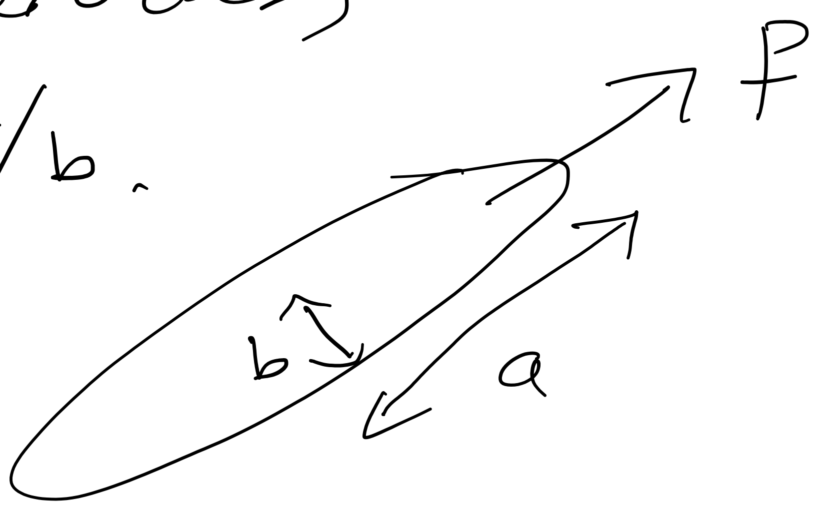
Advanced Fluid Dynamics lecture 6

Sunday, 31 January 2021 20:25

Last time: Jeffery's equation for axisymmetric spheroids, aspect ratio $r = a/b$.



$$\dot{\underline{p}} = \underline{\Omega}^\infty \times \underline{p} + \beta\left(\underline{\underline{E}}^\infty \cdot \underline{p} - \underline{p}\cdot\underline{\underline{E}}^\infty \cdot \underline{p}\,\underline{p}\right)$$

Bretherton parameter $\beta = \dfrac{r^2-1}{r^2+1}$.

Special cases: $\beta = -1$ flat disc

$\beta = 0$ sphere

$\beta = 1$ elongated rods

when $\beta = 0$, $\dot{\underline{p}} = \underline{\Omega}^\infty \times \underline{p}$ so spheres rotate with half the background vorticity.

For $\beta = 1$, $\underline{\underline{E}}^\infty \cdot \underline{p} + \underline{\Omega}^\infty \times \underline{p} = (\nabla \underline{u})^T \cdot \underline{p}$

so $\dot{\underline{p}} = \underline{p}\cdot\nabla\underline{u} - \underline{p}\cdot(\nabla\underline{u})\cdot\underline{p}\,\underline{p}$

Treating $\underline{p}(\underline{x}, t)$ as a vector field for a suspension of many bodies,

$$\dot{\underline{p}} = \underbrace{\partial_t \underline{p} + \underline{u}\cdot\nabla\underline{p} = \underline{p}\cdot\nabla\underline{u}}\ \underbrace{-\,\underline{p}\cdot(\nabla\underline{u})\cdot\underline{p}\,\underline{p}}$$

Evolution equation for material line elements, vorticity in ideal fluids, magnetic field in ideal MHD. Upper convected derivative for vectors.

Extra term preserves $|\underline{p}| = 1$ by making $\frac{d}{dt}\frac{1}{2}|\underline{p}|^2 = \dot{\underline{p}}\cdot\underline{p} = 0$

For general β,

$$\partial_t \underline{p} + \underline{u}\cdot\nabla\underline{p} = \underline{p}\cdot\nabla\underline{u} + (\beta - 1)\,\underline{\underline{E}}\cdot\underline{p} + \beta\,\underline{p}\cdot\underline{\underline{E}}\cdot\underline{p}\,\underline{p}$$

or

$$\partial_t \underline{p} = \nabla\times(\underline{u}\times\underline{p}) - \underline{u}\,\nabla\cdot\underline{p} + (\beta-1)\,\underline{\underline{E}}\cdot\underline{p} + \beta\,\underline{p}\cdot\underline{\underline{E}}\cdot\underline{p}\,\underline{p}$$

Like MHD but with $|\underline{p}| = 1$ instead of $\nabla\cdot\underline{B} = 0$.

Can solve in spherical polars, $\underline{p} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ to get "Jeffery orbits" in shear flow $\underline{u} = \dot{\gamma}\, y\, \hat{\underline{x}}$.

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Dilute suspensions of spheroids

Consider a distribution $\Psi(\underline{p}, \underline{x}, t)$ normalized to (for each $\underline{x}$ & t)

$$\int_S \Psi(\underline{p}, \underline{x}, t)\, dS = 1,$$

integrating dS over the unit sphere covered by $\underline{p}$. Number density n.

"Dilute" is very restrictive for elongated objects as we need $na^3 < 1$ so they can rotate freely.

The volume fraction
$\phi = \frac{4}{3}\pi n a b^2 = O(r^{-2})$.

However, the results apply to the "semidilute" regime $r^{-2} < \phi < r^{-1}$ with minor changes to coefficients.

