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On the Static-Limit Solutions to the Navier–Stokes Equations of Compressible Flow

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Abstract. We consider the zero-velocity stationary problem of the Navier–Stokes equations of compressible isentropic flow describing the distribution of the density ϱ of a fluid in a spatial domain $\Omega \subset \mathrm{R}^N$ driven by a time-independent potential external force $\vec{f} = \nabla F$. We study the structure of the set of all solutions to the stationary problem having a prescribed mass $m > 0$ and a prescribed energy. Cardinality of the solution set depends on m and it is either continuum or at most two. Conditions on m for distinguishing these cases have been found. Uniqueness for the stationary system is also studied.

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1. Introduction

The Navier–Stokes equations for compressible, isentropic flow in N space dimensions can be written in the form:

$$\frac{\partial \varrho}{\partial t} + \mathrm{div}\ (\varrho \vec{u}) = 0,$$

$$\frac{\partial \varrho \vec{u}}{\partial t} + \mathrm{div}\ (\varrho \vec{u} \otimes \vec{u}) - \mu \Delta \vec{u} - (\lambda + \mu) \nabla (\mathrm{div}\ \vec{u}) + \nabla a \varrho^\gamma = \varrho \vec{f},$$

where μ, λ are viscosity coefficients, $a > 0$ and the adiabatic constant $\gamma > 1$.

If the density of the driving force $\vec{f} = \nabla F$ is a gradient of a scalar time independent potential $F = F(x)$, the problem admits a Lyapunov function, namely, the energy (note that the potential F and the energy E are defined up to addition a constant)

$$E(t) = \int_\Omega \frac{1}{2} \varrho(t) \left| \vec{u}(t) \right|^2 + \frac{a}{\gamma - 1} \varrho^\gamma(t) - \varrho(t) F \,\mathrm{d}x.$$

Accordingly, it is plausible to anticipate the $\omega-$limit set of global trajectories to be formed by the solutions of the stationary system $a\nabla\varrho^\gamma = \varrho\nabla F$. This is indeed the case and positive results in this direction, even for global weak solutions, can be found in [5], [8], [9] etc. More specifically, if $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain and $\gamma \geq \frac{9}{5}$, any global finite energy weak solution satisfies

$$\varrho(t) \to \omega[\varrho] \text{ strongly in } \mathrm{L}^\gamma(\Omega), \qquad \sqrt{\varrho(t)}u(t) \to 0 \text{ strongly in } \mathrm{L}^2(\Omega) \text{ as } t \to \infty,$$

where the $\omega-$limit set $\omega[\varrho]$ is a compact (in L^1-topology) and connected subset of the set of solutions of the stationary system:

$$\left.\begin{array}{c} a\partial_i\varrho^\gamma = (\partial_i F)\varrho, \quad i = 1,\dots,N, \quad \text{in } D'(\Omega), \\ a \in (0,\infty),\ \gamma \in (1,\infty), \qquad \varrho \geq 0, \end{array}\right\} \tag{1.1}$$

$$\int_\Omega \varrho(x)\,\mathrm{d}x = m, \tag{1.2}$$

$$\int_\Omega \frac{a}{\gamma-1}\varrho^\gamma - \varrho F\,\mathrm{d}x = e. \tag{1.3}$$

Here, the total mass m and the potential energy $e = \operatorname{ess\,lim}_{t\to\infty} E(t)$ are uniquely determined by the trajectory and thus constant for any function belonging to $\omega[\varrho]$ (see [3], Theorem 13).

Because the ω-limit set is connected, it is of interest to study the topological structure of the set of all solutions to the problem (1.1)–(1.3). In particular, if we knew that the stationary problem admits at most finite number of solutions, it would imply that $\omega[\varrho]$ is a singleton, i.e., the density $\varrho(t)$ in the evolution problem stabilizes for $t \to \infty$ to a certain stationary solution.

At first, we will deal with the uniqueness of the stationary system (1.1)–(1.2) on (not necessarily bounded) domain.

We will show that there exists a critical mass $\widetilde{m}$ such that:

(a) The system (1.1)–(1.2) has at most one solution for the mass $m \in [\widetilde{m},\infty)$.
(b) There is continuum of solutions of the system (1.1)–(1.2) for the mass $m \in (0,\widetilde{m})$.

Later, we will define a critical mass m_c such that:

(a) If $m \in [m_c,\infty)$, then the stationary problem (1.1)–(1.3) admits at most two solutions for each energy $e \in \mathbb{R}$.
(b) If $m \in (0,m_c)$, then there exists an energy $e \in \mathbb{R}$ such that the system (1.1)–(1.3) has continuum of solutions.

The stationary system has been studied by many authors (see [1], [2], [4], [6], [7], [11] etc.). The presented results are generalizing the recent results about the stationary problem obtained in [4] and [6].

2. Uniqueness

Theorem 1. *Let* $\Omega \subset \mathbb{R}^N$ *be a domain,* $F : \Omega \to \mathbb{R}$ *be a locally Lipschitz continuous function on* Ω *and* $-\infty < \inf_\Omega F \leq \sup_\Omega F < \infty$. *Let us suppose* $m \in (0,\infty)$, $a \in (0,\infty)$, $\gamma \in (1,\infty)$ *and define set* $\widetilde{\mathcal{B}} = \{k \in (-\infty, \sup_\Omega F) \,|\, k$ *satisfies that the set* $\{x \in \Omega; F(x) > k\}$ *is not connected*$\}$. *If we define*

$$\widetilde{K} = \begin{cases} \inf \widetilde{\mathcal{B}} & \text{if } \widetilde{\mathcal{B}} \neq \emptyset, \\ \sup_\Omega F & \text{if } \widetilde{\mathcal{B}} = \emptyset, \end{cases} \tag{2.1}$$

$$\widetilde{m} = \int_\Omega \left(\frac{\gamma - 1}{a\gamma} [F(x) - \widetilde{K}]^+ \right)^{\frac{1}{\gamma-1}} \, dx, \tag{2.2}$$

then:

(i) *If* $m \geq \widetilde{m}$, *then there is at most one function* $\varrho \in \mathrm{L}^\infty_{loc}(\Omega)$ *satisfying* (1.1) *and* (1.2).

