

Natural in each variable:

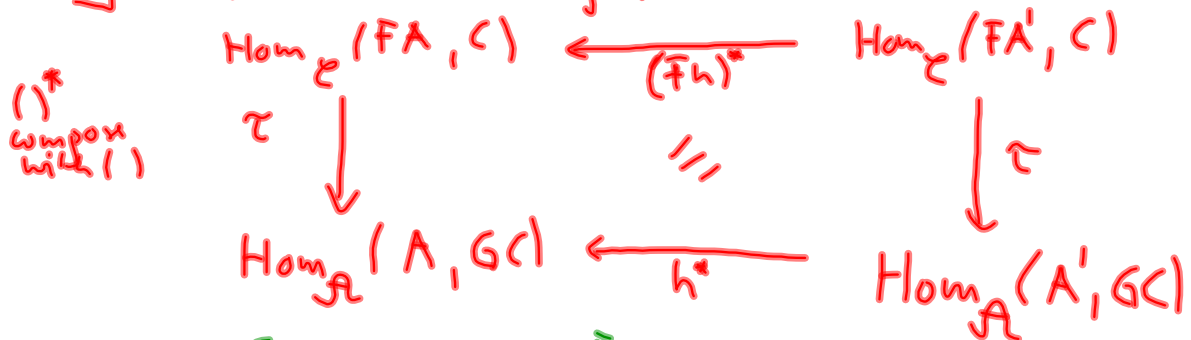
(i) fix C , $\tau_{-,C}$ is a natural transformation.

(ii) fix A , $\tau_{A,-}$ -||-

Spelling out (ii) $\text{Hom}_{\mathcal{C}}(-, C)$ is contravariant.

$$\forall h: A \rightarrow A' \quad Fh: FA \rightarrow FA'$$

\exists commutative diagram



Similarly for (i) [$\text{Hom}_{\mathcal{A}}(A, -)$ covariant]

To check these in Theorem 3.2
 \rightsquigarrow see problem sheet.

$$F : \mathcal{A} \longrightarrow \mathcal{C} \quad (F, G) \text{ adjoint}$$

$$\longleftarrow G$$

$$\tau : \text{Hom}_{\mathcal{C}}(FA, C) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}}(A, GC)$$

Define $\eta : \text{Id}_{\mathcal{A}} \longrightarrow G \circ F$

for $A \in \text{Ob } \mathcal{A}$, take $C = FA$

$$\eta_A := \tau(1_{FA}), A \longrightarrow GFA.$$

Show η is a natural transformation
called
"unit" of adjunction

EXAMPLE $F = B \otimes_R (-)$
 $G = \text{Hom}_S(B, -)$

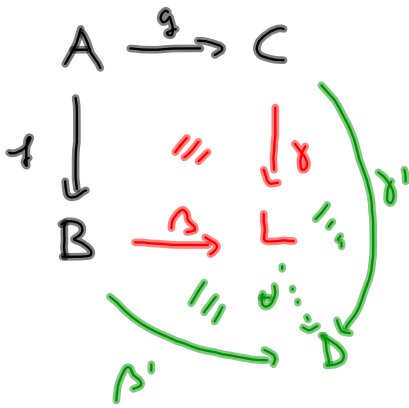
$$A \longrightarrow \text{Hom}_S(B, B \otimes_R A)$$

$$a \longrightarrow [b \longrightarrow ba]$$

Similar idea:

$$\text{get } \varepsilon : F \circ G \longrightarrow \text{Id}_{\mathcal{C}}$$

"counit".



pushout (f, g)
 $= (L, \beta, \gamma)$

such that $\beta f = \gamma g$

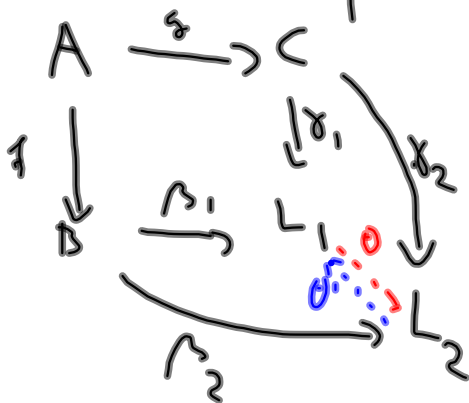
iii $\forall \beta': B \rightarrow D, \gamma': C \rightarrow D$

$\exists! \theta: L \rightarrow D$

such that $\theta \beta = \beta'$
 $\theta \gamma = \gamma'$

Tool to construct
 modules / groups, ...)