As with the bead-spring model we can add a stochastic Brownian torque
$$\underline{T}_{Br} = -\underline{p} \times \frac{\partial}{\partial \underline{p}}\left(k_B T \log \Psi\right)$$
on the LHS of the resistance matrix. This torque is perpendicular to $\underline{p}$, so it's multiplied by the Y^c part of $\underline{\underline{C}}$.

If we take $\underline{\underline{E}}$ and $\underline{\Omega}$ now as the local strain and half vorticity fields we get

$$\dot{\underline{p}} = \underline{\Omega} \times \underline{p} + \beta\left(\underline{\underline{E}} \cdot \underline{p} - \underline{p} \cdot \underline{\underline{E}} \cdot \underline{p}\, \underline{p}\right) - \frac{k_B T}{Y^c}\underbrace{\left(\underline{\underline{I}} - \underline{p}\,\underline{p}\right)}\cdot \frac{\partial}{\partial \underline{p}}\left(\log \Psi\right)$$

projects the orientational part of $\frac{\partial}{\partial \underline{p}}$ to be perpendicular to $\underline{p}$ & preserve $|\underline{p}| = 1$.

The Liouville equation
$$\partial_t \Psi + \underline{u} \cdot \nabla \Psi + \nabla_p \cdot (\dot{\underline{p}}\, \Psi) = 0,$$
where $\nabla_p = \left(\underline{\underline{I}} - \underline{p}\,\underline{p}\right) \cdot \frac{\partial}{\partial \underline{p}}$,

becomes

$$(\partial_t + \underline{u} \cdot \nabla)\Psi + \left(\underline{\Omega} \times \underline{p} - \beta\, \underline{\underline{E}} \cdot \underline{p}\right) \cdot \nabla_p \Psi - 3\beta\, \underline{p} \cdot \underline{\underline{E}} \cdot \underline{p}\, \Psi = D_r \nabla_p^2 \Psi$$

a Fokker-Planck-type equation with rotational Brownian diffusivity $D_r = k_B T / Y^c$.

As before, the translational Brownian diffusion is usually too small to be relevant.

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The stress on the fluid due to particles

In general,

$$\boldsymbol{\sigma}^{P} = \langle \mathbf{m} : \mathbf{E} - \mathbf{h} \cdot \mathbf{T}^{Br} \rangle$$

where $\langle \dots \rangle$ is an average with distribution Ψ. The tensors $\mathbf{m}$ and $\mathbf{h}$ come from inverting the resistance matrix to get the "mobility matrix" formulation for Stokes flow around an arbitrary particle:

$$\begin{pmatrix} \mathbf{U}^{\infty} - \mathbf{U} \\ \boldsymbol{\Omega}^{\infty} - \boldsymbol{\Omega} \\ \mu^{-1} \mathbf{S} \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{b}^T & \tilde{\mathbf{g}} \\ \mathbf{b} & \mathbf{c} & \tilde{\mathbf{h}} \\ \mathbf{g} & \mathbf{h} & \mathbf{m} \end{pmatrix} \begin{pmatrix} \mu^{-1} \mathbf{F} \\ \mu^{-1} \mathbf{T} \\ \mathbf{E}^{\infty} \end{pmatrix}$$

Note $\mathbf{S}$ appears on the LHS in both formulations.

$$\begin{pmatrix} \mathbf{g} & \mathbf{h} \end{pmatrix} = \begin{pmatrix} \mathbf{G} & \mathbf{H} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

and

$$\mathbf{m} = \mathbf{M} - \begin{pmatrix} \mathbf{G} & \mathbf{H} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{G}} \\ \tilde{\mathbf{H}} \end{pmatrix}$$

Again, this all simplifies for axisymmetric objects to give the total stress

$$\begin{aligned} \boldsymbol{\sigma} = &- p \mathbf{I} + 2\mu \mathbf{e} \\ &+ 2\mu \phi \Big(A \langle \mathbf{pppp} \rangle : \mathbf{e} \\ &\quad + B \left(\langle \mathbf{pp} \rangle \cdot \mathbf{e} + \mathbf{e} \cdot \langle \mathbf{pp} \rangle \right) \\ &\quad + C \mathbf{e} + F D_r \langle \mathbf{pp} \rangle \Big) \end{aligned}$$

for scalar constants A, B, C, F determined by the particle shape. (Not simply related to earlier matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ in resistance matrix formulation earlier).

For slender particles $\left(r = \frac{a}{b} \gg 1 \right)$ their asymptotic forms are

$$A \sim \frac{r^2}{2(\log(2r) - 3/2)}$$

$$B \sim \frac{6\log(2r) - 1}{r^2}, \quad C \to 2$$

$$F \sim \frac{3r^2}{\log(2r) - 1/2}.$$

largest

These also hold in the semi-dilute limit, with $\log 2r$ replaced by $-\frac{1}{2} \log \phi$.