(ii) *If* $\infty > \widetilde{m} > 0, 0 < m < \widetilde{m}$, *then there is continuum of solutions of the stationary system* (1.1) – (1.2) *in* $\mathrm{L}^\infty_{loc}(\Omega)$.

Remark. Theorem 1 is generalizing the results obtained in [4]. In [4] the special case of $\widetilde{m} = 0$ is studied.

Proof. (i) Let $\varrho \in \mathrm{L}^\infty_{loc}(\Omega)$ be a function satisfying (1.1), then $\varrho^\gamma \in \mathrm{L}^\infty_{loc}(\Omega)$ and also $\partial_i \varrho^\gamma \in \mathrm{L}^\infty_{loc}(\Omega)$ (it follows from (1.1)). By imbeddings theorems we see that ϱ is continuous on Ω.

Now we can introduce the following notation:

$$\mathcal{R} = \{x \in \Omega : \varrho(x) > 0\}, \tag{2.3}$$

$$\mathcal{F}[c] = \{x \in \Omega \,|\, F(x) > c\}, \qquad \text{where } c \in \mathbb{R}. \tag{2.4}$$

Let $\Theta \subset \Omega$ be a maximal connected component of the open set $\mathcal{R}$. Let us consider a ball B such that $\overline{\mathrm{B}} \subset \Theta$, then there are constants $\underline{\varrho}$, $\overline{\varrho}$ such that

$$0 < \underline{\varrho} \leq \varrho(x) \leq \overline{\varrho} < \infty \qquad \text{for all } x \in \overline{\mathrm{B}}.$$

Then $\varrho \in \mathrm{W}^{1,p}(\mathrm{B})$ for $p \geq 1$ (see [10], Theorem 2.1.11) and we can rewrite equation (1.1) in the ball B:

$$\frac{\partial}{\partial x_i} \left(\frac{a\gamma}{\gamma - 1} \varrho^{\gamma-1} - F \right) = 0, \qquad i = 1, \dots, N, \qquad \text{in } D'(\mathrm{B}).$$

Thus there exists a constant (see [10], Corollary 2.1.9) $k_B \in \mathbb{R}$ such that

$$\frac{a\gamma}{\gamma - 1} \varrho^{\gamma-1} = F - k_B \quad \text{on} \quad \mathrm{B}.$$

Moreover, as ϱ, F are continuous, the constant k_B is independent of B, in other words, there exists k_Θ such that

$$\frac{a\gamma}{\gamma-1}\varrho^{\gamma-1} = F - k_\Theta \quad \text{on} \quad \Theta. \tag{2.5}$$

Hence, $\Theta \subset \mathcal{F}[k_\Theta]$. On the other hand,

$$\frac{a\gamma}{\gamma-1}\varrho^{\gamma-1} = F - k_\Theta = 0 \quad \text{for all} \quad x \in \partial\Theta \cap \Omega,$$

hence

$$\partial\Theta \cap \mathcal{F}[k_\Theta] = \emptyset.$$

Thus the set Θ is a maximal connected component of the open set $\mathcal{F}[k_\Theta]$ and (2.5) is fulfilled. By virtue of (2.5), we see

$$\varrho(x) = \left(\frac{\gamma-1}{a\gamma}[F(x) - k_\Theta]^+\right)^{\frac{1}{\gamma-1}} \quad \text{on } \Theta. \tag{2.6}$$

In the previous paragraphs we proved that for each maximal connected component Θ of the open set $\mathcal{R}$ there exists a constant k_Θ such that (2.6) is fulfilled.

Now, we will prove the following lemma:

Lemma 1. *Let the assumptions of Theorem 1 are fulfilled and $m \geq \widetilde{m}$. Then for each maximal connected component Θ of the open set $\mathcal{R}$ the constant k_Θ fulfills $k_\Theta \leq \widetilde{K}$.*

Proof of Lemma 1. Let us suppose that there exists a maximal connected component Q of the open set $\mathcal{R}$ such that $k_Q > \widetilde{K}$.

We distinguish two cases:

(1) There exists a maximal connected component Θ of the open set $\mathcal{R}$ such that $k_\Theta < \widetilde{K}$. Then by virtue of (2.1), the set $\mathcal{F}[k_\Theta]$ is connected, hence $\Theta = \mathcal{F}[k_\Theta]$. Moreover, $Q \subset \Theta = \mathcal{F}[k_\Theta]$, which follows from $k_\Theta < \widetilde{K} < k_Q$. In particular, the solution ϱ is given by the formula (2.6) with the constant k_Θ on the set Q.

This is in contradiction to the assumption $k_Q > k_\Theta$.

(2) For each maximal connected component Θ of the open set $\mathcal{R}$ the constant k_Θ fulfills $k_\Theta \geq \widetilde{K}$.

Then

$$\int_\Omega \varrho \,\mathrm{d}x = \int_{\Omega - Q} \varrho \,\mathrm{d}x + \int_Q \varrho \,\mathrm{d}x$$

$$\leq \int_{\Omega-Q}\left(\frac{\gamma-1}{a\gamma}[F(x)-\widetilde{K}]^+\right)^{\frac{1}{\gamma-1}} \mathrm{d}x + \int_Q \left(\frac{\gamma-1}{a\gamma}[F(x)-k_Q]^+\right)^{\frac{1}{\gamma-1}} \mathrm{d}x < \widetilde{m} \leq m.$$

This is in contradiction to $\int_\Omega \varrho(x)\,\mathrm{d}x = m$.

