways to attach such a copy of M_j. Therefore, the assumption combined with (17) yields

$$\begin{aligned}\hom(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j) = \big(\hom(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j)\big)^2 \times d^{2^j} n^{|V(M_j)|-2^j} \\ \pm n^{|V(M_j)|-2^{j-1}-2^j} \times \eta_j n^{|V(M_j)|}. \end{aligned} \tag{18}$$

Combining (15), (16), and (18), we obtain, for non-deviant vectors $\boldsymbol{u}_j \in V^{2^{j-1}}$,

$$\big(\hom(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j)\big)^2 = (d^{2^j} \pm (4\sqrt[4]{\eta_j} + \eta_j)/d^{2^j}) n^{|V(M_j)|-2^{j-1}}$$

and, consequently, for sufficiently small choice of η_j (compared to γ_j and d) we have

$$\left|\hom(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j) - d^{2^{j-1}} n^{|V(M_{j-1})|-2^{j-1}}\right| \le \frac{\gamma_j}{2} n^{|V(M_{j-1})|-2^{j-1}}$$

for non-deviant $\boldsymbol{u}_j \in V^{2^{j-1}}$. Summing over all $\boldsymbol{u}_j \in V^{2^{j-1}}$ we get

$$\begin{aligned}\sum_{\boldsymbol{u}_j \in V^{2^{j-1}}} \left|\hom(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j) - d^{2^{j-1}} n^{|V(M_{j-1})|-2^{j-1}}\right| \\ \le \frac{\gamma_j}{2} n^{|V(M_{j-1})|} + \sqrt[4]{\eta_j} n^{|V(M_{j-1})|} \le \gamma_j n^{|V(M_{j-1})|}\end{aligned}$$

as claimed. □

2.5. DEV **implies** CL. In this section we give a direct proof of $\mathrm{DEV}_d \Rightarrow \mathrm{CL}_d$. For that we will introduce another version of DISC_d called FDISC_d, which is motivated by the quasi-random functions introduced by Gowers in [11, see Section 3]. It will turn out that DEV_d implies FDISC_d (see Lemma 16) and the implication from FDISC_d to CL_d will follow by similar arguments to those from [11] (see Lemma 17).

Before we define FDISC_d, we will generalise the weight function w defined in Section 1. For a k-graph H with vertex set V and some $d \in (0,1]$, we define the weight function $w\colon \binom{V}{\leq k} = \bigcup_{j=1}^{k} \binom{V}{j} \to [-1,1]$ as follows: for a set $X \subseteq V$ of cardinality at most k we set

$$w(X) = \begin{cases} 1-d & \text{if } X \in E(H), \\ -d & \text{otherwise.} \end{cases}$$

Our weight function is now applicable also to subsets of cardinality smaller than k. This generalisation will simplify the notation. Moreover, we will again use homomorphism instead of copies of k-graphs. In this section we study the following properties.

$\mathrm{FDISC}_d(\varepsilon)$**:** We say a k-graph H on n vertices has $\mathrm{FDISC}_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$\left| \sum_{\varphi\colon\ [k]\to V(H)} w(\varphi([k])) \prod_{i=1}^{k} g_i(\varphi(i)) \right| \leq \varepsilon n^k$$

for all families of functions $g_i\colon\ V(H) \to [-1,1]$ with $i \in [k]$.

For convenience we will work with the following version of DEV_d.

$\mathrm{DEV}'_d(\varepsilon)$**:** We say a k-graph H_n on n vertices has $\mathrm{DEV}'_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$\left| \sum_{\varphi\colon\ V(M)\to V} \prod_{e\in E(M)} w(e) \right| \leq \varepsilon n^{k2^{k-1}}.$$

This definition, though formally different to the definition of DEV_d, is equivalent to it. For DEV_d we were summing over all labeled copies of M in K_V, and here we sum over all mappings from $V(M)$ to V (note that we extended w to $\binom{V}{\leq k}$ for that). By doing this, we get at most an additional additive error term of $O(n^{k2^{k-1}-1}) = o(n^{k2^{k-1}})$, which is asymptotically negligible.

Lemma 16. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$ exists $\delta > 0$ and n_0 such that the following is true. If H is a k-graph on $n \geq n_0$ vertices that satisfies $\mathrm{DEV}'_d(\delta)$, then H satisfies $\mathrm{FDISC}_d(\varepsilon)$.*

Proof. The assertion $\mathrm{DEV}_d \Rightarrow \mathrm{FDISC}_d$ is a simple generalisation of the proof of Lemma 13. We only have to replace $\mathbb{1}_U(\varphi(x))$ for $x \in X_i(M_j)$ by $\mathbb{1}_U(\varphi(x))\cdot g_i(\varphi(x))$. Thus, applying each time the Cauchy-Schwarz inequality we will square $\mathbb{1}_U(\varphi(x)) \cdot g_i(\varphi(x))$, and we then only have to upper bound $(g_i(\varphi(x)))^2$ by 1. We also now have to sum over all functions $\varphi\colon V(M_j) \to V$ (instead over all homomorphisms $\varphi \in \mathrm{Hom}(M_j, K_V, \mathcal{V})$). With those adjustments the proof works verbatim. □

Let us now consider the implication FDISC $\Rightarrow$ DEV. For that we assume that $\mathrm{FDISC}_d(\delta)$ holds for some $\delta > 0$. Assume now that $\mathrm{DEV}_d(\varepsilon)$ is not true, i.e. the reverse inequality holds. Let $f \in E(M)$ be fixed, thus we have:

$$\Big| \sum_{\varphi: V(M^{(k)}) \to V} \prod_{e \in E(M^{(k)})} w(\varphi(e)) \Big| = \Big| \sum_{\varphi': V(M^{(k)}) \setminus \{f\} \to V} \sum_{\substack{\varphi: V(M^{(k)}) \to V \\ \varphi|_{V(M^{(k)}) \setminus \{f\}} = \varphi'}} \prod_{e \in E(M^{(k)})} w(\varphi(e)) \Big| > \varepsilon n^{k 2^{k-1}}.$$

Using the triangle inequality and a simple averaging argument, we deduce that there must exist some $\varphi' : V(M^{(k)}) \setminus \{f\} \to V$, such that

$$\Big| \sum_{\substack{\varphi: V(M^{(k)}) \to V \\ \varphi|_{V(M^{(k)}) \setminus \{f\}} = \varphi'}} \prod_{e \in E(M^{(k)})} w(\varphi(e)) \Big| > \varepsilon n^{k}$$

holds. Because $M^{(k)}$ is a linear hypergraph, we may further deduce that for this particular φ' all but one term (i.e. $w(\varphi(f))$) in the product $\prod_{e \in E(M^{(k)})} w(\varphi(e))$ depend on at most an image of only one vertex from f under φ (this is simply due to the fact that $|f \cap e| \le 1$). Thus, with every vertex from f we may associate some function g_i (which will be some part of the product). Moreover, this functions g_i are clearly defined because we are summing only over the extensions of φ'. This all shows that taking $\delta := \varepsilon$, we will obtain a contradiction to $\mathrm{DEV}_d(\delta)$.

In a similar way, one could show that FDISC_d is further equivalent to DISC_d. This proof however employs similar ideas as in $\mathrm{FDISC}_d \Longrightarrow \mathrm{DEV}_d$, and we omit it here.

We close this section with the proof of the implication $\mathrm{FDISC}_d \Rightarrow \mathrm{CL}_d$.

