Proof 2.6 finished: $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$
exact

Want:

$$M \otimes_R A \xrightarrow{1\otimes\alpha} M \otimes_R B \xrightarrow{1\otimes\beta} M \otimes_R C \longrightarrow 0$$

is exact.

Done: $1\otimes\beta$ is onto.

Show now $\mathrm{Im}(1\otimes\alpha) = \ker(1\otimes\beta)$.

"$\subseteq$" $(1\otimes\beta)\circ(1\otimes\alpha): m\otimes a \longrightarrow m\otimes\alpha(a) \longrightarrow m\otimes\beta\alpha(a) = 0$, $\beta\alpha = 0$

"$=$" Let $E = \mathrm{Im}(1\otimes\alpha) \subseteq \ker(1\otimes\beta)$

Get

$$\begin{tikzcd} M\otimes_R B \arrow[rr, "1\otimes\beta"] \arrow[dr, "\pi"'] & & M\otimes_R C \\ & M\otimes_R B/E \arrow[ur, "\bar\beta"'] & \end{tikzcd}$$

$\bar\beta: m\otimes b + E \longrightarrow m\otimes\beta(b)$

check $\bar\beta$ is well. defined.

To show $\bar\beta$ is 1-1 [then: kernel of $\bar\beta$ is. $\ker(1\otimes\beta)/E$ is zero

i.e. $\ker(1\otimes\beta) = E$

Construct map

$$\gamma: M\otimes_R C \longrightarrow M\otimes_R B/E$$

To show: the following is well defined

$\gamma(m\otimes c) := m\otimes b + E$ where $\beta(b) = c$

Then $\gamma\circ\bar\beta(m\otimes b + E) =$

$$= \gamma(m\otimes\underbrace{\beta(b)}_{c}) = m\otimes b + E$$

$\gamma\circ\bar\beta = Id$

$\Longrightarrow \bar\beta$ is 1-1.

[Ex 2.7 Hint:

apply Hom$(C, -)$. Look at dimensions of Hom$(C, -)$'s.

Comment on abelian categories: postponed.

Motivation:

Linear algebra $(\ ,\ )$ inner product:

<u>adjoint</u> of linear maps

$$(Av, w) = (v, A^* w)$$

Characters: $H \leqslant G$

χ char of H $\quad (\chi\uparrow^G, \zeta)_G = (\chi, \zeta\downarrow_H)_H$

ζ char. of G

Can view $(\ ,\ )$ of characters

as dimensions of Hom spaces

$\rightsquigarrow$ generalize!

Thm 3.1 (c) $\;{}_RA \quad {}_SB_R \quad {}_SC$

$\mathrm{Hom}_S(B,C)$ is a left R-module:

$f: B \to C$, $r \in R$

$(rf)(b) := f(br)$ (check this works)

Before proving,
Special case: $R \subset S$ subring

Take $B = {}_SS_R$

$B \otimes_R A = S \otimes_R A =$ "A induced to S" $= A\uparrow^S$

$\mathrm{Hom}_S(B,C) = \mathrm{Hom}_S(S,C)$
this is C with action of R from $R \subset S$
i.e. "C restricted to R" $= C\downarrow_R$

3.1 becomes:

$\mathrm{Hom}_S(A\uparrow^S, C) \simeq \mathrm{Hom}_R(A, C\downarrow_R)$

Sometimes called "Frobenius Reciprocity".

Proof 3.1 a)

To define $\tau : \mathrm{Hom}_S(B \otimes_R A, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C))$

Let $f : B \otimes_R A \longrightarrow C$

Fix $a \in A$, define $f_a : B \to C$, $f_a(b) := f(b \otimes a)$

f_a is an S-module hom:

$$f_a(b_1+b_2) = f((b_1+b_2)\otimes a) = f(b_1 \otimes a + b_2 \otimes a) = f(b_1\otimes a) + f(b_2 \otimes a) = f_a(b_1) + f_a(b_2)$$

$$f_a(sb) = f(sb\otimes a) = f(s(b\otimes a)) = sf(b\otimes a) = sf_a(b)$$

Define $\bar{f} : A \longrightarrow \mathrm{Hom}_S(B,C)$

$a \longrightarrow f_a$

Check $\bar{f}$ is an R-module hom $\therefore \bar{f} \in RHS$

<u>Def</u>: $\tau(f) := \bar{f}$.

Check τ is an additive hom. i.e. $\overline{f_1+f_2} = \bar{f}_1 + \bar{f}_2$.

Want: τ is iso.

Construct inverse. $\xleftarrow{\ \phi\ }$

Let $g : A \longrightarrow \mathrm{Hom}_S(B,C)$

Def $g' : B \otimes_R A \longrightarrow C$ by

$$g'(b \otimes a) := g(a)(b)$$

check well-defined...

Define $\phi(g) = g'$

Check $\phi \circ \tau = \mathrm{id}$, $\tau \circ \phi = \mathrm{id}$

3.1 in terms of functors:

$F = B \otimes_R -$ $\qquad G = \mathrm{Hom}_S(B, -)$

3.1 becomes

$$\mathrm{Hom}_S(FA, C) \simeq \mathrm{Hom}_R(A, GC)$$

Looks formally same as "adjoint linear maps"

Two more examples: K field

$U: K\text{-Mod} \longrightarrow \mathcal{S}$ forgetful functor

v-spaces $\qquad$ sets

$\xleftarrow[F]{}$

X set, $F(X) :=$ the K-vector space with basis X

$X \xrightarrow{f} Y \quad F(X) \xrightarrow{F(f)} F(Y)$

the linear map defined on basis $x \to f(x) \in Y \subseteq F(Y)$

Can show

$$\tau : \mathrm{Hom}_K(FX, V) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}}(X, U(V))$$

$\cup$

$\alpha : FX \longrightarrow V$

$\tau(\alpha) : X \longrightarrow U(V)$

$\qquad\qquad\qquad\quad$ $= V$ as set

$x \longrightarrow \alpha(x)$

i.e. $\tau(\alpha) = \alpha\Big|_{\text{set } X}$

Ex

$$\mathcal{A}b \xrightarrow{U} \mathcal{G} \qquad U(A) = A$$

abelian groups — all groups

$$\xleftarrow[F]{}$$

$$G/G' \longleftarrow G$$

$$\exists \tau : \mathrm{Hom}_{Ab}(F(G), A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{G}}(G, U(A))$$

(adjoint)