3.5 $F : \mathfrak{A} \longrightarrow \mathcal{C}$, F has right adjoint

Then F preserves $\varinjlim$

Given

$$\begin{tikzcd} & \varinjlim A_i \\ & & A_i \arrow[ul, "\alpha_i"'] \arrow[d, "\varphi^i_j"] \\ & & A_j \arrow[uul, bend left, "\alpha_j"] \end{tikzcd}$$

Apply F

$$\begin{tikzcd} F(\varinjlim A_i) \arrow[rr, dotted, "\text{To find ?}"] & & X \\ & FA_i \arrow[ul, "F\alpha_i"'] \arrow[ur, "g_i"] \arrow[d, "F(\varphi^i_j)"] \\ & FA_j \arrow[uul, bend left, "F\alpha_j"] \arrow[uur, bend right, "g_j"'] \end{tikzcd}$$

Apply G, use counit

$$\begin{tikzcd} A_i \arrow[r, "\eta_{A_i}"] \arrow[d, "\varphi^i_j"'] \arrow[dr, phantom, "\cong (\eta \text{ is nat trans})"] & GFA_i \arrow[r, "Gg_i"] \arrow[d, "GF(\varphi^i_j)"] & GX \\ A_j \arrow[r, "\eta_{A_j}"'] & GFA_j \arrow[ur, "Gg_j"'] \end{tikzcd}$$

From (1)

$$Gg_i \circ \eta_{A_i} = \tau(g_i)$$

$$\|$$

$$=\quad Gg_j \circ \eta_{A_j} \circ \varphi^i_j \overset{(1)}{=} \tau(g_j) \circ \varphi^i_j$$

$$\begin{tikzcd} \varinjlim A_i \arrow[rr, dotted, "\exists \beta"] & & GX \\ & A_i \arrow[ul] \arrow[ur, "\tau g_i"] \arrow[d] \\ & A_j \arrow[uul, bend left] \arrow[uur, bend right, "\tau g_j"'] \end{tikzcd}$$

$\varinjlim A_i$ is limit get β

Apply τ^{-1}, $\tau^{-1}(\beta) : F(\varinjlim A_i) \to X$

this will do

Cor: $F = B \otimes_R (-)$ has right ajoint

$\Rightarrow$ preserves limits.

$L =$ Pull-back: (L, β, γ)

$$\begin{tikzcd}
D \arrow[dr, "\theta"] \arrow[drr, bend left, "\gamma'"] \arrow[ddr, bend right, "\beta'"'] & & \\
& L \arrow[r, "\gamma"] \arrow[d, "\beta"'] & C \arrow[d, "g"] \\
& B \arrow[r, "f"] & A
\end{tikzcd}$$

$g\gamma' = f\beta' \Rightarrow \exists!\ \theta$ with $\beta\theta = \beta'$, $\gamma\theta = \gamma'$

If $\exists$ then unique (see pushout proof)

Existence (for modules)

Take $\{ (b, c) \in B \oplus C : f(b) = g(c) \}$

$\beta(b, c) = b$, $\gamma(b, c) = c$

Check.

$(I, \leq)$ quasi ordered

Inverse system = <u>contravariant</u> functor $I \longrightarrow \mathcal{C}$

$\forall i \,,\ F_i \in \mathrm{Ob}\,\mathcal{C}$

$i \leq j \qquad \psi_i^j : F_j \longrightarrow F_i$

$\psi_i^i = \mathrm{id}\,,$

$i \leq j \leq k \ : \ \psi_i^j \circ \psi_j^k = \psi_i^k$

<u>Ex</u>: $I = \{1, 2, 3\} \quad 1 \leq 3,\ 1 \leq 2$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 3 \\ 2 & \end{tikzcd} \quad \overset{F}{\rightsquigarrow} \quad \begin{tikzcd} F_1 & F_3 \arrow[l] \\ F_2 \arrow[u] & \end{tikzcd}$$

(see pull-back!)