Proof uniqueness:



$\beta_i f = \gamma_i g$

L_1 in pushout

$\exists \theta: \theta \beta_1 = \beta_2 \quad \theta \gamma_1 = \gamma_2$

L_2 in pushout

$\exists \theta': L_2 \rightarrow L_1$
 $\theta' \beta_2 = \beta_1 \quad \theta' \gamma_2 = \gamma_1$

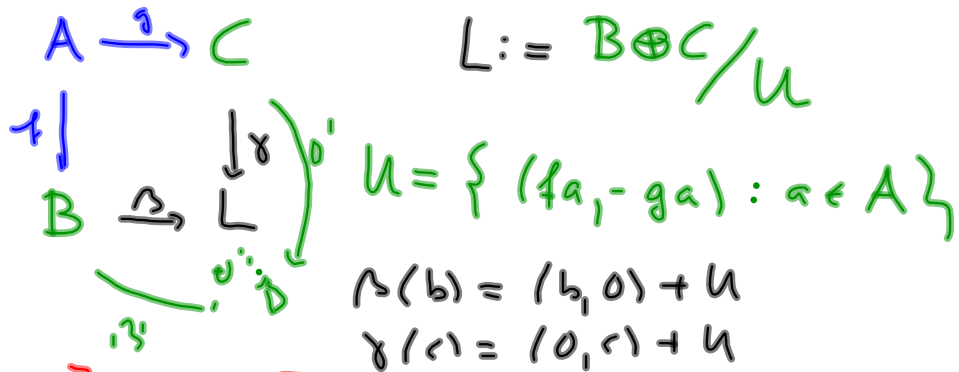
$$\beta_1 = \theta' \beta_2 = \theta' \theta \beta_1$$

$$\gamma_1 = \theta' \theta \gamma_1$$

L_1 in pushout, uniqueness. $\beta_1 = \theta' \beta_2$

Similarly, $\theta \theta' = \beta_2$

3.4 For modules, pushouts exist.



(i) \exists L_* = \exists ?

$$\begin{aligned}
 \beta \circ f(a) &= (fa, 0) + U & (fa, 0) - (0, ga) &= (fa, -ga) \in U \\
 \gamma \circ g(a) &= (0, ga) + U & & \\
 \therefore \beta \circ f(a) &= \gamma \circ g(a).
 \end{aligned}$$

(ii) Let $\beta' : B \rightarrow D$, $\gamma' : C \rightarrow D$
 and $\beta' \circ f = \gamma' \circ g$

Define $\theta : L \rightarrow D$

$$\theta((b, c) + U) := \beta'(b) + \gamma'(c)$$

Well-defined?

$$\text{If } (b, c) = (fa, -ga)$$

$$\Rightarrow \beta'(fa) + \gamma'(-ga) = (\beta'f - \gamma'g)(a) = 0$$

Check hom.

$$\begin{aligned}
 \exists \beta' = \theta \circ \beta? & \quad \theta(\beta(b)) = \theta((b, 0) + U) \\
 & = \beta'(b) + 0 \checkmark
 \end{aligned}$$

Unique - exercise.

Follows Rotman.

[Other versions in the literature. Rotman easiest]

(I, \leq) quasi-ordered: \leq transitive and reflexive

View as category: \mathcal{I}
 Objects I $\text{Hom}_{\mathcal{I}}(x, y) = \begin{cases} \{i_y^x\} & \text{if } x \leq y \\ \emptyset & \text{else.} \end{cases}$