Lemma 17. *For every integer $k \ge 2$, every $d > 0$, every linear k-graph F on ℓ vertices, and every $\varepsilon > 0$, there exists $\delta > 0$ and n_0 such that the following is true. If H is a k-graph on $n \ge n_0$ vertices that satisfies $\mathrm{FDISC}_d(\delta)$, then H satisfies $\mathrm{CL}_d(F, \varepsilon)$.*

Proof. We may assume $E(F) \neq \emptyset$ and let us fix an edge $f \in E(F)$. It suffices to verify an estimate on

$$\hom(F, H) = \sum_{\varphi \in \mathrm{Hom}(F, K_V, \mathcal{V})} \prod_{e \in E(F)} \mathbb{1}_{E(H)}(\varphi(e)). \tag{19}$$

the number of homomorphism from F into H. Here, again, we may further enlarge the sum by going over all functions $\varphi\colon V(F) \to V$. However, for every φ which is not a homomorphism, there will be an $f \in E(F)$ with $|\varphi(f)| < k$, and thus φ will contribute 0 to the total sum. Noting furthermore that $\mathbb{1}_{E(H)}(\varphi(e)) = w(\varphi(e)) + d$ for every $\varphi(e) \in \binom{V}{\le k}$ we may rewrite (19) as

$$\begin{aligned} \hom(F, H) &= \sum_{\varphi\colon V(F) \to V} \prod_{e \in E(F)} (w(\varphi(e)) + d) \\ &= \sum_{\varphi': V(F) \setminus \{f\} \to V} \sum_{\substack{\varphi: V(F) \to V \\ \varphi|_{V(F) \setminus \{f\}} = \varphi'}} \prod_{e \in E(F)} (w(\varphi(e)) + d). \end{aligned}$$

Now we may concentrate on the inner sum. We first multiply out the product $\prod_{e \in E(F)} (w(\varphi(e)) + d)$, and consider the inner sum. We obtain the leading term

$d^{e(F)}n^k$, while the other terms from the product can be interpreted as functions g_i (for every vertex i of f since F is linear) and we can apply $\mathrm{FDISC}_d(\delta)$ to obtain an estimate for the inner sum. Therefore, setting $\delta = \varepsilon/2^{k+1}$, we have shown that the inner sum is $d^{e(F)}n^k \pm \varepsilon n^k/2$ and, hence,

$$\hom(F, H) = d^{e(F)}n^\ell \pm \varepsilon n^\ell \,,$$

which implies $\mathrm{CL}_d(F, \varepsilon)$ for sufficiently large n. $\square$

3. Proof of Theorem 5

In this section we present the proof of Theorem 5. We have to show that for every $k \geq 2$, every linear k-graph F with at least one edge and $V(F) = [\ell]$ for some integer ℓ, every $d > 0$, and every vector $\boldsymbol{\alpha} \in (0,1]^\ell$ the properties DISC_d, $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$ and $\mathrm{HCL}_{d,F}$ are equivalent. In Section 3.1 we show the simple implication

$$\mathrm{HCL}_{d,F,\boldsymbol{\alpha}} \xRightarrow{\text{Fact 18}} \mathrm{HCL}_{d,F} \;\;.$$

The main part of this section is devoted to the proof of $\mathrm{HCL}_{d,F} \Rightarrow \mathrm{DISC}_d$. For that we will introduce another property REG_d, which will turn out to be equivalent to DISC_d and we then show $\mathrm{HCL}_{d,F} \Rightarrow \mathrm{REG}_d$ in Section 3.2

$$\mathrm{HCL}_{d,F} \xRightarrow{\text{Lemma 25}} \mathrm{REG}_d \xLeftrightarrow{\text{Fact 24}} \mathrm{DISC}_d \;\;.$$

Finally, in Section 3.3 we verify

$$\mathrm{DISC}_d \xRightarrow{\text{Fact 27}} \mathrm{HCL}_{d,F,\boldsymbol{\alpha}} \;\;.$$

3.1. $\mathrm{HCL}_{d,F,\alpha}$ implies $\mathrm{HCL}_{d,F}$.

The following observation yields the implication from $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$ to $\mathrm{HCL}_{d,F}$.

Fact 18. *For every integer $k \geq 2$, every $d > 0$, every linear k-graph F with at least one edge and $V(F) = [\ell]$ for some integer ℓ, all vectors $\boldsymbol{\alpha} \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \alpha_i < 1$, and every $\varepsilon > 0$, there exists $\delta > 0$ and n_0 such that the following is true. If H is a k-graph on $n \geq n_0$ vertices that satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\delta)$, then, for all $\boldsymbol{\beta} \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \beta_i < 1$, H satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\beta}}(\varepsilon)$.*

Proof. Note that it suffices to consider the case when $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\ell)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_\ell)$ differ in at most one entry, i.e., there is an $i \in [\ell]$ such that $\alpha_i \neq \beta_i$ and for all $j \neq i$ we have $\alpha_j = \beta_j$. Without loss of generality we may assume that $i = \ell$. For given k, d, F, $\boldsymbol{\alpha}$, and $\varepsilon > 0$ we set $\delta = \varepsilon \min\{\alpha_\ell, 1 - \sum_{i\in[\ell]} \alpha_i\}/7$ and let n_0 be sufficiently large. We then verify the fact for given $\boldsymbol{\beta} \in (0,1)^\ell$.

First, we prove the claim for all $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{\ell-1}, \gamma)$ with $\gamma \geq \alpha_\ell$. Let $U_1, \dots, U_\ell \subseteq V(H)$ be subsets satisfying $|U_i| = \lfloor \beta_i n \rfloor$ for $i \in [\ell - 1]$, $|U_\ell| = \lfloor \gamma n \rfloor$ and $\mathcal{P} = \{W \subset U_\ell \colon |W| = \lfloor \alpha_\ell n \rfloor\}$. Since H satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\delta)$ and $\beta_j = \alpha_j$ for all $j \in [\ell - 1]$ we infer

$$N_F(U_1, \dots, U_{\ell-1}, W) = d^{e(F)} \lfloor \alpha_\ell n \rfloor \prod_{i\in[\ell-1]} |U_i| \pm \delta n^\ell$$

for all $W \in \mathcal{P}$. Hence, having each copy of F counted $\binom{\lfloor \gamma n\rfloor -1}{\lfloor \alpha_\ell n\rfloor -1}$ times, we obtain, for $n \geq 1/\alpha_\ell$,

$$\begin{aligned} N_F(U_1,\dots,U_\ell) &= \binom{\lfloor \gamma n\rfloor -1}{\lfloor \alpha_\ell n\rfloor -1}^{-1} \sum_{W\in\mathcal{P}} N_F(U_1,\dots,U_{\ell-1},W) \\ &= \binom{\lfloor \gamma n\rfloor -1}{\lfloor \alpha_\ell n\rfloor -1}^{-1} \binom{\lfloor \gamma n\rfloor}{\lfloor \alpha_\ell n\rfloor} d^{e(F)} \left(\lfloor \alpha_\ell n\rfloor \prod_{i\in[\ell]} |U_i| \pm \delta n^\ell \right) \\ &= d^{e(F)} \prod_{i\in[\ell]} |U_i| \pm \frac{2\delta}{\alpha_\ell} n^\ell\,, \end{aligned}$$

which by our choice of δ yields the fact for this case.

Suppose $\beta_\ell < \alpha_\ell$. Without loss of generality we may assume that $\sum_{i\in[\ell]} \beta_i + \alpha_\ell < 1$. (Otherwise, first choose $\beta'_\ell = (1-\sum_{i\in[\ell]}\alpha_i)/2$ and then use the proof from above to finish the claim for β_ℓ.) Let $U_1,\dots,U_\ell \subseteq V(H)$ be pairwise disjoint with $|U_i| = \lfloor \beta_i n\rfloor$, $i\in[\ell]$. Considering $W \subseteq V\setminus U_\ell$ of size $|W| = \lfloor \alpha_\ell n\rfloor$ we infer from $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\delta)$ and the case considered above

$$N_F(U_1,\dots,U_{\ell-1},U_\ell\dot\cup W) = d^{e(F)}(\lfloor \alpha_\ell n\rfloor + \lfloor \beta_\ell n\rfloor) \prod_{i\in[\ell-1]} |U_i| \pm \frac{2\delta}{\alpha_\ell} n^\ell$$

and

$$N_F(U_1,\dots,U_{\ell-1},W) = d^{e(F)} \lfloor \alpha_\ell n\rfloor \prod_{i\in[\ell-1]} |U_i| \pm \delta n^\ell.$$

Hence, we have

$$\begin{aligned} N_F(U_1,\dots,U_\ell) &= N_F(U_1,\dots,U_{\ell-1},U_\ell\dot\cup W) - N_F(U_1,\dots,U_{\ell-1},W) \\ &= d^{e(F)} \prod_{i\in[\ell]} |U_i| \pm \frac{3\delta}{\alpha_\ell} n^\ell\,, \end{aligned}$$

which concludes the proof of the fact by the choice of δ. □

3.2. $\mathrm{HCL}_{d,F}$ **implies** DISC_d. In this section we verify the implication from $\mathrm{HCL}_{d,F}$ to DISC_d. The proof is based on ideas of Shapira and Yuster [20], the main tools being the theorem of Gottlieb [10] on the rank of the inclusion matrices and the weak regularity lemma for hypergraphs. In the next section, Section 3.2.1, we introduce the result of Gottlieb and its consequences. In Section 3.2.2 we introduce the weak regularity lemma for hypergraphs and another quasi-random property REG_d, which is equivalent to DISC_d. Finally, in Section 3.2.3 we prove that $\mathrm{HCL}_{d,F}$ implies REG_d.