Q.E.D. {Lemma 1}

Now, we will continue the proof of Theorem 1. As Lemma 1 is proved, we can distinguish two cases:

(a) $m > \widetilde{m}$. Then, by virtue of Lemma 1, there exists the maximal connected component Q of the open set $\mathcal{R}$ such that $k_Q < \widetilde{K}$.

Then Q is the maximal connected component of the open set $\mathcal{F}[k_Q]$, which is, by virtue of (2.1), connected.

In particular, $Q = \mathcal{F}[k_Q]$. As

$$\mathcal{F}[k_1] \subset \mathcal{F}[k_2] \text{ for } k_1 > k_2, \tag{2.7}$$

we see that $\mathcal{R}$ is connected,

$$\mathcal{R} = \mathcal{F}[k_Q]$$

and there exists the unique constant k_Q such that

$$\varrho(x) = \left(\frac{\gamma-1}{a\gamma}[F(x) - k_Q]^+\right)^{\frac{1}{\gamma-1}} \text{ on } \Omega.$$

(b) $m = \widetilde{m}$. Then, by virtue of Lemma 1 and the case (a), $k_Q = \widetilde{K}$ for each maximal connected component Q of the open set $\mathcal{R}$.

In particular,

$$\mathcal{R} = \mathcal{F}[\widetilde{K}]$$

and

$$\varrho(x) = \left(\frac{\gamma-1}{a\gamma}[F(x) - \widetilde{K}]^+\right)^{\frac{1}{\gamma-1}} \text{ on } \Omega.$$

The first part of Theorem 1 has been proved.

(ii) Let $0 < m < \widetilde{m} < \infty$. We define a function $S: [\widetilde{K}, \sup_\Omega F] \to [0, \widetilde{m}]$ by the formula

$$S(k) = \int_\Omega \left(\frac{\gamma-1}{a\gamma}[F(x) - k]^+\right)^{\frac{1}{\gamma-1}} \mathrm{d}x.$$

It is easy to verify that S is a continuous, decreasing function, $S(\sup_\Omega F) = 0$, $S(\widetilde{K}) = \widetilde{m}$, thus there exists $k_m \in (\widetilde{K}, \sup_\Omega F)$ such that $S(k_m) = m$.

By virtue of (2.1), there exists a constant l such that

$$\widetilde{K} < l < k_m \qquad \& \qquad \mathcal{F}[l] \text{ is not connected.}$$

Then the open set $\mathcal{F}[l]$ has at least two components. As $m > 0$, there exists a component Θ of the open set $\mathcal{F}[l]$ such that

$$s = \int_{\Omega-\Theta} \left(\frac{\gamma-1}{a\gamma}[F(x) - k_m]^+\right)^{\frac{1}{\gamma-1}} \mathrm{d}x > 0.$$

Now, we can define a function $T: [l, \sup_\Omega F] \to [0, \widetilde{m})$ by the formula

$$T(k) = \int_\Theta \left(\frac{\gamma - 1}{a\gamma} [F(x) - k]^+ \right)^{\frac{1}{\gamma-1}} \mathrm{d}x.$$

Then T is continuous, nonincreasing function, $T(l) > T(k_m) = m - s \geq 0$, thus there exist constants p and q such that:

$$l < p < q < k_m, \quad \& \quad m - \frac{s}{2} > T(p) > T(q) > 0, \quad \& \quad T \text{ is decreasing on } (p, q).$$

Now, for each $c \in (p, q)$ we will find the function ϱ_c such that $\int_\Omega \varrho_c(x)\,\mathrm{d}x = m$ and (1.1) holds.

First, we define the function ϱ_c by the formula (2.6) with the constant c on Θ. The function $S - T$ is continuous on $[k_m, \sup_\Omega F]$ and $(S - T)(k_m) > m - T(c) > s/2 > 0 = (S - T)(\sup_\Omega F)$, thus there exists a constant $d \in (k_m, \sup_\Omega F)$ such that $S(d) - T(d) = m - T(c)$.

Hence, we can define the function ϱ_c by the formula (2.6) with the constant d on $\Omega - \Theta$. Then $\varrho_c \in \mathrm{L}^\infty_{loc}(\Omega)$, (1.1) holds and $\int_\Omega \varrho_c(x)\,\mathrm{d}x = S(d) - T(d) + T(c) = m$. Moreover, the functions ϱ_c, $c \in (p, q)$, are pairwise distinct.

Q.E.D.

Remark. The hypothesis $\varrho \in \mathrm{L}^\infty_{loc}(\Omega)$ in part (i) of Theorem 1 may be omitted. It is sufficient to suppose only $\varrho^\gamma \in \mathrm{L}^1_{loc}(\Omega)$. This hypothesis is necessary for equation (1.1) to make sense in distributions and, by standard bootstrap argument, it implies the hypothesis $\varrho \in \mathrm{L}^\infty_{loc}(\Omega)$ (see [4] for details).

3. On the stationary system with finite number of solutions

We will study the number of solutions of the stationary problem (1.1)–(1.3).

In the following, we will suppose that Ω is a bounded domain, $F: \Omega \to \mathbb{R}$ is a locally Lipschitz continuous function on Ω and $-\infty < \inf_\Omega F \leq \sup_\Omega F < \infty$, $\gamma \in (1, \infty)$ and $a \in (0, \infty)$.

