R M R-module $\Rightarrow$ $\exists$ injective module E and $M \hookrightarrow E$.

Sketch for $R = \mathbb{Z}$ [the general case is outlined in notes

Def M, $m \in R$ is divisible by $r \in R$ if $m = r m'$ (for $m' \in M$)

M is divisible $\Longleftrightarrow$ every $m \in M$ is divisible by a non-zero divisor

Ex: $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module:

$m \in \mathbb{Q}$, $z \in \mathbb{Z}$, $z \neq 0$

$m = z m'$ $\quad m' = \frac{m}{z} \in \mathbb{Q}$

4.14

[Recall (Baer) E injective

$$\Longleftrightarrow \quad \begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "="] & R \arrow[dl, dotted, "\exists"] \\ & E & \end{tikzcd}$$

Let $m \in E$, $r_0 \in R$ non-zero divisor.

(want $m = r_0 m'$ $m' \in E$)

$$\begin{tikzcd} 0 \arrow[r] & I = Rr_0 \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "="] & R \arrow[dl, dotted, "g"] \\ & E & \end{tikzcd}$$

$f(rr_0) := rm$

[if $rr_0 = 0$

r_0 non-zero divisor

$\Rightarrow r = 0$

$rm = 0$

$m = f(r_0) = g(r_0) = r_0 g(1)$

$m' = g(1) \in E \qquad m = r_0 m'$ ✓

<u>4.15</u> R PID

D = R-module, is injective $\Longleftrightarrow$ divisible.

$\Longrightarrow$ 4.14

"$\Longleftarrow$"

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, "f"'] & R \arrow[dl, dotted, "?\tilde{f}"] \\ & D & \end{tikzcd}$$

, I left ideal

wlog $I \neq 0$

$I = Rr_0$

$\Rightarrow$ r_0 non-zero divisor.

$f(r_0) = r_0 d$ (some $d \in D$)

Since D is divisible.

Define $\tilde{f}(r) := rd$, well-defined hom

$\tilde{f} \circ \iota (s r_0) = s \tilde{f}(r_0) = s r_0 d$

$f(sr_0) = sf(r_0) = sr_0 d$.

4.16 $G = \mathbb{Z}$-module.

$G = F/S$ F free $\simeq \sum_{J} \mathbb{Z}$

$F \subset \sum_{J} \mathbb{Q}$

$G \subset \left(\sum_{J} \mathbb{Q}\right)/S$

$\mathbb{Q}$ divisible $\underset{\text{exercises}}{\Longrightarrow}$ $\left(\sum_{J} \mathbb{Q}\right)/S$ divisible

$\mathbb{Z}$ is ID, so by 4.15 injective.

4.19 given $0 \longrightarrow X \xrightarrow{\alpha} Y$ $\quad X, Y \in R\text{-Mod}$

to show $R \otimes_R X \xrightarrow{1 \otimes \alpha} R \otimes Y$ is 1-1.

$r \otimes x \mapsto rx$

$$\begin{tikzcd}
R \otimes_R X \arrow[r, "1\otimes\alpha"] \arrow[d, "\phi_X \simeq"'] \arrow[dr, phantom, "///"] & R \otimes Y \arrow[d, "\phi_Y \simeq"] \\
X \arrow[r, "\alpha"] & Y
\end{tikzcd}$$

where $\Phi : R \otimes_R (-) \longrightarrow \mathrm{Id}$ natural tran

[sheet Ex 7]

natural isomorphism

commutative square, vertical $\simeq$

bottom 1-1

$\Rightarrow$ top 1-1.

4.20 and 4.19

$\Longrightarrow \sum_j R$ flat.

Direct summands of flat are flat $\Rightarrow$ projective are flat.

free $\Rightarrow$ projective
$\not\Leftarrow$ in general

R PID $\Leftarrow$

Ex: $R = \mathbb{Z}$

$\mathbb{Q}$ is <u>not</u> free ($\Rightarrow$ not proj)

if free basis $\{x_j\}$

$x_j \neq x_k \quad \mathbb{Z}x_j \cap \mathbb{Z}x_k$
should be zero

$x_j = p_j/q_j \quad x_k = p_k/q_k$

$\exists\, z x_j = z' x_k \neq 0$ ⚡ and $\mathbb{Z}x \subsetneq \mathbb{Q}$

proj $\Rightarrow$ flat
$\not\Leftarrow$ ex $\mathbb{Q}$ see next sheet.

$$
\begin{tikzcd}
 & \ker\alpha \arrow[r, "f|"] \arrow[d, hook] & \ker\beta \arrow[r, "p|"] \arrow[d, hook] & \ker\gamma \arrow[d, hook] \arrow[ddll, bend left, "\partial"] & \\
 & A' \arrow[r, "f"] \arrow[d, "\alpha"] & A \arrow[r, "p"] \arrow[d, "\beta"] & A'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & C' \arrow[r, "i"] \arrow[d, two heads] & C \arrow[r, "g"] \arrow[d, two heads] & C'' \arrow[d, two heads] & \\
 & C'/\operatorname{Im}\alpha \arrow[r, "\bar{i}"] & C/\operatorname{Im}\beta \arrow[r, "\bar{g}"] & C''/\operatorname{Im}\gamma &
\end{tikzcd}
$$

$a'' \in \ker\gamma \qquad \partial a'' := i^{-1}\beta p^{-1}(a'') + \operatorname{Im}\alpha$

Does this make sense?

$a'' \in \ker\gamma$, p onto, so $a'' = pa$

Is $\beta a \in \operatorname{im} i = \ker g$? ✓

$g\beta a = \gamma p(a) = \gamma a'' = 0$

$\beta a = i c'$ for a unique $c' \in C'$

$\partial a'' := c' + \operatorname{im}\alpha$

Well defined: Let also $a'' = p(a_1)$

$\beta(a_1) = i(c'_1)$

$p(a - a_1) = pa - pa_1 = a'' - a'' = 0$

So $a - a_1 = f(a')$ $\quad a' \in A'$

$\beta(a - a_1) = \beta f(a') = i\alpha(a')$

$= i(c' - c'_1)$

$c' - c'_1 = \alpha(a')$

$c' + \operatorname{im}\alpha = c'_1 + \operatorname{Im}\alpha$ ✓.