3.2.1. *Tools from linear algebra.* For positive integers $r \geq \ell \geq k$ the inclusion matrix $I(r,\ell,k)$ is an $\binom{r}{\ell} \times \binom{r}{k}$ matrix defined as follows. For $L \in \binom{[r]}{\ell}$ and $K \in \binom{[r]}{k}$ the entry of $I_{L,K}$ is given by

$$I_{L,K} = \begin{cases} 1 & \text{if } K \subset L \\ 0 & \text{otherwise} \end{cases}$$

Note that we implicitly assume fixed orderings on the set of subgraphs $\binom{[r]}{\ell}$ and on the edge set $\binom{[r]}{k}$. This does not effect the rank of $I(r,\ell,k)$ which is at most $\binom{r}{k}$ and in fact it was shown by Gottlieb [10], that $I(r,\ell,k)$ has full rank if $r \geq \ell + k$.

Theorem 19 (Gottlieb)**.** *For all positive integers $\ell \geq k$ and $r \geq \ell + k$ the inclusion matrix $I(r, \ell, k)$ has rank $\binom{r}{k}$.* □

Note that the rows of $I(r, \ell, k)$ can be interpreted as incidence vectors of the edges of copies of the complete k-graph K_ℓ in K_r. For our purposes, it will be convenient to consider a similar matrix, where the rows correspond to incidence vectors of the edges of the given k-graph F. To this end, for a k-graph F on ℓ vertices, we define the matrix $A(r, F, k)$ as follows. The rows of $A(r, F, k)$ are indexed by the labelled copies of F in K_r and the columns are indexed, as above, by the k-element subsets of $[r]$. Now for a labeled copy $\tilde{F}$ of F in K_r and a k-set $e \in \binom{[r]}{k}$ the entry $A_{\tilde{F},e}$ is given by

$$A_{\tilde{F},e} = \begin{cases} 1 & \text{if } e \in E(\tilde{F}) \\ 0 & \text{otherwise.} \end{cases}$$

Thus $A(r, F, k)$ is a $N_F(K_r) \times \binom{[r]}{k}$ and Theorem 19 determines the rank of $A(r, F, k)$.

Corollary 20. *For all positive integers $\ell \geq k$, $r \geq \ell + k$ and all non-empty k-graphs F on ℓ vertices the matrix $A(r, F, k)$ has rank $\binom{r}{k}$.*

Proof. The proof of Corollary 20 is identical to the proof of Lemma 3.1 in [19] and follows from the observation that the rows of $A(r, F, k)$ span the rows of $I(r, \ell, k)$. Indeed, summing all rows of $A(r, F, k)$ that correspond to copies $\tilde{F}$ of F with the same vertex set $L \in \binom{[r]}{\ell}$ we obtain a multiple of the row in $I(r, \ell, k)$ indexed by L. □

From Corollary 20 we deduce the key lemma of this section, Lemma 21 below. In Lemma 21 we consider complete, weighted k-graphs on r vertices. Let $w\colon E(K_r) \to (0, 1]$ be an arbitrary weight function and F be a fixed k-graph on ℓ vertices. We set the weight of a labeled copy $\tilde{F}$ of F in K_r, as before, to the product of the weights of the edges of $\tilde{F}$, i.e.,

$$w(\tilde{F}) = \prod_{e \in E(\tilde{F})} w(e)\,.$$

Lemma 21 states that if $w(\tilde{F})$ is "almost" the same for all copies of F, then w must be almost constant.

Lemma 21. *For all integers $\ell \geq k \geq 2$ and $r \geq \ell + k$, every $d > 0$, every k-graph F on ℓ vertices with at least one edge, and every $\varepsilon > 0$, there exists $\delta > 0$ such that if $w\colon E(K_r) \to (0, 1]$ satisfies*

$$w(\tilde{F}) = d^{e(F)} \pm \delta$$

for all labeled copies $\tilde{F}$ of F in K_r, then $w(e) = d \pm \varepsilon$ for all $e \in E(K_r)$.

Proof. Let ℓ, k, r, d, F, and ε be given. Due to the continuity of the function 2^x we can choose $\varepsilon' > 0$ such that if $|x - \log_2 d| \leq \varepsilon'$ then $|2^x - d| \leq \varepsilon$. Next we fix an ordering $e_1, \dots, e_m$, $m = \binom{r}{k}$ of the edges of the K_r and an ordering $\tilde{F}_1, \dots, \tilde{F}_t$ for $t = r(r-1)\dots(r-\ell+1)$ of all labeled copies of F in K_r. This defines the matrix $A = A(r, F, k)$ which, by Corollary 20, has rank $\binom{r}{k}$. Thus $A\colon \mathbb{R}^{\binom{r}{k}} \to \mathbb{R}^t$ is an injective and linear function and consequently there exists a $\delta' > 0$ such that the following holds: if $A\boldsymbol{y} = \boldsymbol{b}$ and $A\boldsymbol{x} = \boldsymbol{c}$ with $\|\boldsymbol{b} - \boldsymbol{c}\|_\infty \leq \delta'$ then $\|\boldsymbol{y} - \boldsymbol{x}\|_\infty \leq \varepsilon'$. Further, due to the continuity of the function $\log_2 x$ we can choose $\delta > 0$ such that if $|2^b - d^{e(F)}| \leq \delta$, then $|b - e(F)\log_2 d| \leq \delta'$ and we fix the δ for Lemma 21 this way.

Now let $w\colon E(K_r) \to (0,1]$ satisfy the assumption of the lemma. Therefore, we have for every copy $\tilde{F}$ of F in K_r

$$\sum_{e\in E(\tilde{F})} \log w(e) = \log_2(d^{e(F)} \pm \delta)\,. \tag{20}$$

Let $\boldsymbol{y} = ((y(e_1), \dots, y(e_m)) \in \mathbb{R}^m$ be given by

$$y(e_i) = \log_2 w(e_i)$$

for $i = 1, \dots, m$. Then (20) is equivalent to $A\boldsymbol{y} = \boldsymbol{b}$ where $\boldsymbol{b} = (b_1, \dots, b_t)$ with $b_i = \log_2(d^{e(F)} \pm \delta)$ for all $i \in [t]$.

On the other hand, by Corollary 20 we know that A has rank $\binom{r}{k}$ and, hence, the system of linear equations $A\boldsymbol{x} = \boldsymbol{c}$ for $\boldsymbol{c} = (e(F)\log d)\mathbf{1}_t$ for the all ones vector $\mathbf{1}_t = \{1\}^t$ has at most one solution. Since the everywhere $\log d$ vector $(\log_2 d)\mathbf{1}_m$ is a solution to this system of equations, it must be the unique solution $\boldsymbol{x}$.

From our choice of δ we infer $\|\boldsymbol{b}-\boldsymbol{c}\|_\infty \leq \delta'$ and, consequently, due to the choice of δ' we have $\|\boldsymbol{y} - \boldsymbol{x}\|_\infty \leq \varepsilon'$. In other words, $|\log_2(w(e_i)) - \log_2(d)| \leq \varepsilon'$ for every $i = 1, \dots, m$ and the choice of ε' yields $|w(e) - d| \leq \varepsilon$ for all edges $e \in E(K_r)$. □

3.2.2. *Weak hypergraph regularity lemma.* For the proof of $\mathrm{HCL}_{d,F} \Rightarrow \mathrm{DISC}_d$ we will use the so-called weak regularity lemma for k-graphs, which is a straightforward extension of Szemerédi's regularity lemma for graphs [27]. Roughly speaking, the property $\mathrm{HCL}_{d,F}$ will imply that for the weighted cluster-hypergraph of a regular partition the assumption of Lemma 21 hold. Consequently, the densities of all k-tuples of the regular partition will be close to d and from this we will infer DISC_d. Below we introduce the weak hypergraph regularity lemma and a few related results.