For each open set A we can define a function $M_A: \mathbb{R} \to \mathbb{R}$ by the formula

$$M_A(c) = \int_A \left(\frac{\gamma - 1}{a\gamma} [F(x) - c]^+ \right)^{\frac{1}{\gamma-1}} \mathrm{d}x. \tag{3.1}$$

Our goal is to define the critical mass m_c such that the stationary problem (1.1)–(1.3) admits at most two solutions for the mass $m \in [m_c, \infty)$ and for each energy $e \in \mathbb{R}$, and, on the other hand, if $m \in (0, m_c)$, then there exists an energy $e \in \mathbb{R}$ such that the system (1.1)–(1.3) has continuum of solutions.

Definition. Let Ω be a bounded domain, $F : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function on Ω and $-\infty < \inf_\Omega F \le \sup_\Omega F < \infty$. Putting

$$\hat{\mathcal{B}} = \left\{ \begin{array}{ll} k \in (-\infty, \sup_\Omega F) \,| & k \text{ satisfies that there are domains } \Theta_1,\, \Theta_2 \\ & \text{such that } \overline{\Omega} = \overline{\Theta_1} \cup \overline{\Theta_2}, \Theta_1 \cap \Theta_2 = \emptyset \text{ and} \\ & \forall\, c \in (-\infty, k) \text{ the set } \{x \in \Theta_i; F(x) > c\} \\ & \text{is connected in } \Theta_i,\ i = 1, 2 \end{array} \right\},$$

we can define the constant $\hat{K}$ by the formula

$$\hat{K} = \sup \hat{\mathcal{B}}. \tag{3.2}$$

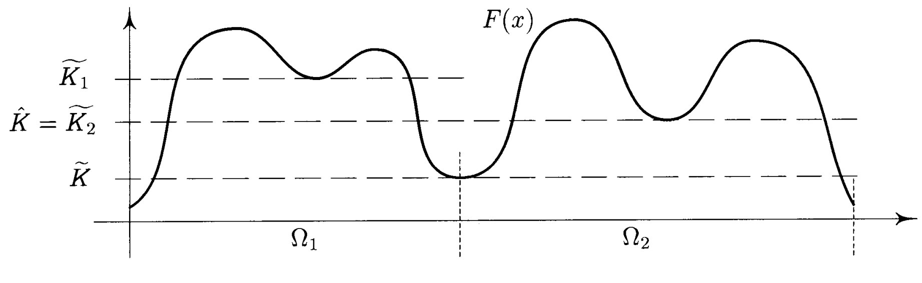
By virtue of the previous definition, there exist two domains Ω_1, Ω_2 such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\forall\, c \in (-\infty, \hat{K})$ the set $\{x \in \Omega_i; F(x) > c\}$ is connected in Ω_i, $i = 1, 2$.

Definition. Let $\hat{K} > \widetilde{K}$, then, for $i = 1, 2$, we define

$$\widetilde{\mathcal{B}_i} = \{k \in (-\infty, \sup_\Omega F) \,|\, k \text{ satisfies that the set } \{x \in \Omega_i; F(x) > k\} \text{ is not connected}\},$$

$$\widetilde{K_i} = \begin{cases} \inf \widetilde{\mathcal{B}_i} & \text{if } \widetilde{\mathcal{B}_i} \ne \emptyset, \\ \sup_\Omega F & \text{if } \widetilde{\mathcal{B}_i} = \emptyset. \end{cases} \tag{3.3}$$

The reader can see the picture for better understanding of the previous definitions. The picture is in one dimension, thus the domain Ω is an interval. You can see the function $F : \Omega \to \mathbb{R}$ and the corresponding constants $\widetilde{K}$, $\hat{K}$, $\widetilde{K_1}$ and $\widetilde{K_2}$.



For brevity, we will denote the function M_{Ω_i} by the symbol M_i, $i = 1, 2$, thus $M_i : \mathbb{R} \to \mathbb{R}$,

$$M_i(c) = \int_{\Omega_i} \left(\frac{\gamma - 1}{a\gamma} [F(x) - c]^+ \right)^{\frac{1}{\gamma - 1}} \,\mathrm{d}x \quad \text{for } i = 1, 2 \tag{3.4}$$

(compare with (3.1)).

We also define $\widetilde{m}$ by the formula (2.2) and we will use the notation from (2.3) and (2.4).

Definition. If $\hat{K} = \widetilde{K}$, then we define the critical mass $m_c = \widetilde{m}$.

If $\hat{K} > \widetilde{K}$, then we define the critical mass m_c in the following way (we distinguish four possible cases):

If $\sup_\Omega F = \widetilde{K_1} = \widetilde{K_2}$, then $m_c = 0$.

If $\widetilde{K_1} = \sup_\Omega F > \widetilde{K_2}$, then $m_c = M_1(\widetilde{K}) + M_2(\widetilde{K_2})$.

If $\widetilde{K_1} < \sup_\Omega F = \widetilde{K_2}$, then $m_c = M_1(\widetilde{K_1}) + M_2(\widetilde{K})$.

If $\sup_\Omega F > \widetilde{K_1}$ and $\sup_\Omega F > \widetilde{K_2}$, then

$$m_c = \max\left\{M_1(\widetilde{K}) + M_2(\widetilde{K_2}), M_1(\widetilde{K_1}) + M_2(\widetilde{K})\right\}.$$

Remark. If $\hat{K} > \widetilde{K}$, then we can rewrite the previous definition to the formula

$$m_c = \max\left\{\begin{array}{c}\left(M_1(\widetilde{K}) + M_2(\widetilde{K_2})\right)\cdot \operatorname{sign}\left(\sup_\Omega F - \widetilde{K_2}\right)\\ \left(M_1(\widetilde{K_1}) + M_2(\widetilde{K})\right)\cdot \operatorname{sign}\left(\sup_\Omega F - \widetilde{K_1}\right)\end{array}\right\}. \tag{3.5}$$

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $F : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function on Ω and $-\infty < \inf_\Omega F \le \sup_\Omega F < \infty$. Let $m \in [m_c, \infty)$ and $e \in \mathbb{R}$. Then there exist at most two functions $\varrho \in \mathrm{L}^\infty_{loc}(\Omega)$ such that* (1.1), (1.2) *and* (1.3) *hold.*

Remark. Theorem 2 is generalizing the results obtained in [6]. In [6] the special case of $m_c = 0$ is studied.

