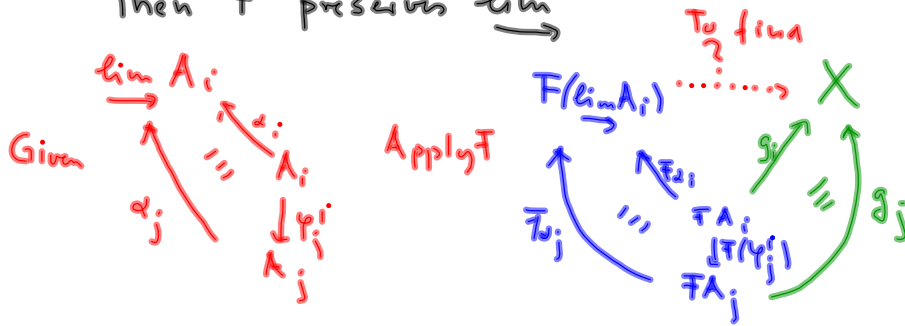
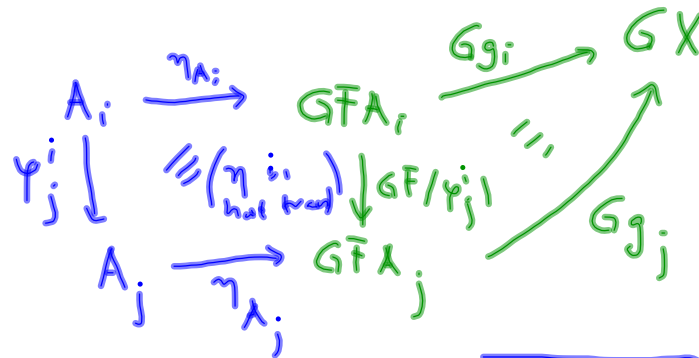


3.5  $F : \mathcal{A} \rightarrow \mathcal{C}$ ,  $F$  has right adjoint

Then  $F$  preserves  $\lim$



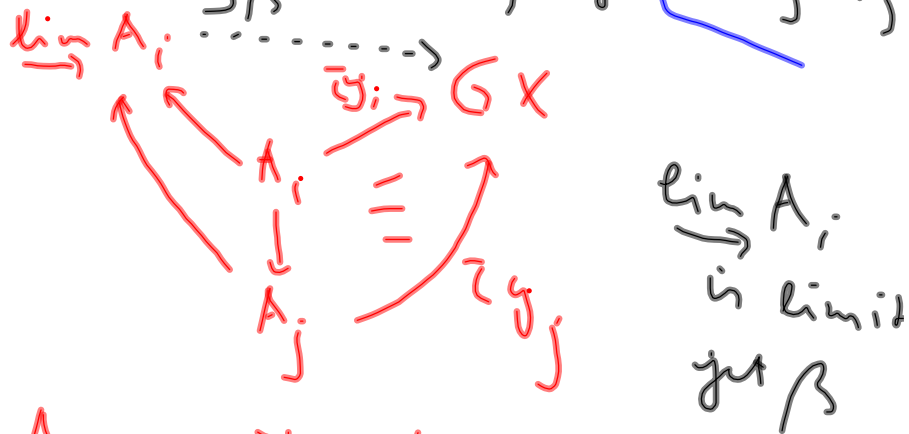
Apply  $G$ , use counit



From (1)

$$Gg_i \circ \eta_{A_i} = \tau(g_i)$$

$$\stackrel{||}{=} Gg_j \circ \eta_{A_i} \circ \varphi_j^i \stackrel{(1)}{=} \tau(g_j) \circ \varphi_j^i$$

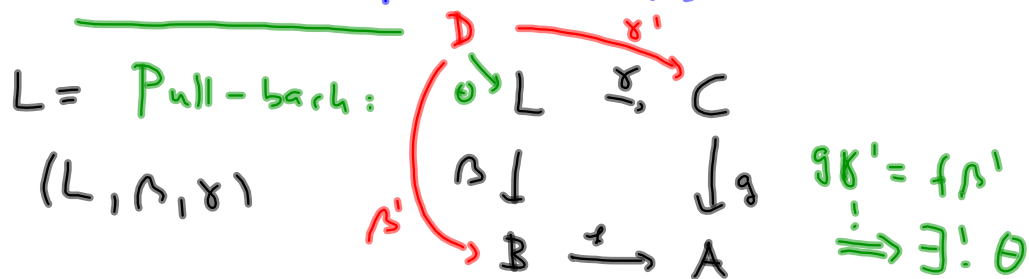


Apply  $\tau^{-1}, \tau^{-1}(\beta) : F(\lim A_i) \rightarrow X$

this will do

Cov:  $F = B \otimes_R (-)$  has right adjoint

$\Rightarrow$  preserves limits



$$\begin{aligned}
 g\gamma' &= f\beta' \\
 \Downarrow & \\
 \exists! \theta & \\
 \text{with} & \\
 \beta\theta &= \beta' \\
 \gamma\theta &= \gamma'
 \end{aligned}$$

If  $\exists$  then unique (no pushout proof)  
Existence (for modules)

Take  $\left\{ (b, c) \in B \oplus C : f(b) = g(c) \right\}$

$\beta(b, c) = b, \gamma(b, c) = c$   
Check.

