

4. Free, projective, injective, flat modules

$R =$ ring.

DEF $F \in R - \text{Mod}$ is **free** $\Leftrightarrow F$ is isomorphic to a sum of copies of R . That is $F \cong \sum_{j \in J} F_j$ where $F_j \cong R$.

$F_j \cong R \Leftrightarrow F_j = Rx_j$ for some $x_j \in F_j$ with $rx_j = 0$ if and only if $r = 0$.

If so then $X := \{x_j : j \in J\}$ is called an R -basis of F . Every $x \in F$ has a **unique** expression $x = \sum r_i x_i$ with $r_i \in R$ and almost all $r_i = 0$.

Universal property:

4.1 Theorem *Let F be free with basis X . Given any module B and any function f from the the [set](#) X to B , there is a unique R -module homomorphism $\tilde{f} : F \rightarrow B$ extending f .*

Proof

4.2 Lemma *Every $M \in R - \text{Mod}$ is a homomorphic image of a free module.*

Proof Take F free, with basis indexed by all elements of M , say $X = \{x_m : m \in M\}$ (such F exists...).

Define $f : X \rightarrow M$ (a map on a set) by $f(x_m) := m$. This is already onto.

Take $\tilde{f} : F \rightarrow M$ be the R -homomorphism extending f .

DEF A **free resolution** of M is an exact sequence

$$\dots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$$

in which each F_i is a free R -module.

4.2.1 Every M has a free resolution:

4.3 Proposition Suppose $B \xrightarrow{\beta} C \rightarrow 0$ is exact. If F is free and $\alpha : F \rightarrow C$ is any R -module homomorphism then there exists $\gamma : F \rightarrow B$ with $\alpha = \beta\gamma$.

[γ not unique in general.]

Proof Let $X = \{x_j\}_{j \in J}$ be an R -basis for F .

For each j there is some $b_j \in B$ such that $\beta(b_j) = \alpha(x_j)$.

Then there is a map (on sets) $\phi : X \rightarrow B$ such that $\phi(x_j) = b_j$ [by axiom of choice]. By 4.1 there is an R -module homomorphism $\gamma : F \rightarrow B$ with $\gamma(x_j) = \phi(x_j)$ for all j .

$\alpha = \beta\gamma$: check on basis X .

4.4 Corollary F free \Rightarrow the functor $\text{Hom}_R(F, -)$ is exact.

Proof Left exact by 2.5.

To show right exact. Let $B \xrightarrow{\beta} C \rightarrow 0$ be exact, to show

$\beta_* : \text{Hom}(F, B) \rightarrow \text{Hom}(F, C)$ is onto.

Let $f \in \text{Hom}(F, C)$, by 4.3 there is $g \in \text{Hom}(F, B)$ with $\beta \circ g = f$. By definition $\beta \circ g = \beta_*(g)$.

Question For which modules P is $\text{Hom}_R(P, -)$ exact?

For which E is $\text{Hom}_R(-, E)$ exact?

Projective modules

DEF A module P is **projective** \Leftrightarrow for any surjection $\beta : B \rightarrow C$ and any $\alpha : P \rightarrow C$ there is $\gamma : P \rightarrow B$ such that $\alpha = \beta\gamma$.

Example Any free module is projective (by 4.3).

4.5 Theorem P is projective $\Leftrightarrow \text{Hom}(P, -)$ is exact.

Proof

4.6 Lemma Given a SES

$$0 \rightarrow A \xrightarrow{\iota} B \xrightarrow{\beta} C \rightarrow 0$$

and $\alpha : X \rightarrow C$. This can be completed to a commutative diagram with exact rows (with D the pull-back)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & D & \xrightarrow{p_X} & X & \longrightarrow & 0 \\ & & \text{Id} \downarrow & & p_B \downarrow & & \alpha \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

Proof

Exercise Formulate and prove the dual of this. [With push-out]

4.7 Lemma P is projective \Leftrightarrow every SES $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$ is split.

4.8 Theorem (a) Every summand of a projective is projective.
(b) P is projective $\Leftrightarrow P$ is isomorphic to a summand of a free module.

Proof of 4.8 (a)

(b) \Rightarrow By 4.2 there is a SES

$$0 \rightarrow A \xrightarrow{\iota} F \rightarrow P \rightarrow 0$$

with F free. By 4.7, this is split. By Lemma (Ch 2) $F \cong P \oplus A$.

\Leftarrow 4.3 and (a)

Examples (1) $R = KG$, where $G = \langle g \rangle$ order 2, and $\text{char}(K) \neq 2$. Then $1_R = e_0 + e_1$ where

$$e_0 = \frac{1}{2}(v_1 + v_g), \quad e_1 = \frac{1}{2}(v_1 - v_g)$$

Then $e_i^2 = e_i$ and $e_0e_1 = 0$. Therefore as R -modules

$$R = Re_0 \oplus Re_1.$$

Then Re_i is projective **but not free**. (Too small, the smallest projective module is R and $\dim R = 2$ but $\dim Re_i = 1$).