Proof. We proved in Theorem 1 that the stationary system (1.1)–(1.3) has at most one solution for $m \in [\widetilde{m}, \infty)$. Thus it is sufficient to deal only with the case $\widetilde{m} > m_c \ge 0$ and $m \in [m_c, \widetilde{m})$.

Let us suppose that we have a solution of the problem (1.1)–(1.3) and we will show the necessary conditions for it.

As in the proof of Theorem 1, the functions satisfying (1.1) are given by the formula (2.6), in particular, for each component Θ of the open set $\mathcal{R}$ there exists the constant k_Θ such that ϱ is given by the formula

$$\varrho(x) = \left(\frac{\gamma - 1}{a\gamma}[F(x) - k_\Theta]^+\right)^{\frac{1}{\gamma-1}} \text{ on } \Theta, \tag{3.6}$$

moreover, Θ is also the component of the open set $\mathcal{F}[k_\Theta]$.

At first, we will prove the following auxiliary lemma:

Lemma 2. *Let the assumptions of Theorem 2 are fulfilled. Then for each maximal connected component* Θ, $\Theta \cap \Omega_1 \neq \emptyset$, *of the open set* $\mathcal{R}$, *the constant* k_Θ *fulfills* $k_\Theta \le \widetilde{K_1}$.

Proof of Lemma 2. Let us suppose that there exists a maximal connected component Q of the open set $\mathcal{R}$ such that $Q \cap \Omega_1 \neq \emptyset$ and $k_Q > \widetilde{K_1}$.

We distinguish two cases:

(1) There exists a maximal connected component Θ of the open set $\mathcal{R}$ such that $\Theta \cap \Omega_1 \neq \emptyset$ and $k_\Theta < \widetilde{K_1}$.

Then by virtue of (3.3), the set $\{x \in \Omega_1 \,|\, F(x) > k_\Theta\}$ is connected, hence the solution $\varrho(x)$ is given by the formula (3.6) with the constant k_Θ on the whole set Ω_1, in particular, on Q. This is in contradiction to the assumption $k_Q > \widetilde{K_1} > k_\Theta$.

(2) For each maximal connected component Θ, $\Theta \cap \Omega_1 \neq \emptyset$, of the open set $\mathcal{R}$, the constant k_Θ fulfills $k_\Theta \ge \widetilde{K_1}$.

Because of $k_Q > \widetilde{K_1}$, we get

$$\int_{\Omega_1} \varrho(x)\,\mathrm{d}x < M_1(\widetilde{K_1}). \tag{3.7}$$

Let E be a maximal connected component of the open set $\mathcal{R}$ such that $E \cap \Omega_2 \neq \emptyset$. If $k_E < \widetilde{K}$ then, by virtue of the definition (2.1), the set $\mathcal{F}[k_E]$ is connected, hence the solution $\varrho(x)$ is given by the formula (3.6) with the constant k_E on the whole set Ω, in particular, on Q, which is impossible. Thus we have $k_E \ge \widetilde{K}$, hence

$$\int_{\Omega_2} \varrho(x)\,\mathrm{d}x \le M_2(\widetilde{K}). \tag{3.8}$$

Adding up (3.7) and (3.8), we get

$$\int_{\Omega} \varrho(x)\,\mathrm{d}x < M_1(\widetilde{K_1}) + M_2(\widetilde{K}) \le m_c \le m.$$

This is in contradiction to (1.2).

Q.E.D. {Lemma 2}

Now, we will continue the proof of Theorem 2. Symmetrically to Lemma 2, an analogous lemma for the domain Ω_2 is fulfilled, in particular, for each maximal connected component Θ, $\Theta \cap \Omega_2 \neq \emptyset$, of the open set $\mathcal{R}$, the constant k_Θ fulfills $k_\Theta \le \widetilde{K_2}$.

By virtue of Lemma 2 and the definition of $\widetilde{K_1}$, $\widetilde{K_2}$, we get that one of the following three cases is satisfied:

(a)

$$\varrho(x) = \left(\frac{\gamma - 1}{a\gamma} [F(x) - c_1]^+ \right)^{\frac{1}{\gamma-1}} \quad \text{for } x \in \Omega_1,$$

$$\varrho(x) = \left(\frac{\gamma - 1}{a\gamma} [F(x) - c_2]^+ \right)^{\frac{1}{\gamma-1}} \quad \text{for } x \in \Omega_2, \tag{3.9}$$

(b)

$$\varrho(x) = \left(\frac{\gamma - 1}{a\gamma} [F(x) - c_1]^+ \right)^{\frac{1}{\gamma-1}} \quad \text{for } x \in \Omega_1 \qquad \& \qquad \varrho(x) = 0 \text{ for } x \in \Omega_2, \tag{3.10}$$

(c)

$$\varrho(x) = 0 \text{ for } x \in \Omega_1 \qquad \& \qquad \varrho(x) = \left(\frac{\gamma - 1}{a\gamma} [F(x) - c_2]^+ \right)^{\frac{1}{\gamma-1}} \quad \text{for } x \in \Omega_2, \tag{3.11}$$

where $c_1 \in (-\infty, \widetilde{K_1}]$, $c_2 \in (-\infty, \widetilde{K_2}]$ are constants. The values of the constants c_1 and c_2 will be specified by the hypothesis (1.2) and (1.3).