Let $H = (V, E)$ be a k-graph and let $U_1, \dots, U_k$ be pairwise disjoint non-empty subsets of V. Recall that $e(U_1, \dots, U_k)$ denotes the number of edges with one vertex in each U_i, $i \in [k]$ and the *density* of $(U_1, \dots, U_k)$ is defined to be

$$d(U_1, \dots, U_k) = \frac{e(U_1, \dots, U_k)}{|U_1| \cdot \ldots \cdot |U_k|}\,.$$

We say the k-tuple $(V_1, \dots, V_k)$ of pairwise disjoint subsets $V_1, \dots, V_k \subseteq V$ is ε-*regular* if

$$|d(U_1, \dots, U_k) - d(V_1, \dots, V_k)| \leq \varepsilon$$

for all k-tuples of subsets $U_1 \subset V_1, \dots, U_k \subset V_k$ satisfying $|U_1| \geq \varepsilon|U_1|, \dots |U_k| \geq \varepsilon|V_k|$.

Though the notion of weak regularity is not sufficient to imply a general counting lemma it was shown in [14] that it is strong enough to imply a counting lemma for linear k-graphs:

Lemma 22 (Counting lemma for linear hypergraphs)**.** *For all integers $\ell \geq k \geq 2$ and every γ, there exist $\varepsilon = \varepsilon(\ell, k, \gamma) > 0$ and $m_0 = m_0(\ell, k, \gamma)$ so that the following holds.*

Let $F = ([\ell], E(F))$ be a linear k-graph and let $H = (V_1\dot\cup \ldots \dot\cup V_\ell, E)$ be an ℓ-partite, k-graph where $|V_1|, \dots, |V_\ell| \geq m_0$. Suppose, moreover, that for all edges $f \in E(F)$, the k-tuple $(V_i)_{i\in f}$ is (ε, d_f)-regular. Then the following holds:

$$N_F(V_1, \dots, V_\ell) = \prod_{f\in E(F)} d_f \prod_{i\in[\ell]} |V_i| \pm \gamma \prod_{i\in[\ell]} |V_i|\,.$$

□

A partition $V_1\dot\cup\dots\dot\cup V_t$ of $V(H)$ will be called a *t-equipartition* if $|V_1| \leq |V_2| \leq \dots \leq |V_t| \leq |V_1| + 1$ and such an equipartition will be called *ε-regular* if all but at most $\varepsilon\binom{t}{k}$ of the k-tuples $(V_{i_1}, \dots, V_{i_k})$ are ε-regular. The proof of the following theorem follows the lines of the original proof of Szemerédi (see, e.g., [3, 8, 26]).

Theorem 23 (Weak hypergraph regularity lemma)**.** *For all k, $t_0 \in \mathbb{N}$ and all $\varepsilon > 0$ there is a $T_0 = T_0(t_0, \varepsilon)$ and an n_0 such that for all $n \geq n_0$ and all k-graphs H on n vertices there is an ε-regular, t-equipartition of H with t satisfying $t_0 \leq t \leq T_0$.* □

In case of graphs, it was noted by Simonovits and Sós [22] that there is a close relationship between quasi-randomness and the Szemerédi regular partition. Indeed, it is easily shown that a graph G is quasi-random in the sense of Theorem 1 if and only if G permits a partition such that almost all pairs of partition classes are regular and have roughly the same density. This generalises to k-graphs in a straightforward manner.

It will be convenient to consider the property REG_d defined as follows.

$\mathrm{REG}_d(\varepsilon)$**:** We say a k-graph H on n vertices has $\mathrm{REG}_d(\varepsilon)$ for d, $\varepsilon > 0$, if there exists an ε-regular, t-equipartition $V(H) = V_1\dot\cup\dots\dot\cup V_t$ of H with $g(d, \varepsilon) \geq t \geq 1/\varepsilon$ for some arbitrary function $g(d, \varepsilon) \geq 1/\varepsilon$ independent of H and n such that $d(V_{i_1}, \dots, V_{i_k}) = d \pm \varepsilon$ for all but at most εt^k tuples $\{i_1, \dots, i_k\} \in \binom{[t]}{k}$.

It is easy to see that DISC_d and REG_d are equivalent (see, e.g. [3]) and we omit the proof here.

Fact 24. *For every integer $k \geq 2$ and every $d > 0$ the properties DISC_d and REG_d are equivalent.* □

3.2.3. $\mathrm{HCL}_{d,F}$ *implies* REG_d. In this section we deduce REG_d from $\mathrm{HCL}_{d,F}$ by proving the following lemma.

Lemma 25. *For every integer $k \geq 2$, every $d > 0$, every linear k-graph F containing at least one edge, and every $\varepsilon > 0$, there exists $\delta > 0$ and n_0 such that the following is true. If H is a k-graph on $n \geq n_0$ vertices that satisfies $\mathrm{HCL}_{d,F}(\delta)$, then H satisfies $\mathrm{REG}_d(\varepsilon)$.*

Besides the results from Sections 3.2.1 and 3.2.2 we will also need the following consequence of a packing result of Rödl [17].

Lemma 26. *For all integers $r \geq k \geq 2$ and every $\gamma > 0$ there exists an integer t_0 such that for all $t \geq t_0$ the following holds. If R is a k-graph on t vertices with $e(R) \geq (1-\gamma)\binom{t}{k}$ edges, then there exist at least $(1-\gamma r^k)\binom{t}{k}$ edges in R each of which belong to at least one copy of K_r in R.*

Proof. We choose t_0 large enough to guarantee that the packing result of Rödl [17] is applicable for $t \geq t_0$ and r, k, and γ. Given a k-graph R on t vertices which contains at least $(1-\gamma)\binom{t}{k}$ edges we first consider the complete k-graph K_t on the same vertex set. From Rödl's theorem we infer that K_t contains at least $(1-\gamma)\binom{t}{k}/\binom{r}{k}$ edge disjoint copies of the K_r. Taking the same copies of K_r we see that at most $\gamma\binom{t}{k} = \gamma\binom{r}{k}\binom{t}{k}/\binom{r}{k}$ of them fail to be a subgraph of R since R contains at least $(1-\gamma)\binom{t}{k}$ edges. This implies that R contains at least $(1-\gamma-\gamma\binom{r}{k})\binom{t}{k}/\binom{r}{k}$ edge disjoint copies of K_r which implies that all but at most $\gamma r^k\binom{t}{k}$ edges of R are contained in a copy of a K_r in R. □

Proof of Lemma 25. For given k, d, linear k-graph F with at least one edge and $V(F)=[\ell]$, and $\varepsilon>0$, we first apply Lemma 21 with ℓ, k, and $r=\ell+k$, d, F, and ε and obtain $\delta_{\rm GL}>0$. Then we apply the counting lemma, Lemma 22, with ℓ, k, and $\gamma_{\rm CL}=\delta_{\rm GL}/2$ to obtain $\varepsilon_{\rm CL}$ and $m_{\rm CL}$. Further, we apply Lemma 26 with r, k and $\gamma_{\rm PL}=\varepsilon/(2r^k)$ to obtain $t_{\rm PL}$. Applying the weak regularity lemma, Theorem 23, with

$$\varepsilon_{\rm RL}=\min\{\varepsilon_{\rm CL},\varepsilon/(2r^k)\}\quad\text{and}\quad t_0=\max\{1/\varepsilon_{\rm RL},t_{\rm PL}\}$$

we obtain T_0. Finally, we choose $\delta=\delta_{\rm GL}d^{e(F)}/(2^{\ell+2}T_0^\ell)$ and $n_0\geq T_0 m_{\rm CL}$ sufficiently large to satisfy the equations needed.

Let H be a k-graph on n vertices with $n\geq n_0$ which satisfies $\mathrm{HCL}_{d,F}(\delta)$. We have to show that there exists a partition $V_1\dot\cup\dots\dot\cup V_t=V(H)$ such that

(i) $1/\varepsilon\leq t\leq T_0$ (note that $T_0=T_0(d,\varepsilon,F)$ is independent of H and n),
(ii) $||V_i|-|V_j||\leq 1$ for all $i,j\in[t]$
(iii) all but at most εt^k k-tuples $(V_{i_1},\dots V_{i_k})$ are ε-regular and have density $d\pm\varepsilon$.

