

R M R -module $\Rightarrow \exists$ injective
module E and $M \hookrightarrow E$.

Sketch for $R = \mathbb{Z}$ [the general case
is outlined in notes

Def M , $m \in R$ is divisible by $r \in R$
if $m = r m'$ (for $m' \in M$)

M is divisible \Leftrightarrow every $m \in M$ is
divisible by a
non-zero divisor

Ex : \mathbb{Q} is a divisible \mathbb{Z} -module :

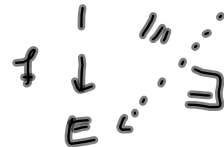
$$m \in \mathbb{Q}, z \in \mathbb{Z}, z \neq 0$$

$$m = z m' \quad m' = \frac{m}{z} \in \mathbb{Q}$$

4.14 [Recall (Baer) E injection

$$\Leftrightarrow 0 \rightarrow I \rightarrow R$$

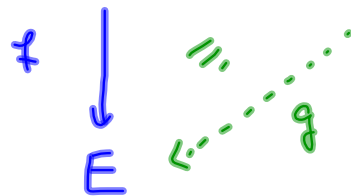
Let $m \in E$, $r_0 \in R$
 non-zero divisor.



(want $m = r_0 m'$ $m' \in E$)

$$0 \rightarrow I = Rr_0 \rightarrow R$$

$f(r_0) = r m$
 if $r_0 = 0$
 r_0 non-zero divisor
 $\Rightarrow r = 0$
 $\vee m = 0$



$$m = f(r_0) = g(r_0) = r_0 g(1)$$

$$m' = g(1) \in E \quad m = r_0 m' \quad \checkmark$$

4.16 $G = \mathbb{Z}$ -module.

$$G = F/S \quad F \text{ free} \approx \sum_j \mathbb{Z}$$

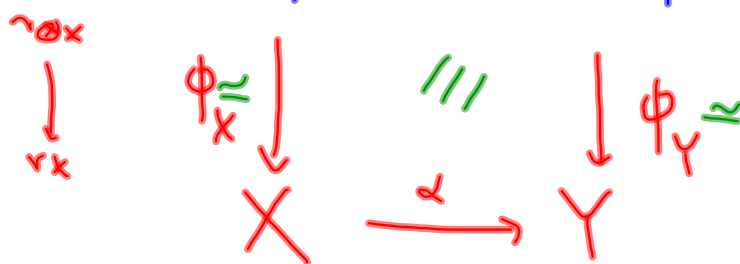
$$F \subset \sum_j \mathbb{Q}$$

$$G \subset \left(\sum_j \mathbb{Q} \right) / S$$

① divisible \implies Exercises $\left(\sum_j \mathbb{Q} \right) / S$ divisible
 \mathbb{Z} in ID, so by 4.15 \implies injective.

4.19 given $0 \rightarrow X \xrightarrow{\alpha} Y \quad X, Y \in R\text{-Mod}$

to show $R \otimes_R X \xrightarrow{1 \otimes \alpha} R \otimes Y$ is 1-1.



where $\phi : R \otimes_R (-) \rightarrow \text{Id}$ natural trans [sheet Ex 7]

natural isomorphs
 commutative square, vertical \cong
 bottom 1-1
 \Rightarrow top 1-1.

4.20 and 4.19

$\Rightarrow \sum_j R$ flat.

Direct summands of flat are flat \Rightarrow projective are flat.

free \Rightarrow projection
 \nLeftarrow in general

\mathbb{Z} PID \Leftarrow

Ex: $R = \mathbb{Z}$

\mathbb{Q} is not free (\Rightarrow not proj)

if free basis $\{x_j\}$

$x_j \neq x_k \quad \mathbb{Z}x_j \cap \mathbb{Z}x_k$
 should be zero

$x_j = \frac{p_j}{q_j} \quad x_k = \frac{p_k}{q_k}$

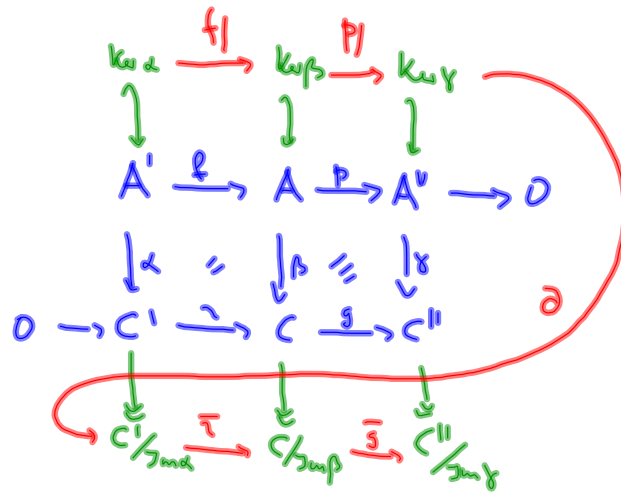
$\exists \mathbb{Z}x_j = \mathbb{Z}'x_k \neq 0$

proj \Rightarrow flat

~~\nLeftarrow~~ ex \mathbb{Q}

and $\mathbb{Z}x \subsetneq \mathbb{Q}$

see next sheet.



$$a'' \in k_{w\gamma} \quad \partial a'' := \gamma^{-1}(\beta p^{-1}(a'')) + \text{im } \alpha$$

Does this make sense?

$$a'' \in k_{w\gamma}, p \text{ onto, so } a'' = pa$$

$$\exists \beta a \in \text{im } \gamma = k_{w\gamma} \quad \checkmark$$

$$\gamma \beta a = \gamma p^{-1}(a'') = \gamma a'' = 0$$

$$\hookrightarrow \gamma a = 0$$

$$\beta a = \gamma^{-1}(0) \text{ for a unique } c' \in C'$$

$$\partial a'' := c' + \text{im } \alpha$$

Well defined: Let also $a'' = p(a_1)$

$$\beta(a_1) = \gamma^{-1}(c'_1)$$

$$p(a - a_1) = pa - pa_1 = a'' - a'' = 0$$

$$\text{So } a - a_1 = f(a') \quad a' \in A'$$

$$\beta(a - a_1) = \beta f(a') = \gamma^{-1}(c'_1)$$

$$= \gamma^{-1}(c' - c'_1) \quad \hookrightarrow$$

$$c' - c'_1 = \alpha(a')$$

$$c' + \text{im } \alpha = c'_1 + \text{im } \alpha \quad \checkmark$$