Now, we define the numbers

$$\vartheta_i = \inf\{c \in \mathbb{R};\ M_i(c) = 0\} \qquad \text{for } i = 1, 2, \tag{3.12}$$

where M_i, $i = 1, 2$, are functions defined by (3.4).

Then $M_i(c)$ is decreasing function on $(-\infty, \vartheta_i]$, hence, there exist inverse decreasing functions $G_i \equiv M_i^{-1}$, $i = 1, 2$, in particular, G_i maps the interval $[M_i(\vartheta_i), \infty) = [0, \infty)$ on the interval $(-\infty, \vartheta_i]$.

Therefore, we can substitute $c_i = G_i(m_i)$, $i = 1, 2$, in (3.9), (3.10) and (3.11), where m_1 and m_2 are nonnegative constants.

Let us note that the case (3.10) (resp. (3.11)) can be fulfilled only if $M_2(\widetilde{K_2}) = 0$ (resp. $M_1(\widetilde{K_1}) = 0$), in particular, the case (3.10) (resp. (3.11)) is a special case of (3.9) with $c_2 = \vartheta_2$ (resp. $c_1 = \vartheta_1$).

Hence, the solutions ϱ of the system (1.1)–(1.2) are necessarily given by the formula

$$\varrho(x) = \left(\frac{\gamma - 1}{a\gamma} [F(x) - G_1(m_1)]^+ \right)^{\frac{1}{\gamma-1}} \quad \text{for } x \in \Omega_1,$$

$$\varrho(x) = \left(\frac{\gamma - 1}{a\gamma} [F(x) - G_2(m_2)]^+ \right)^{\frac{1}{\gamma-1}} \quad \text{for } x \in \Omega_2,$$

where $m_1 \in [0, \infty)$, $m_2 \in [0, \infty)$ and (1.2) implies

$$m_1 + m_2 = m.$$

Now we will consider the equation (1.3). It will give other necessary conditions on parameters m_1 and m_2.

We define auxiliary functions

$$F_i(c) = \frac{1 - \gamma}{\gamma} \left(\frac{\gamma - 1}{a\gamma} \right)^{\frac{1}{\gamma-1}} \int_{\Omega_i} \left([F(x) - c]^+ \right)^{\frac{\gamma}{\gamma-1}} \mathrm{d}x \quad \text{for } i = 1, 2.$$

The functions F_i, $i = 1, 2$, are nonpositive and nondecreasing on $\mathbb{R}$. It is easy to verify that $F_i \in \mathrm{C}^1(\mathbb{R})$ and

$$F_i'(c) = M_i(c) \ \text{ for } i = 1, 2.$$

By virtue of the definitions of the functions F_i and M_i, we can rewrite the energy of the solution ϱ (ϱ is given by the formula (3.9)) in the form

$$E(c_1, c_2) = \int_\Omega \frac{a}{\gamma - 1}\varrho^\gamma - \varrho F \,\mathrm{d}x = F_1(c_1) - M_1(c_1) \cdot c_1 + F_2(c_2) - M_2(c_2) \cdot c_2.$$

If we substitute $c_i = G_i(m_i)$ for $i = 1, 2$, we get

$$E(m_1, m_2) = F_1(G_1(m_1)) - m_1 \cdot G_1(m_1) + F_2(G_2(m_2)) - m_2 \cdot G_2(m_2).$$

Therefore, the hypothesis (1.2) and (1.3) imply

$$m_1 + m_2 = m \quad \& \quad E(m_1, m_2) = e.$$

In the final part of the proof, we shall show that there are at most two $m_1 \in [0, m]$ such that $E(m_1, m - m_1) = e$.

Let us define the function $e : [0, m] \to \mathbb{R}$ by the formula $e(m_1) = E(m_1, m - m_1)$, in particular,

$$e(m_1) = F_1(G_1(m_1)) - m_1 \cdot G_1(m_1) + F_2(G_2(m - m_1)) - (m - m_1) \cdot G_2(m - m_1).$$

It is easy to verify that $e \in \mathrm{C}^1[0, m]$ and

$$e'(m_1) = G_2(m - m_1) - G_1(m_1) \qquad \text{for } m_1 \in [0, m].$$

(Precisely, the result $e \in \mathrm{C}^1[0, m]$ is obvious for $\gamma \in (1, 2)$. If $\gamma \in [2, \infty)$ then, by virtue of the definiton, the function G_i, $i = 1, 2$, is Lipschitz continuous in (δ, m) for arbitrary $0 < \delta < \frac{m}{2}$. Thus $e \in \mathrm{C}^1(\delta, m - \delta)$ for arbitrary small $\delta > 0$.)

Now, the equation $e'(m_1) = 0$ is equivalent to the equation $G_1(m_1) = G_2(m - m_1)$, which has at most one solution (on the left side there is a decreasing function, on the right side there is an increasing function). Hence, the equation $e(m_1) = e$ has at most two solutions in the interval $[0, m]$, in particular, Theorem 2 has been proved.

Q.E.D.

Remarks. (i) The hypothesis $\varrho \in \mathrm{L}^\infty_{loc}(\Omega)$ in Theorem 2 may be omitted. It is sufficient to suppose only that $\varrho^\gamma \in \mathrm{L}^1_{loc}(\Omega)$ (compare with the remark after Theorem 1).

(ii) It is not difficult to check the lower estimate $e > -m \sup_\Omega F$. Thus we can suppose $e \in (-m \sup_\Omega F, \infty)$ in Theorem 2.