To this end, we first apply Theorem 23 with $\varepsilon_{\rm RL}$ and t_0 to obtain a partition $V(H)=V_1\dot\cup\dots\dot\cup V_t$, which already satisfies (i) and (ii) and the first part of (iii), i.e., all but at most $\varepsilon_{\rm RL}\binom{t}{k}\leq\frac12\varepsilon t^k$ k-tuples $(V_{i_1},\dots V_{i_k})$ are ε-regular. Thus, it remains to show that all but at most $\frac12\varepsilon t^k$ of the k-tuples $(V_{i_1},\dots V_{i_k})$ have density $d\pm\varepsilon$.

We consider the reduced (or cluster) k-graph R, i.e., the k-graph on the vertex set $\{1,\dots,t\}$ with $\{i_1,\dots,i_k\}$ being an edge if and only if $(V_{i_1},\dots,V_{i_k})$ is $\varepsilon_{\rm RL}$-regular. Then R is a k-graph on t vertices which contains at least $(1-\varepsilon_{\rm RL})\binom{t}{k}$ edges and we assign to each edge $\{i_1,\dots,i_k\}$ the weight

$$w(i_1,\dots,i_k)=d(V_{i_1},\dots,V_{i_k})\,.$$

Applying Lemma 26 to R we know that all but at most $\gamma_{\rm PL}r^k\binom{t}{k}<\frac12\varepsilon t^k$ edges belong to a copy of K_r in R. Thus, it is sufficient to show that every edge contained in a copy of K_r has weight $d\pm\varepsilon$.

For that fix a copy of K_r in R and without loss of generality we may assume that $V_1,\dots,V_r$ are the vertices of that copy. Recall that H satisfies $\mathrm{HCL}_{d,F}(\delta)$ and as a consequence we have for every injective map $\varphi\colon[\ell]\to[r]$

$$N_F(V_{\varphi(1)},\dots,V_{\varphi(\ell)})=d^{e(F)}\prod_{i\in[\ell]}|V_{\varphi(i)}|\pm\delta n^\ell\,.$$

Since each set $V_{\varphi(j)}$ has size at least $n/(2T_0)$ and $\delta=\delta_{\rm GL}/(2^{\ell+2}T_0^\ell)$, we obtain

$$N_F(V_{\varphi(1)},\dots,V_{\varphi(\ell)})=\Big(d^{e(F)}\pm\delta_{\rm GL}/2\Big)\prod_{i\in[\ell]}|V_{\varphi(i)}|\,. \tag{21}$$

On the other hand, applying the counting lemma, Lemma 22, we obtain

$$N_F(V_{\varphi(1)},\dots,V_{\varphi(\ell)})=\left(\prod_{e\in E(F)}w(\varphi(e))\pm\gamma_{\rm CL}\right)\prod_{i\in[\ell]}|V_{\varphi(i)}|\,. \tag{22}$$

Combining (21) and (22) with the choice of $\gamma_{\rm CL}=\delta_{\rm GL}/2$ we conclude that

$$\prod_{e\in E(F)}w(\varphi(e))=d^{e_F}\pm\delta_{\rm GL}$$

for all injective mappings $\varphi\colon [\ell] \to [r]$. By applying Lemma 21 we derive that all edges $\{i_1, \dots, i_k\}$ have weight $d \pm \varepsilon$ and, therefore, $d(V_{i_1}, \dots, V_{i_k}) = d \pm \varepsilon$ which finishes the proof of Lemma 25. $\square$

3.3. DISC_d **implies** $\mathrm{HCL}_{d,F,\alpha}$**.** In this section we deduce $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$ from DISC_d by proving the following lemma.

Fact 27. *For every integer $k \geq 2$, every $d > 0$, every linear k-graph F with at least one edge and $V(F) = [\ell]$ for some integer ℓ, and every vector $\boldsymbol{\alpha} \in (0,1]^\ell$, there exists $\delta > 0$ and n_0 such that the following is true. If H is k-graph on $n \geq n_0$ vertices that satisfies $\mathrm{DISC}_d(\delta)$, then H satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\varepsilon)$.*

Proof. The fact is a simple consequence of the counting lemma, Lemma 22. Indeed for given k, $d > 0$, F, $\boldsymbol{\alpha} \in (0,1]^\ell$, and $\varepsilon > 0$, set δ to be sufficiently small, so that $\mathrm{DISC}_d(\delta)$ implies $\mathrm{DISC}_{d,k}(\delta')$ (see Theorem 6) for $\delta' = (\delta_{\mathrm{CL}} d \min_{i \in \ell} \alpha_i)^k$, where δ_{CL} is given by Lemma 22 applied for F and $\gamma_{\mathrm{CL}} = \varepsilon/2$ and we may assume $\delta_{\mathrm{CL}} \leq \varepsilon/2$. Let n_0 be sufficiently large and H be a k-graph on $n \geq n_0$ vertices which satisfies $\mathrm{DISC}_d(\delta)$.

Let $U_1, \dots, U_\ell \subseteq V(H)$ with $|U_i| = \lfloor \alpha_i n \rfloor$ be pairwise disjoint sets. We consider the induced ℓ-partite k-graph $H[U_1, \dots, U_\ell]$. Since H satisfies $\mathrm{DISC}_d(\delta)$, by Theorem 6 we infer that H satisfies $\mathrm{DISC}_{d,k}(\delta')$. Moreover, since $(\delta')^{1/k} / \min_{i \in [\ell]} \alpha_i \leq \delta_{\mathrm{CL}}$ we have that $(U_{i_1}, \dots, U_{i_k})$ is δ_{CL}-regular with density $d \pm \delta_{\mathrm{CL}}$ for every choice $1 \leq i_i < \dots < i_k \leq \ell$. Consequently, Lemma 22 implies

$$N_F(U_1, \dots, U_\ell) = (d^{e(F)} \pm (\delta_{\mathrm{CL}} + \gamma_{\mathrm{CL}})) \prod_{i \in [\ell]} |U_i| = d^{e(F)} \prod_{i \in [\ell]} |U_i| \pm \varepsilon n^\ell\,,$$

which concludes the proof of the fact. $\square$

4. Proof of Theorem 6

This section concerns the proof of Theorem 6. We have to show that for $k \geq \ell \geq 2$, every (ℓ, k)-function τ, and every $d > 0$ the properties DISC_d, $\mathrm{DISC}_{d,\ell}$, and $\mathrm{DISC}_{d,\tau}$ are equivalent. The equivalence will follow from the implication

$$\mathrm{DISC}_{d,\ell} \xRightarrow{\text{Fact 28}} \mathrm{DISC}_{d,\ell+1}\ ,$$

which holds for every $\ell = 1, \dots, k-1$ and the equivalence

$$\mathrm{DISC}_{d,k} \xRightarrow{\text{Fact 32}} \mathrm{DISC}_{d,\tau} \xRightarrow{\text{Fact 30}} \mathrm{DISC}_{d,k}\ ,$$

which holds for every $\ell = 1, \dots, k$ and every (ℓ, k)-function τ. Theorem 6 then follows, since Fact 28 applied for all $\ell = 1, \dots, k-1$ gives

$$\mathrm{DISC}_d = \mathrm{DISC}_{d,1} \Rightarrow \dots \Rightarrow \mathrm{DISC}_{d,\ell} \Rightarrow \mathrm{DISC}_{d,\ell+1} \Rightarrow \dots \Rightarrow \mathrm{DISC}_{d,k}$$

and Fact 32 applied for the unique $(1,k)$-function $\tau = (1)$ gives

$$\mathrm{DISC}_{d,k} \Rightarrow \mathrm{DISC}_{d,(1)} = \mathrm{DISC}_d\,.$$

Finally, due to Fact 30 and Fact 32 we have

$$\mathrm{DISC}_{d,k} \Leftrightarrow \mathrm{DISC}_{d,\tau}$$

for every $\ell = 1, \dots, k$ and every (ℓ, k)-function τ. We prove Fact 28, Fact 30, and Fact 32 in the next section.

4.1. **Equivalence of different versions of** DISC**.** We first deduce $\mathrm{DISC}_{d,\ell+1}$ from $\mathrm{DISC}_{d,\ell}$ in a straightforward way.