$(I, \leq)$  quasi ordered

Inverse system = contravariant functor  $I \rightarrow \mathcal{C}$

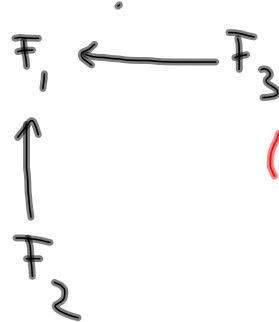
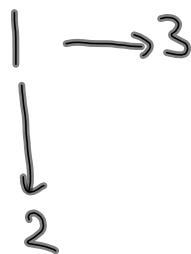
$\forall i, F_i \in \text{Ob } \mathcal{C}$

$i \leq j \quad \psi_i^j : F_j \rightarrow F_i$

$\psi_i^i = \text{id}$

$i \leq j \leq k : \psi_i^j \circ \psi_j^k = \psi_i^k$

Ex:  $I = \{1, 2, 3\} \quad 1 \leq 3, 1 \leq 2$

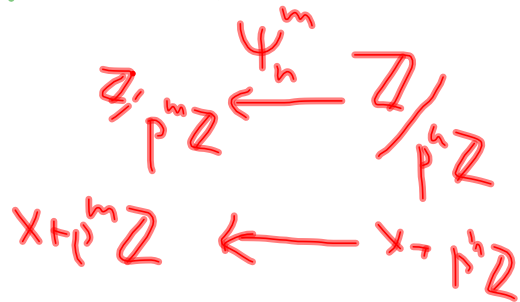


(see pull-back!)

Ex:  $I = \mathbb{N}$

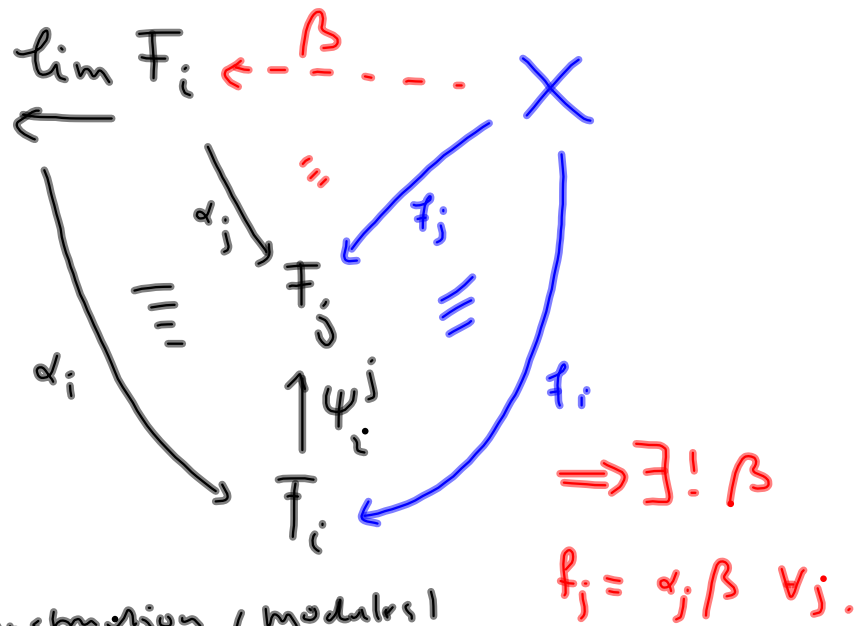


$F_n := \mathbb{Z}/p^n\mathbb{Z} \quad n \leq m$   
 ( $R = \mathbb{Z}$ )



check: inverse syst

Inverse limit



Construction, modules

$$\varprojlim F_i \cong \{ (a_i) \in \prod F_i : a_i = \psi_i^j a_j \text{ for } i \leq j \}$$

3.8 If  $(F, G)$  adjoint pair  
 $\Rightarrow G$  preserves  $\varprojlim$ .  
 [e.g.  $\text{Hom}_S(B, -)$ ]

$$R \text{ free } \Leftrightarrow F \simeq \sum F_j, \quad F_j \simeq R$$

$$F_j = Rx_j, \quad (rx_j = 0 \ (r \in R) \Leftrightarrow r = 0)$$

$X := \{x_j\}_{j \in J}$  in  $R$ -basis for  $F$ .

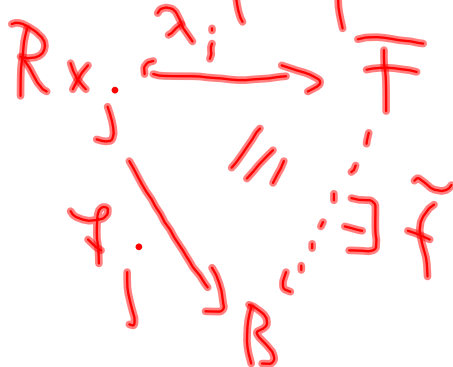
$$\forall x \in F \quad x = \sum r_j x_j \quad \text{unique} \\ (\text{almost all } r_j = 0)$$

Proof:  $F = \sum Rx_j$   $B$   $R$ -module

Given  $f: X \rightarrow B$  map for  $rx_j$

Define  $f_j: Rx_j \rightarrow B$  by  $f_j(rx_j) = rb_j$   
 $R$ -module hom. where  $b_j = f(x_j)$

By universal property for  $\sum Rx_j$





$$\text{Hom}_R(F, -)$$

$\text{Hom}(*, -)$  is always left exact

$F$  free  $\Rightarrow$  also right exact.

$$B \xrightarrow{\beta} C \rightarrow 0$$

want:  $\text{Hom}_R(F, B) \xrightarrow{\beta_*} \text{Hom}_R(F, C) \ni \alpha$   
onto.

$$F \xrightarrow{\alpha} C$$

By 4.3  $\alpha = \beta \circ g$  for some  $g: F \rightarrow B$   
 $= \beta_* (g)$

$\mathcal{P}$  projection :