4. The optimality of the critical mass $\mathbf{m_C}$

We shall show in this section that the result of Theorem 2 is optimal. We shall deal with the system (1.1)–(1.3) for mass from the interval $(0, m_c)$ and we will prove that there exists an energy e such that the stationary system (1.1)–(1.3) has continuum of solutions.

The main theorem in this section is the following one:

Theorem 3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $F : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function on Ω and $-\infty < \inf_\Omega F \le \sup_\Omega F < \infty$. Let $m \in (0, m_c)$. Then there exists an energy $e \in \mathbb{R}$ such that there is continuum of functions $\varrho_\xi \in \mathrm{L}^\infty_{loc}(\Omega)$ such that* (1.1), (1.2) *and* (1.3) *hold.*

Proof. We will distinguish two cases:

(a) $\hat{K} > \widetilde{K}$. According to the symmetry of the definition (3.5), we can suppose without lost of generality that

$$m_c = M_1(\widetilde{K}) + M_2(\widetilde{K_2}), \tag{4.1}$$

where $M_1(\widetilde{K}) > 0$ and $M_2(\widetilde{K_2}) > 0$.

If we define for the functions M_i, $i = 1, 2$, the constants ϑ_i by the formula (3.12), we see that the function M_1 is continuous and decreasing on the interval $[\widetilde{K}, \vartheta_1]$,

$$M_1(\widetilde{K}) > M_1(\widetilde{K}) \cdot \frac{m}{m_c} > 0 = M_1(\vartheta_1),$$

hence, there exists the constant $k_1 \in (\widetilde{K}, \vartheta_1)$ such that $M_1(k_1) = M_1(\widetilde{K}) \cdot \frac{m}{m_c}$.

The function M_2 is continuous and decreasing on the interval $[\widetilde{K_2}, \vartheta_2]$,

$$M_2(\widetilde{K_2}) > M_2(\widetilde{K_2}) \cdot \frac{m}{m_c} > 0 = M_2(\vartheta_2),$$

thus, there exists the constant $l_1 \in (\widetilde{K_2}, \vartheta_2)$ such that $M_2(l_1) = M_2(\widetilde{K_2}) \cdot \frac{m}{m_c}$.

By virtue of $l_1 > \widetilde{K_2}$ and the definition $\widetilde{K_2}$ there exists the constant l_2, $\widetilde{K_2} \le l_2 < l_1$, such that $\{x \in \Omega_2 \,|\, F(x) > l_2\}$ is not connected. Let Θ_1 be a maximal connected component of the open set $\{x \in \Omega_2 \,|\, F(x) > l_2\}$. Let us denote $\Theta_2 = \Omega_2 - \Theta_1$ and define $\psi_i = \inf\{c \in \mathbb{R};\ M_{\Theta_i}(c) = 0\}$, $i = 1, 2$.

The function M_{Θ_i}, $i = 1, 2$, is continuous and decreasing on the interval (l_2, ψ_i),

$$M_{\Theta_i}(l_2) > M_{\Theta_i}(l_2) \cdot \frac{M_2(l_1)}{M_2(l_2)} > 0 = M_{\Theta_i}(\psi_i),$$

thus there exist the constants $n_i \in (l_2, \psi_i)$, $i = 1, 2$, such that $M_{\Theta_i}(n_i) = M_{\Theta_i}(l_2) \cdot \frac{M_2(l_1)}{M_2(l_2)}$.

If $n_1 = n_2$, then we can find $\delta > 0$ sufficiently small and the constants $\widetilde{n_i} \in (l_2, \psi_i)$, $i = 1, 2$, such that $\widetilde{n_1} \neq \widetilde{n_2}$ and $M_{\Theta_i}(\widetilde{n_i}) = M_{\Theta_i}(l_2) \cdot \frac{M_2(l_1)}{M_2(l_2)} + (-1)^i \cdot \delta$. If $n_1 \neq n_2$, then we put $\widetilde{n_i} = n_i$, $i = 1, 2$.

Now, for better understanding, we put $Q_1 = \Omega_1$, $Q_2 = \Theta_1$, $Q_3 = \Theta_2$, $k_2 = \widetilde{n_1}$, $k_3 = \widetilde{n_2}$,

$$\varepsilon_1 = \min\left\{\frac{\vartheta_1 - k_1}{2}, \frac{k_1 - \widetilde{K}}{2}\right\},$$

$$\varepsilon_2 = \min\left\{\frac{\psi_1 - k_2}{2}, \frac{k_2 - l_2}{2}\right\},$$

$$\varepsilon_3 = \min\left\{\frac{\psi_2 - k_3}{2}, \frac{k_3 - l_2}{2}\right\}.$$

Then $\Omega = \bigcup_{i=1}^{3} Q_i$, the sets Q_i are pairwise distinct and $\varepsilon_i > 0$, $i = 1, 2, 3$.

If $\varrho(x)$ is given by the formula

$$\varrho(x) = \left(\frac{\gamma - 1}{a\gamma}[F(x) - k_i]^+\right)^{\frac{1}{\gamma-1}} \quad \text{on } Q_i,\ i = 1, 2, 3,$$

then ϱ is the solution of the system (1.1)–(1.2), which follows from

$$\int_\Omega \varrho(x)\,\mathrm{d}x = \sum_{i=1}^{3} M_{Q_i}(k_i) = M_1(k_1) + M_{\Theta_1}(\widetilde{n_1}) + M_{\Theta_2}(\widetilde{n_2})$$

$$= M_1(k_1) + M_2(l_1) = \left(M_1(\widetilde{K}) + M_2(\widetilde{K_2})\right) \cdot \frac{m}{m_c} = m.$$

Next, we put

$$e = \int_\Omega \frac{a}{\gamma - 1}\varrho^\gamma - \varrho F\,\mathrm{d}x.$$

We will show that there is continuum of solutions of the system (1.1)–(1.3) with the mass m and the energy e.