Fact 28. *For all integers $1 \leq \ell < k$, every $d > 0$, and every $\varepsilon > 0$ the following holds. If H is a k-graph that satisfies $\mathrm{DISC}_{d,\ell}(\varepsilon/3)$, then H satisfies $\mathrm{DISC}_{d,\ell+1}(\varepsilon)$.*

Proof. Let $U_1, \dots, U_{\ell+1} \subset V(H)$ be pairwise disjoint sets. Then

$$\begin{aligned}\mathrm{vol}(U_1, \dots, U_{\ell-1}, U_\ell, U_{\ell+1}) &= \mathrm{vol}(U_1, \dots, U_{\ell-1}, U_\ell \dot\cup U_{\ell+1}) \\ &\quad - \mathrm{vol}(U_1, \dots, U_{\ell-1}, U_\ell) - \mathrm{vol}(U_1, \dots, U_{\ell-1}, U_{\ell+1}).\end{aligned}$$

and

$$\begin{aligned}e(U_1, \dots, U_{\ell-1}, U_\ell, U_{\ell+1}) &= e(U_1, \dots, U_{\ell-1}, U_\ell \dot\cup U_{\ell+1}) \\ &\quad - e(U_1, \dots, U_{\ell-1}, U_\ell) - e(U_1, \dots, U_{\ell-1}, U_{\ell+1}).\end{aligned}$$

Since H satisfies $\mathrm{DISC}_{d,\ell}(\varepsilon/3)$ we have

$$e(U_1, \dots, U_{\ell-1}, X) = d\mathrm{vol}(U_1, \dots, U_{\ell-1}, X) \pm \varepsilon n^k/3$$

for all $X \in \{U_\ell, U_{\ell+1}, U_\ell \dot\cup U_{\ell+1}\}$ and, consequently

$$e(U_1, \dots, U_\ell, U_{\ell+1}) = d\mathrm{vol}(U_1, \dots, U_\ell, U_{\ell+1}) \pm \varepsilon n^k.$$

□

We continue with the following observation, which is a direct consequence of the principle of inclusion and exclusion.

Fact 29. *Let t, ℓ, and k be positive integers with $t + \ell \leq k + 1$ and let $\tau \in T(\ell, k)$ be an (ℓ, k)-function with $\tau(\ell) = t$. Let τ' be the $(\ell + t - 1, k)$-function given by*

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i < \ell \\ 1 & \text{if } i \geq \ell. \end{cases}$$

Then for every k-graph H and all $\ell + t - 1$ pairwise disjoint sets $U_1, \dots, U_{\ell-1}$, $U_\ell^1, \dots U_\ell^t \subset V(H)$ we have

$$e_{\tau'}(U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t) = \textstyle\sum_{\emptyset \neq J \subseteq [t]} (-1)^{t-|J|} e_\tau\big(U_1, \dots, U_{\ell-1}, \bigcup_{j \in J} U_\ell^j\big)\,.$$

Proof. Let $K \subset \dot\bigcup_{j \in [\ell-1]} U_j \dot\cup \dot\bigcup_{j \in [t]} U_\ell^j$ be a set of size k such that $K \cap U_i = \tau(i)$ for all $i < \ell$ and let $I_K = \{i \colon |K \cap U_\ell^i| > 0\}$. Note that K appears in $e_{\tau'}(U_1, \dots, U_{\ell-1}, U_\ell^1, \dots, U_\ell^t)$ if and only if $|I_K| = t$. Moreover, the contribution of K to the right-hand side is

$$\sum_{I_K \subseteq J \subseteq [t]} (-1)^{t-|J|} = \sum_{j=0}^{t-|I_K|} \binom{t - |I_K|}{j} (-1)^{t-(|I_K|+j)} = \begin{cases} 1 & \text{if } |I_K| = t \\ 0 & \text{otherwise.} \end{cases}$$

□

Fact 30. *For all integers $1 \leq \ell \leq k$, every $d > 0$, every (ℓ, k)-function τ, and every $\varepsilon > 0$ the following holds. If H is a k-graph that satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon/2^{k^2/2})$, then H satisfies $\mathrm{DISC}_{d,k}(\varepsilon)$.*

Proof. Recall first that $\mathrm{DISC}_{d,k}(\varepsilon) = \mathrm{DISC}_{d,\sigma}(\varepsilon)$ if σ is the everywhere 1-function or equivalently the unique (k,k)-function. For a given τ we call $|\{i\colon \tau(i) \geq 2\}|$ the defect of τ. Since the everywhere 1-function σ is the only (ℓ,k)-function, for any ℓ, with defect 0, the fact follows from at most $\lfloor k/2 \rfloor$ applications of the following claim. □

Claim 31. *Suppose τ is an (ℓ,k)-function with defect $s \geq 1$. Then there is a τ' with defect $s-1$ such that if H satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon/2^k)$, then H satisfies $\mathrm{DISC}_{d,\tau'}(\varepsilon)$.*

Proof. Claim 31 follows from Fact 29. For a given $\tau \in T(\ell,k)$ with defect $s \geq 1$ we may assume without loss of generality that $\tau(\ell) = t \geq 2$. We define the $(\ell+t-1,k)$-function τ' by

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i < \ell \\ 1 & \text{if } i \geq \ell. \end{cases} \tag{23}$$

Then τ' has defect $s-1$ and from Fact 29 we infer

$$e_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t) = \textstyle\sum_{\emptyset\neq J\subseteq[t]}(-1)^{t-|J|}e_\tau\big(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j\big)$$

and

$$\mathrm{vol}_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t) = \textstyle\sum_{\emptyset\neq J\subseteq[t]}(-1)^{t-|J|}\mathrm{vol}_\tau\big(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j\big)$$

for any choice of pairwise disjoint sets $U_1,\dots,U_{\ell-1}$, $U_\ell^1,\dots U_\ell^t \subset V(H)$. Since H satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon/2^k)$ we have

$$e_\tau\big(U_1,\dots,U_{\ell-1},\textstyle\bigcup_{j\in J}U_\ell^j\big) = d\,\mathrm{vol}_\tau\big(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j\big) \pm \varepsilon n^k/2^k$$

for all $\emptyset \neq J \subseteq [t]$ and, hence,

$$\begin{aligned} &e_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t) \\ &\qquad= \sum_{\emptyset\neq J\subseteq[t]}(-1)^{t-|J|}(d\mathrm{vol}_\tau\Big(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j\Big) \pm \varepsilon n^k/2^k) \\ &\qquad= d\sum_{\emptyset\neq J\subseteq[t]}(-1)^{t-|J|}\mathrm{vol}_\tau\Big(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j\Big) \pm 2^{t-k}\varepsilon n^k \\ &\qquad\qquad= d\mathrm{vol}_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t) \pm \varepsilon n^k. \end{aligned}$$

□

The last observation in this section reverses the implication of Fact 30.

Fact 32. *For all integers $1 \leq \ell \leq k$, every $d > 0$, every (ℓ,k)-function τ, and every $\varepsilon > 0$ there is an n_0 such that the following holds. If H is a k-graph on $n \geq n_0$ vertices that satisfies $\mathrm{DISC}_{d,k}(\varepsilon/3^{k^2})$, then H satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon)$.*

Proof. We choose n_0 sufficiently large and by induction on $\ell = k,\dots,1$ we prove that if H satisfies $\mathrm{DISC}_{d,k}(\varepsilon/3^{(k-\ell)k})$ then H also satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon)$ for an arbitrary (ℓ,k)-function τ.

For $\ell = k$ there is only one (ℓ,k)-function τ which is the everywhere 1-function. Then $\mathrm{DISC}_{d,\tau}(\varepsilon) = \mathrm{DISC}_{d,k}(\varepsilon)$ and the implication is obviously true.

So suppose by induction that for every $(\ell+1,k)$-function τ' every k-graph H on n vertices with the property $\mathrm{DISC}_{d,k}(\varepsilon/3^{(k-\ell)k})$ also satisfies $\mathrm{DISC}_{d,\tau'}(\varepsilon/3^k)$.