(2) $R = \mathbb{Z}/6\mathbb{Z}$, then (by 'Chinese Remainder Theorem')

$R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ as R -modules. So $\mathbb{Z}/2\mathbb{Z}$ is projective as R -module. Not free (too small).

4.9 Proposition Every module is the homomorphic image of a projective module.

Proof By 4.2, and since free \Rightarrow projective.

Injective modules

DEF An R -module E is **injective** if for every exact sequence

$$0 \rightarrow A \xrightarrow{\iota} B$$

and every $f : A \rightarrow E$ there is $g : B \rightarrow E$ such that $g\iota = f$.

4.10 Theorem E is injective \Leftrightarrow the functor $\text{Hom}(-, E)$ is exact.

[analogous to 4.5.]

To get examples, use Baer!

4.11 Theorem (Baer criterion) Let $E \in R - \text{Mod}$. Then E is injective \Leftrightarrow

- every homomorphism $f : I \rightarrow E$ for I a left ideal of R can be extended to $\tilde{f} : R \rightarrow E$.

Proof

Example Let R be a finite-dimensional algebra over some field K . Let P be a finite-dimensional **projective** right R -module. Then

$$D(P) := \text{Hom}_K(P, K)$$

is a left R -module, **It is injective.**

Proof (! thanks to Baer)

4.12 Proposition E is injective \Leftrightarrow
every SES $0 \rightarrow E \rightarrow B \rightarrow C \rightarrow 0$ is split.

Aim Every module is a submodule of some injective module.

Much more work than the 'dual' version for projective modules. We'll give a sketch.

4.13 Lemma A module E is injective if and only if every SES

$$0 \rightarrow E \rightarrow B \rightarrow C \rightarrow 0$$

with C cyclic, splits.

Proof

DEF Let M be some R -module, $m \in M$.

(a) m is **divisible** by $r \in R$ if $m = rm'$ for some $m' \in M$.

(b) M is **divisible** \Leftrightarrow every $m \in M$ is divisible by every non-zero-divisor in R .

Example $(\mathbb{Q}, +)$ is a divisible \mathbb{Z} -module.

4.14 Theorem Every injective module E is divisible.

Proof Let $m \in E$ and $r_0 \in R$ be a non-zero divisor. Define $f : Rr_0 \rightarrow E$ by

$$f(rr_0) := rm$$

(well-defined!) Since E injective $\Rightarrow \exists g : R \rightarrow E$ extending f . Hence $m = f(r_0) = g(r_0) = r_0g(1)$.

Exercise Show that

- (a) a quotient of a divisible module is divisible.
- (b) the sum of divisible modules is divisible.

4.15 Theorem Assume R is a PID. Then an R -module D is injective $\Leftrightarrow D$ is divisible.

Proof

4.16 Theorem Every \mathbb{Z} -module G (= 'abelian group') can be embedded into an injective \mathbb{Z} -module.

Proof G is a quotient of a free \mathbb{Z} module, say $G = F/S$ and $F = \sum_j \mathbb{Z}$. Embed each copy of \mathbb{Z} into \mathbb{Q} , then $G \subseteq (\sum_j \mathbb{Q})/S$.
Apply Exercise and the previous Theorem.

4.17 Theorem Suppose D is a divisible abelian group, and R any ring then $\text{Hom}_{\mathbb{Z}}(R, D)$ is an injective left R -module.

(The proof this needs adjoint functors. Exercise).

So eg $\text{Hom}_{\mathbb{Z}}(R, \mathbb{Q})$ is an injective left R -module.

4.18 Theorem Every $M \in R\text{-Mod}$ can be embedded into an injective module.

$\Rightarrow M$ has an injective resolution.

Proof View M as abelian group. By 4.16 we have a 1-1 \mathbb{Z} -module homomorphism, $\iota : M \rightarrow D$, with D an injective \mathbb{Z} -module.

For $m \in M$, define $f_m : R \rightarrow M$ by $r \mapsto rm$. Define $\varphi : M \rightarrow \text{Hom}_{\mathbb{Z}}(R, D)$ by

$$m \mapsto \iota \circ f_m$$

Check: This is an R -module homomorphism, and is 1-1.

Flat modules

DEF A right module $M \in \text{Mod} - R$ is **flat** if the functor $M \otimes_R (-)$ is exact. [Similarly for left modules]

4.19 Theorem R is flat (both as left and as right R -module).

Proof

4.20 Theorem Let $\{B_j : j \in J\}$ be a family of right R -modules. Then $\sum_j B_j$ is flat \Leftrightarrow each B_j is flat.

Proof

Corollary Every projective module is flat.