These solutions will be given by the formula

$$\varrho_\xi(x) = \left(\frac{\gamma - 1}{a\gamma}[F(x) - c_i]^+\right)^{\frac{1}{\gamma-1}} \quad \text{on } Q_i,\ i = 1, 2, 3, \tag{4.2}$$

where $c_i \in (k_i - \varepsilon_i, k_i + \varepsilon_i)$, $i = 1, 2, 3$. Then, the equation (1.1) is fulfilled.

The function $M_{Q_i}(c)$, $i = 1, 2, 3$, (see (3.1) for the definition) is decreasing on the interval $c \in (k_i - \varepsilon_i, k_i + \varepsilon_i)$. Then we can define the inverse function $G_i = M_{Q_i}^{-1}$ on these intervals. Substituting $c_i = G_i(m_i),\ i = 1, 2, 3$, in (4.2), we get

$$\varrho(x) = \left(\frac{\gamma - 1}{a\gamma}[F(x) - G_i(m_i)]^+\right)^{\frac{1}{\gamma-1}} \quad \text{for } x \in Q_i,\ i = 1, 2, 3, \tag{4.3}$$

where $m_i \in (M_{Q_i}(k_i + \varepsilon_i), M_{Q_i}(k_i - \varepsilon_i))$, $i = 1, 2, 3$.

Moreover, the condition (1.2) is equivalent to the condition

$$m_3 = m - m_1 - m_2.$$

Now, we define the auxiliary functions

$$F_i(c) = \frac{1-\gamma}{\gamma} \left(\frac{\gamma - 1}{a\gamma} \right)^{\frac{1}{\gamma-1}} \int_{Q_i} \left([F(x) - c]^+ \right)^{\frac{\gamma}{\gamma-1}} \, \mathrm{d}x \ \text{ for } i = 1, 2, 3.$$

Then the condition (1.3) is equivalent to the equation $E(m_1, m_2, m_3) = e$, where the energy of the solution ϱ (see (4.3)) is given by the formula

$$E(m_1, m_2, m_3) = \sum_{i=1}^{3} \Big(F_i(G_i(m_i)) - m_i \cdot G_i(m_i) \Big),$$

in particular, if $m_i \in (M_{Q_i}(k_i + \varepsilon_i), M_{Q_i}(k_i - \varepsilon_i))$ satisfies the hypothesis

$$E(m_1, m_2, m_3) = e \qquad \& \qquad m_3 = m - m_1 - m_2,$$

then there exists the solution of the system (1.1)–(1.3). Thus, it is sufficient to prove following Lemma 3:

Lemma 3. *There is continuum of pairs of numbers m_1, m_2, $m_i \in (M_{Q_i}(k_i + \varepsilon_i), M_{Q_i}(k_i - \varepsilon_i))$, $i = 1, 2$, such that*

$$e(m_1, m_2) - e = 0, \tag{4.4}$$

where the energy $e(m_1, m_2)$ is given by the formula

$$e(m_1, m_2) = F_3(G_3(m - m_1 - m_2)) - (m - m_1 - m_2) \cdot G_3(m - m_1 - m_2) + \sum_{i=1}^{2} \Big(F_i(G_i(m_i)) - m_i \cdot G_i(m_i) \Big).$$

Proof of Lemma 3. The function $e(m_1, m_2)$ is continuously differentiable and

$$\frac{\partial e}{\partial m_2}(M_{Q_1}(k_1), M_{Q_2}(k_2)) = k_3 - k_2 \neq 0$$

($k_2 \neq k_3$ because of the definition of k_2 and k_3). Thus we can use the standard implicit function theorem for equation $e(m_1, m_2) - e = 0$ in the point $[M_{Q_1}(k_1), M_{Q_2}(k_2)]$.

Q.E.D. {Lemma 3}

Thus, in the case (a), the proof is finished.

(b) $\hat{K} = \widetilde{K}$. Let us consider the function $M_\Omega(c) : [\hat{K}, \sup_\Omega F] \to \mathbb{R}$ given by the formula

$$M_\Omega(c) = \int_\Omega \left(\frac{\gamma - 1}{a\gamma} [F(x) - c]^+ \right)^{\frac{1}{\gamma-1}} \mathrm{d}x.$$

It is a continuous function, $M_\Omega(\hat{K}) = m_c$, $M_\Omega(\sup_\Omega F) = 0$, thus, there exists $s_1 \in (\hat{K}, \sup_\Omega F)$ such that $M_\Omega(s_1) = m$. By virtue of $s_1 > \hat{K}$, there exists $s_2 \in \mathbb{R}$, $\hat{K} < s_2 < s_1$, such that the open set $\mathcal{F}[s_2]$ has at least three components. Let Θ be one of the components of the open set $\mathcal{F}[s_2]$.

Now, we put $\underline{\Omega_1} = \Theta$, $\underline{\Omega_2} = \Omega - \Theta$, $\underline{\hat{K}} = \underline{\widetilde{K}} = \underline{\widetilde{K_1}} = \underline{\widetilde{K_2}} = s_2$, $\underline{m_c} = M_\Omega(s_2)$, $\underline{M_1} \equiv M_{\underline{\Omega_1}}$, $\underline{M_2} \equiv M_{\underline{\Omega_2}}$. Then $m \in (0, \underline{m_c})$,

$$\underline{m_c} = \underline{M_1}(\underline{\widetilde{K}}) + \underline{M_2}(\underline{\widetilde{K_2}}).$$

Hence, if we consider the underlined objects instead of the original ones, we are in the situation (4.1) and we can repeat our arguments used in the proof of the case (a).

Q.E.D.

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