For sufficiently elongated particles we need only keep A & F. If they're also large enough to be non-Brownian we can drop F too, giving

$$\boldsymbol{\sigma} = - p \mathbf{I} + 2\mu \left(\mathbf{e} + N \langle \mathbf{pppp} \rangle : \mathbf{e} \right).$$

The "non-Newtonian" parameter

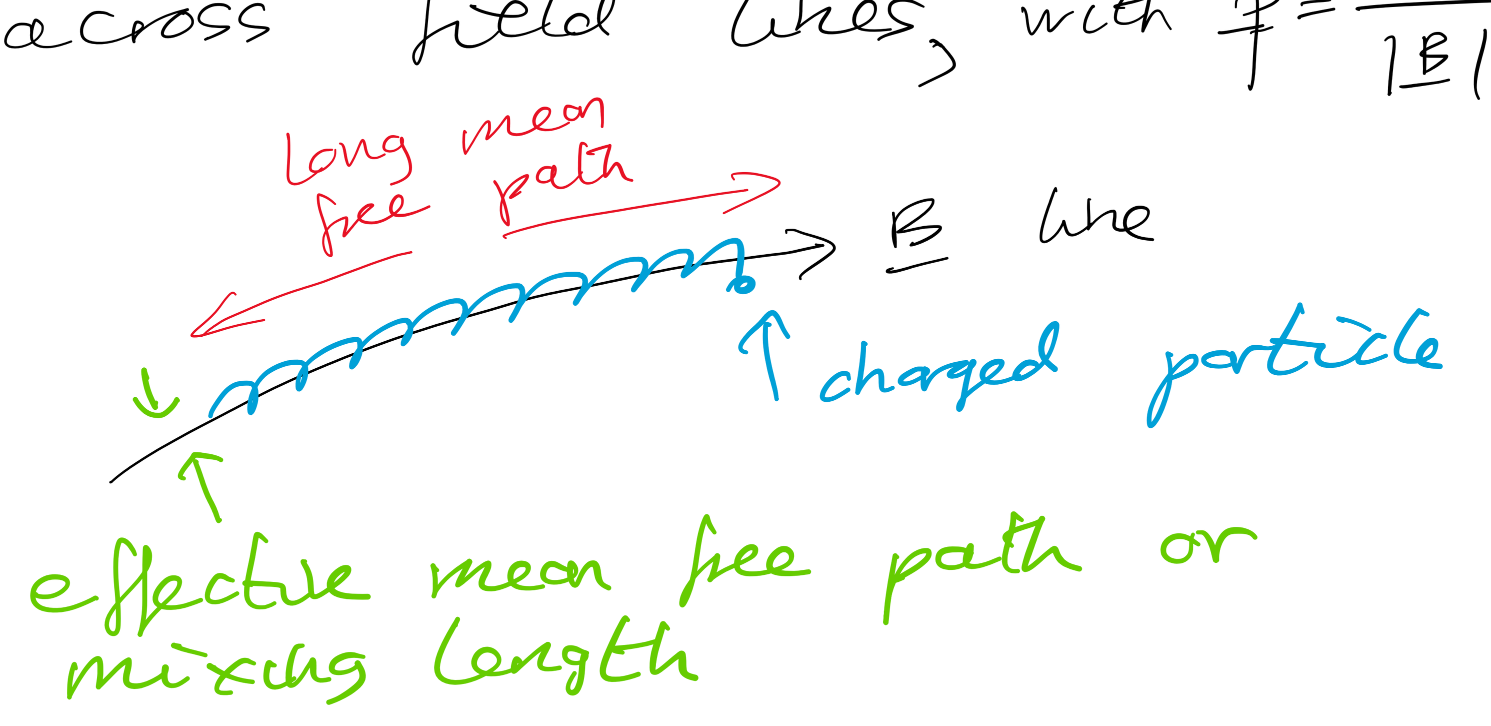
$$N = \phi A \sim \frac{\phi r^2}{\log r}$$

can be significant even when ϕ is small.

If the particles are aligned, so we can drop $\langle \dots \rangle$ to get

$$\boldsymbol{\sigma} = - p \mathbf{I} + 2\mu \left(\mathbf{e} + N \mathbf{pp} (\mathbf{pp}) : \mathbf{e} \right)$$

This is the stress in a common approximation to "Braginskii MHD" where the magnetic field suppresses viscous momentum transport across field lines, with $\mathbf{p} = \frac{\mathbf{B}}{|\mathbf{B}|}$.



We can also form evolution equations for moments, e.g.

$$\mathbf{Q} = \langle \mathbf{pp} - \tfrac{1}{3} \mathbf{I} \rangle,$$

which vanishes for uniformly distributed orientations, evolves according to

$$\overset{\nabla}{\mathbf{Q}} = (\partial_t + \mathbf{u} \cdot \nabla) \mathbf{Q} - \mathbf{Q} \cdot (\nabla \mathbf{u}) - (\nabla \mathbf{u})^T \cdot \mathbf{Q}$$

$$= -6 D_r \mathbf{Q} + \tfrac{2}{3} \mathbf{e} - 2 \langle \mathbf{pppp} \rangle : \mathbf{e}$$

Unlike for beads & springs, this equation isn't closed as it evolves $\langle \mathbf{pppp} \rangle$. There are models for $\langle \mathbf{pppp} \rangle$ in terms of $\langle \mathbf{pp} \rangle$ that can be used to model fibre suspensions.