Let τ be an arbitrary (ℓ,k)-function and let $U_1,\dots,U_\ell \subseteq V(H)$ be pairwise disjoint sets. Without loss of generality we assume that $\tau(\ell)=t\geq 2$ and we define an $(\ell+1,k)$-function τ' by

$$\tau'(i)=\begin{cases}\tau(i) & \text{if } i<\ell\\ \tau(i)-1 & \text{if } i=\ell\\ 1 & \text{if } i=\ell+1.\end{cases} \tag{24}$$

Further let $\mathcal{P}(U_\ell)$ be the family of all ordered bipartitions of U_ℓ into two equitable sets, i.e. all pairs (W_1,W_2) with $U_\ell=W_1\dot\cup W_2$ and $|W_1|=\lfloor |U_\ell|/2\rfloor = w$. Then

$$\mathrm{vol}_{\tau'}(U_1,\dots,U_{\ell-1},W_1,W_2)=\binom{w}{t-1}(|U_\ell|-w)\prod_{i\in[\ell-1]}\binom{|U_i|}{\tau(i)}$$

holds for all bipartitions $(W_1,W_2)\in\mathcal{P}(U_\ell)$. Since H satisfies $\mathrm{DISC}_{d,\tau'}(\varepsilon/3^k)$ we have

$$e_{\tau'}(U_1,\dots,U_{\ell-1},W_1,W_2)=d\mathrm{vol}_{\tau'}(U_1,\dots,U_{\ell-1},W_1,W_2)\pm\varepsilon n^k/3^k\,.$$

Summing over all bipartitions in $\mathcal{P}(U_\ell)$ every edge in $E_\tau(U_1,\dots,U_\ell)$ is counted exactly $t\binom{|U_\ell|-t}{w-(t-1)}$ times. Thus, we infer

$$\begin{aligned}e_\tau(U_1,\dots,U_\ell)&=\frac{1}{t\binom{|U_\ell|-t}{w-(t-1)}}\sum_{(W_1,W_2)\in\mathcal{P}(U_\ell)}e_{\tau'}(U_1,\dots,U_{\ell-1},W_1,W_2)\\&=\frac{|\mathcal{P}(U_\ell)|}{t\binom{|U_\ell|-t}{w-(t-1)}}\left(d\binom{w}{t-1}(|U_\ell|-w)\prod_{i\in[\ell-1]}\binom{|U_i|}{\tau(i)}\pm\varepsilon n^k/3^k\right).\end{aligned}$$

With $|\mathcal{P}(U_\ell)|=\binom{|U_\ell|}{w}$ and

$$\frac{|\mathcal{P}(U_\ell)|}{t\binom{|U_\ell|-t}{w-(t-1)}}\binom{w}{t-1}(|U_\ell|-w)=\binom{|U_\ell|}{t}$$

and since $|\mathcal{P}(U_\ell)|\leq 3^k t\binom{|U_\ell|-t}{w-(t-1)}$ we obtain

$$e_\tau(U_1,\dots,U_\ell)=d\prod_{i\in[\ell]}\binom{|U(i)|}{\tau(i)}\pm\varepsilon n^k.$$

□

5. Concluding remarks

5.1. **Extension of P_3.** For Theorem 3 we extended properties P_1,P_2,P_4,P_6, and P_7. While the extension of P_5 is straightforward and its equivalence to DISC_d follows similarly to Fact 18, we did not find an interesting generalisation of P_3 for k-graphs and leave this open.

5.2. **Uniform edge distribution with respect to i-sets.** We studied quasi-random properties equivalent to uniform edge distribution of k-graphs with respect to large vertex sets. A natural generalisation concerns the edge distribution with respect to large subsets of i-tuples.

i-DISC$_d(\varepsilon)$: We say a k-graph $H = (V, E)$ on n vertices has i-DISC$_d(\varepsilon)$ for $1 \le i \le k-1$, d, $\varepsilon > 0$, if

$$|E(H) \cap \mathcal{K}_k(G^{(i)})| = d|\mathcal{K}_k(G^{(i)})| \pm \varepsilon n^k ,$$

for any i-graph $G^{(i)}$ with vertex set V, where $\mathcal{K}_k(G^{(i)})$ denotes the set of all k-sets K in $\binom{V}{k}$ which span a copy of $K_k^{(i)}$ (the complete i-graph on k vertices) in $G^{(i)}$.

Clearly, i-DISC$_d$ for $i = 1$ coincides with DISC$_d$ and for $i = k-1$ this is the central concept of quasi-randomness studied in [15]. The general notion i-DISC$_d$ was first studied by Frankl and Rödl [8] and Chung [2, 3]. We believe that Theorem 3 can be extended for general i. As 1-DISC$_d$ is characterised by the subgraph frequencies of linear k-graphs, i-DISC$_d$ is closely related to the appearance of partial Steiner $(i+1, k)$-systems, i.e., k-graphs for which every two hyperedges intersect in at most i vertices. In this context the natural generalisation of the "doubling" operation from Section 1.1 seems to be the following. Let A be a k-partite k-graph with vertex classes $X_1, \dots, X_k$ and let $I \in \binom{[k]}{i}$ be an i-set, then the doubling $\mathrm{db}_I(A)$ of A is obtained by taking two copies of A and identifying the vertices in the classes X_i for all $i \in I$. Again starting with a single edge and applying consecutively db_I for every $I \in \binom{[k]}{i}$ (in some arbitrary order) we will get a k-partite k-graph, which seems likely to be of similar importance for i-DISC$_d$ as M had in Theorem 3. In fact, for $i = k-1$, this way we obtain the k-graph of the octahedron $K^{(k)}_{2,\dots,2}$ which was already studied in connection with $(k-1)$-DISC$_d$ in [4, 15].

A related line of research concerns the connection to extensions of Szemerédi's regularity lemma. While there is a regularity lemma which decomposes any given k-graph into relatively few "blocks" such that most of them satisfy a k-partite version 1-DISC$_d$ (i.e., DISC$_{d,k}$), for $i \ge 2$ the notion of i-DISC seems too strong and likely no regularity lemma compatible for this notion exists. Instead, one needs to work with "relative" versions of i-DISC. For $i = k-1$, this notion of quasi-randomness was introduced in the work on hypergraph regularity by Rödl et al. [9, 18] and Gowers [11, 12], and for $k = 3$ the equivalence was studied in [16]. It would be interesting to further investigate those connections for general i and we intend to return to this in the near future.

5.3. **Extension of Corollary 4.** In Corollary 4 we showed that for every $k \ge 2$ the complete graph K_k and the line graph of the k-dimensional hypercube $M(k)$ (which alternatively can be obtained from the k-graph M_k by replacing every hyperedge of M_k with a graph clique K_k) is a quasi-random pair. The construction of $M(k)$ can be easily extended from cliques to arbitrary graphs F. For a graph F with vertex set $[k]$ let $M(F)$ be the graph obtained from the k-graph M_k with vertex classes $X_1, \dots, X_k$ by replacing every hyperedge by a copy of F such that the vertex representing vertex $i \in [k] = V(F)$ lies in X_i. It seems possible that $(F, M(F))$ is a quasi-random pair for every graph F. Indeed the following observation supports this belief.

While the notion of quasi-random pairs is closely related to the property MIN$_d$, we may also consider the following version of DEV$_d$ for graphs.

$\mathrm{DEV}_{d,F}(\varepsilon)$**:** We say a graph $G=(V,E)$ on n vertices has $\mathrm{DEV}_{d,F}(\varepsilon)$ for a graph F with vertex set $[k]$ and d, $\varepsilon>0$, if

$$\left|\sum_{\tilde{M}}\prod_{\tilde{F}\subseteq\tilde{M}}\left(\left(\prod_{e\in E(\tilde{F})}\mathbb{1}_E(e)\right)-d^{e(F)}\right)\right|\le\varepsilon n^{k2^{k-1}},$$

where the sum runs over all copies $\tilde{M}$ of $M(F)$ in the complete graph K_V on vertex set V and the outer product runs over the 2^k copies $\tilde{F}$ of F (corresponding to the hyperedges of M_k).

Following closely the lines of the proof of Lemma 13 it can be shown that for every $d>0$ and every graph F with at least one edge, a graph G satisfying $\mathrm{DEV}_{d,F}(\varepsilon)$ also satisfies the assumptions of Theorem 2 and consequently such graphs are quasi-random with density d.

5.4. **Strengthening of Theorem 5.** It would be interesting to strengthen Theorem 5. We believe the partite assumption of $\mathrm{HCL}_{d,F}$ is not needed and it suffices that a given k-graph H contains approximately the "right" number of copies of F on every subset $U\subseteq V(H)$. Indeed, for graphs Theorem 2 and for 3-graphs the recent work of Dellamonica and Rödl [7] imply such an assertion.