Direct system :=

a functor $\bar{F}: \mathcal{I} \rightarrow \mathcal{C}$

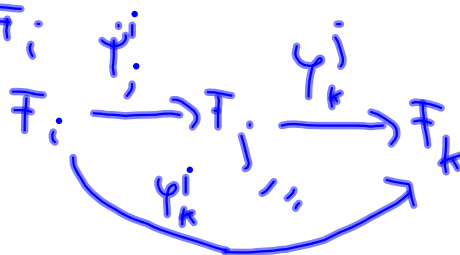
i.e. $\forall i \in I, \bar{F}_i \in \mathcal{C}$

if $i \leq j$, have $\varphi_j^i: \bar{F}_i \rightarrow \bar{F}_j$

$$\varphi_i^i = \text{id}_{\bar{F}_i}$$

$i \leq j \leq k:$

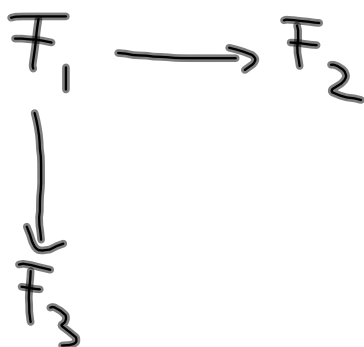
$$\varphi_k^i = \varphi_k^j \circ \varphi_j^i$$

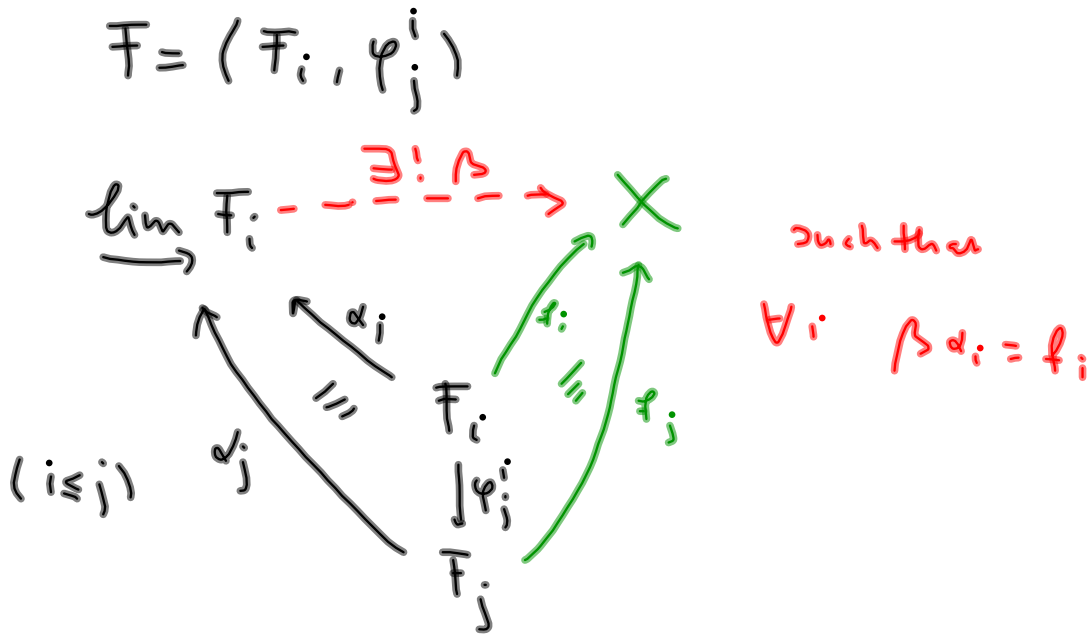


Ex (1) I with $i \leq j \iff i = j$
direct system is just a family of [modules]
.....

(2) $I = \{1, 2, 3\}$ $1 < 2$, $1 < 3$

A direct system is a diagram





Thm 3.4: \exists (sketch) for modules:

$$\sum_j F_j = \{ (a_j) : \text{almost all } a_i = 0 \}$$

$U =$ submodule generated by $(0, \dots, 0, a_i, \dots, -\varphi_j^i(a_i), 0, \dots)$
 $i \leq j$

Show $\sum_j F_j / U \cong \varinjlim F_i$

analogous to pushout.

$R_k : \varinjlim F_i$ very large

eg $F_i = 1\text{-dim} \in K\text{-Mod}$
field

(I, \leq) trivial quasi order

then $\varinjlim \bar{F}_i = \sum \bar{F}_i$

∞ dim,

all sequences with

\leq finite # nonzero terms.

$R\text{-Mod}$ has \varinjlim by 3.4

but $R\text{-mod}$ does not.