<u>Ex:</u> $I = \mathbb{N} \qquad 1 \to 2 \to 3 \dots$

$F_n := \mathbb{Z}/p^n\mathbb{Z} \qquad n \leq m$

$(R = \mathbb{Z})$

$$\mathbb{Z}/p^m\mathbb{Z} \xleftarrow{\psi_n^m} \mathbb{Z}/p^n\mathbb{Z}$$

$$x + p^m\mathbb{Z} \longleftarrow x + p^n\mathbb{Z}$$

check: inverse syst

<u>Inverse limit</u>

$$\begin{tikzcd}
\varprojlim F_i \arrow[dr, "\alpha_j"'] \arrow[ddr, bend right, "\alpha_i"'] & & X \arrow[ll, dashed, "\beta"'] \arrow[dl, "f_j"] \arrow[ddl, bend left, "f_i"] \\
& F_j \arrow[ul, phantom, "="] \arrow[ur, phantom, "="] & \\
& F_i \arrow[u, "\psi_i^j"'] &
\end{tikzcd}$$

$\Rightarrow \exists!\, \beta$

$f_j = \alpha_j \beta \quad \forall j.$

Construction (modules)

$$\varprojlim F_i \cong \left\{ (a_i) \in \prod F_i \,:\, a_i = \psi_i^j a_j \text{ for } i \leq j \right\}$$

<u>3.8</u> If (F, G) adjoint pair $\Rightarrow$ G preserves $\varprojlim$.

[e.g. $\mathrm{Hom}_S(B, -)$]

R $\quad F$ free $\Longleftrightarrow$ $F \simeq \sum F_j$, $F_j \simeq R$

$F_j = Rx_j$, $\left(r x_j = 0 \ (r \in R) \Leftrightarrow r = 0 \right)$

$X := \{x_j\}_{j \in J}$ is R-basis for F.

$\forall\, x \in F \quad x = \sum r_j x_j$ unique
(almost all $r_j = 0$)

<u>Proof</u>: $F = \sum R x_j$ $\quad B$ R-module

Given $f : X \longrightarrow B$ map for sets

Define $f_j : Rx_j \longrightarrow B$ by $f_j(r x_j) = r b_j$
R-module hom. where $b_j = f(x_j)$

By universal property for $\sum R x_j$

$$\begin{tikzcd} Rx_j \arrow[r, "\lambda_i"] \arrow[dr, "f_j"'] & F \arrow[d, dotted, "\exists \tilde{f}"] \arrow[dl, phantom, "///"] \\ & B \end{tikzcd}$$

M R-module

$X = \{ x_m : m \in M \}$ set

F free with basis X

Define set map $f : X \to M$, $f(x_m) = m$

By 4.1, $\exists \tilde{f} : F \to M$ extending f
R-module map.

f onto $\Rightarrow \tilde{f}$ is onto.

4.2.1 M has free resolution:

By 4.2

$$\begin{tikzcd}
 & & F_0 \arrow[r, "d_0"] & M \arrow[r] & 0 \\
F_1 \arrow[urr, "d_1"] \arrow[r] & \ker d_0 \arrow[ur, hook, "\cong"] \arrow[r] & 0 & &
\end{tikzcd}$$

F_0 free

F_1 free

repeat

4.3

$$\begin{tikzcd}
 & F \arrow[d, "\alpha"] \arrow[dl, dashed, "\exists \gamma"'] & \\
B \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}$$

Proof: Let $X = \{x_j\}$ be R-basis of F

$\alpha(x_j) = \beta(b_j)$ for some $b_j \in B$

Define set map $f : X \to B$ $f(x_j) = b_j$

4.1 $\Rightarrow \gamma = \tilde{f} : F \to B$ R-module hom extending f.

$$\beta \circ \gamma \left(\sum r_j x_j\right) = \beta\left(\sum r_j \gamma(x_j)\right)$$
$$= \beta\left(\sum r_j f(x_j)\right)$$
$$= \beta\left(\sum r_j b_j\right) = \sum r_j \beta(b_j)$$
$$= \sum r_j \alpha(x_j)$$
$$= \alpha\left(\sum r_j x_j\right)$$

$\mathrm{Hom}_R(F,-)$

$\mathrm{Hom}(*,-)$ is always left exact

F free $\Rightarrow$ also right exact.

$$B \xrightarrow{\beta} C \longrightarrow 0$$

Want: $\mathrm{Hom}_R(F,B) \xrightarrow{\beta_*} \mathrm{Hom}_R(F,C) \ni \alpha$ onto.

$$F \xrightarrow{\alpha} C$$

By 4.3

$\alpha = \beta \circ g$ for some $g : F \to B$

$= \beta_*(g)$

P projection:

$$\begin{tikzcd} & P \arrow[d, "\alpha"] \arrow[dl, dotted, "\exists \gamma"'] & \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$