5.5. **Algorithmic considerations.** Since DEV_d, MIN_d, and MDEG_d can be easily checked in polynomial time, in fact in $O(n^{k2^{k-1}})$, we obtain by Theorem 3 an efficient algorithm which can approximately check whether a given k-graph has DISC_d. More precisely, for any given d and $\varepsilon>0$ there exists some positive $\varepsilon'<\varepsilon$ such that the algorithm can distinguish in polynomial time, whether a given k-graph H satisfies $\mathrm{DISC}_d(\varepsilon')$ or fails to satisfy $\mathrm{DISC}_d(\varepsilon)$. In some sense we cannot hope for an efficient algorithm, which decides $\mathrm{DISC}_d(\varepsilon)$ precisely, since it was shown in [1] that deciding $\mathrm{DISC}_d(\varepsilon)$ for graphs is co-NP complete.

Likely such an approximation algorithm can be used for an algorithmic version of the weak hypergraph regularity lemma, Theorem 23. Such an algorithm would find an ε-regular partition in $O(n^{k2^{k-1}})$. However, a more efficient algorithm, with running time $O(n^{2k-1}\log^2 n)$ was found by Czygrinow and Rödl [6].

Moreover, since the proof of the implication $\mathrm{DEV}_d\Rightarrow\mathrm{DISC}_d$, Lemma 13 extends to sparse k-graphs, i.e., for the case $d=o(1)$ as long as $d\gg n^{-(k-1)/2}$, we obtain a sufficient, efficiently verifiable condition for checking DISC_d for sparse k-graphs. We believe it would be interesting to investigate this problem further. For example, we are not aware of a property which is equivalent to DISC_d as long as $d\gg n^{-k+1}$ and which can be verified in polynomial time.

5.6. **Non quasi-random pairs.** In this section we show that there exists no minimal configuration for 3-graphs with 6 or less vertices. In other words for 3-graphs the 3-graph M from property MIN_d with 8 edges and 12 vertices can not be replaced by a 3-graph on at most 6 vertices. Hence, for every linear 3-graph F on six vertices we have to construct 3-graphs of density $d>0$ such that they contain the right number of copies of F, but fail to be weak quasi-random, i.e., fail to satisfy DISC_d. There are, up to isomorphism, 6 such 3-graphs F: the one with no edge, with a single edge, with two disjoint edges, with two edges sharing a vertex, the $(6,3)$-configuration (the unique linear 3-graph with 3 edges on six vertices), and the Pasch-configuration (the unique linear 3-graph with 4 edges on six vertices). It is simple to see that for F being one of the first four of those configuration the property that H contains $\sim (2/9)^{e(F)}n^{|V(F)|}$ labeled copies of F does not imply that H has $\mathrm{DISC}_{2/9}$ as for example the complete, 3-partite 3-graph on vertex classes of size $n/3$ shows. Hence we will focus on the $(6,3)$- and the Pasch-configuration.

5.6.1. *The* $(6,3)$*-configuration.* We denote by C the $(6,3)$-configuration, which is the 3-graph with $V(C) = [6]$ and $E(C) = \{\{1,2,3\},\{3,4,5\},\{5,6,1\}\}$. We consider the complete 3-partite 3-graph $H = H(\alpha)$ on n vertices with vertex classes V_1, V_2, V_3 such that $|V_1| = |V_2| = (1-\alpha)n/2$ and $|V_3| = \alpha n$ for some $\alpha \in (0, 1/3]$. The density of H is $\frac{3}{2}\alpha(1-\alpha)^2 - o(1)$, while simple calculations show that

$$N_C(H) = \left(\frac{3}{8}\alpha^2(1-\alpha)^4 + o(1)\right)n^6\,,$$

since any copy of C in H must distribute the copies of the vertices $1, 3, 5$ over all three distinct classes, and after fixing the vertex classes of the copies of 1, 3, and 5 the vertex classes of the other three vertices are fixed. Now we need to chose $\alpha > 0$ in such a way that

$$f(\alpha) = \left(\frac{3}{2}\alpha(1-\alpha)^2\right)^3 - \frac{3}{8}\alpha^2(1-\alpha)^4$$

is close to 0, as this would yield that $H = H(\alpha)$ contains the "right" number of copies of C, but clearly H would not satisfy $\mathrm{DISC}_{3\alpha(1-\alpha)^2/2}$. Solving $f(\alpha) = 0$ is equivalent to solving $g(\alpha) = \alpha(1-\alpha)^2$ equals $1/9$. Since $g(0) = 0$ and $g(1/3) = 4/27$, we infer that there exists an $\hat{\alpha} \in (0, 1/3]$ such that $f(\hat{\alpha}) = 0$ (indeed $\hat{\alpha} \approx 0.16$). Hence, $H(\hat{\alpha})$ has the desired properties. Moreover, we obtain other 3-graphs with the same properties (having the right number of copies of C, but failing to have DISC_d) for other densities d, if we consider random sub-hypergraphs of $H(\hat{\alpha})$.

5.6.2. *The Pasch-configuration.* Again we will construct a 3-graph H of density d which violates DISC_d, but has $\sim d^4 n^6$ labeled copies of the Pasch-configuration P. For that we first construct a graph G and then consider its triangles to be the hyperedges of H, i.e., $H = \mathcal{K}_3(G)$. Let $G = G(\alpha)$ be the complete, 5-partite graph with vertex classes $V_1 \dot{\cup} \dots \dot{\cup} V_5 = V(G)$ and $|V_1| = |V_2| = |V_3| = |V_4| = (1-\alpha)n/4$ and $|V_5| = \alpha n$. The number of labeled triangles of G satisfies

$$N_{K_3}(G) = \left(\frac{3}{8}(1-\alpha)^3 + \frac{9}{4}(1-\alpha)^2\alpha + o(1)\right)n^3$$

while for the number of labeled $K_{2,2,2}$ in G we have

$$N_{K_{2,2,2}}(G) = \left(\frac{(1-\alpha)^4}{128}\big(3(1-\alpha)^2 + 126\alpha^2 + 54\alpha(1-\alpha)\big) + o(1)\right)n^6.$$

As above, we are interested in a solution to

$$\left(\frac{3}{8}(1-\alpha)^3 + \frac{9}{4}(1-\alpha)^2\alpha\right)^4 = \frac{(1-\alpha)^4}{128}\big(3(1-\alpha)^2 + 126\alpha^2 + 54\alpha(1-\alpha)\big)\,,$$

with $\alpha \in (0, 1/5]$. Since for $\alpha = 0$ the left-hand side is smaller than the right-hand side, while for $\alpha = 1/5$ the inequality switches, there must be an $\hat{\alpha} \in (0, 1/5]$ such that both sides equal.

Let $H = H(\hat\alpha) = \mathcal{K}_3(G(\hat\alpha))$, i.e., H is the 3-graph whose hyperedges correspond to the triangles of $G(\hat\alpha)$. It follows that the number of edges of H equals the number of triangles in G, i.e., for $d_{\hat\alpha} = \frac{3}{8}(1-\hat\alpha)^3 + \frac{9}{4}(1-\hat\alpha)^2\hat\alpha$

$$e(H) = (d_{\hat\alpha} + o(1))\binom{n}{3}\,.$$

On the other hand, every labeled copy of $K_{2,2,2}$ in G gives rise to a labeled $K^{(3)}_{2,2,2}$ in H, which gives rise to exactly one labeled Pasch-configuration (note, that in fact a copy of $K^{(3)}_{2,2,2}$ contains exactly two Pasch-configurations, however, those correspond to two different labelings of the same unlabeled copy of $K^{(3)}_{2,2,2}$). Moreover, every labeled copy of the Pasch-configuration P in H corresponds to a $K_{2,2,2}$ in G and, consequently,

$$N_P(H) = N_{K_{2,2,2}}(G) = (d^4_{\hat\alpha} + o(1))n^6\,,$$

due to the choice of $\hat\alpha$. Obviously, $H = H(\hat\alpha)$ is 5-partite and does not satisfy $\mathrm{DISC}_{d_{\hat\alpha}}$, which shows that it has the desired properties.

Moreover, we remark that the graph $G = G(\hat\alpha)$ from above has the properties

$$N_{K_3}(G) = (d_{\hat\alpha} + o(1))n^3 \quad \text{and} \quad N_{K_{2,2,2}}(G) = (d^4_{\hat\alpha} + o(1))n^6$$

while it obviously fails to satisfy $\mathrm{DISC}_{d_{\hat\alpha}}$ for graphs. This answers a question of Shapira and Yuster from [21].

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