Natural in each variable:

(i) fix C, $\tau_{-,C}$ is a natural transformation.

(ii) fix A, $\tau_{A,-}$ —"— .

Spelling out (i) $\mathrm{Hom}_{\mathcal{C}}(-,C)$ is contravariant.

$\forall\ h : A \longrightarrow A'$ $\quad Fh : FA \longrightarrow FA'$

$\exists$ commutative diagram

$( )^*$ compose with $( )$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(FA,C) \arrow[d, "\tau"'] & \mathrm{Hom}_{\mathcal{C}}(FA',C) \arrow[l, "(Fh)^*"'] \arrow[d, "\tau"] \\
\mathrm{Hom}_{\mathcal{A}}(A,GC) & \mathrm{Hom}_{\mathcal{A}}(A',GC) \arrow[l, "h^*"]
\end{tikzcd}$$

Similarly for (ii) [$\mathrm{Hom}(A,-)$ covariant]

To check these in Theorem 3.2

$\rightsquigarrow$ see problem sheet.

$$F : \mathcal{A} \underset{G}{\overset{}{\rightleftarrows}} \mathcal{C} \qquad (F, G) \text{ adjoint}$$

$$\tau : \mathrm{Hom}_{\mathcal{C}}(FA, \overset{FA}{C}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(A, \overset{GFA}{GC})$$

Define $\eta : \mathrm{Id}_{\mathcal{A}} \longrightarrow G \circ F$

for $A \in \mathrm{Ob}\,\mathcal{A}$, take $C = FA$

$$\eta_A := \tau(1_{FA}), \; A \longrightarrow GFA.$$

Show η is a natural transformation

Called "unit" of adjunction

EXAMPLE $\quad F = B \otimes_R (\;)$
$\quad G = \mathrm{Hom}_S(B, -)$

$A \longrightarrow \mathrm{Hom}_S(B, B \otimes_R A)$

$a \longrightarrow [\, b \longrightarrow b \otimes a \,]$

Similar idea:
get $\varepsilon : F \circ G \longrightarrow \mathrm{Id}_{\mathcal{C}}$
"counit".

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & C \arrow[d, "\gamma"] \arrow[ddr, bend left, "\gamma'"] & \\
B \arrow[r, "\beta"] \arrow[drr, bend right, "\beta'"'] & L \arrow[dr, dotted, "\theta"] & \\
& & D
\end{tikzcd}$$

pushout (f, g)

$= (L, \beta, \gamma)$

such that (i) $\beta f = \gamma g$

(ii) $\forall \beta' : B \to D, \gamma' : C \to D$

$\exists ! \ \theta : L \to D$

such that $\theta \beta = \beta'$

$\theta \gamma = \gamma'$

Tool to construct modules (groups, ...)

Proof uniqueness:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\gamma_1"] \arrow[ddr, bend left, "\gamma_2"] & \\
B \arrow[r, "\beta_1"] \arrow[drr, bend right, "\beta_2"'] & L_1 \arrow[dr, dotted, shift left, "\theta"] & \\
& & L_2 \arrow[ul, dotted, shift left, "\theta'"]
\end{tikzcd}$$

$\beta_i f = \gamma_i f$

L_1 is pushout

$\exists \theta : \theta \beta_1 = \beta_2 \quad \theta \gamma_1 = \gamma_2$

L_2 is pushout

$\exists \theta' : L_2 \to L_1$

$\theta' \beta_2 = \beta_1 \quad \theta' \gamma_2 = \gamma_1$

$$1_{L_1} \beta_1 = \theta' \beta_2 = \theta' \theta \beta_1$$

$$1_{L_1} \gamma_1 = \theta' \theta \gamma_1$$

L_1 is pushout, use uniqueness: $1_{L_1} = \theta' \theta$

Similarly, $\theta \theta' = 1_{L_2}$.

3.4 For modules, pushouts exist.

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\gamma"] \arrow[ddr, bend left, "\gamma'"] & \\
B \arrow[r, "\beta"] \arrow[drr, bend right, "\beta'"'] & L \arrow[dr, dotted, "\theta"] & \\
 & & D
\end{tikzcd}$$

$L := B \oplus C / U$

$U = \{ (fa, -ga) : a \in A \}$

$\beta(b) = (b, 0) + U$

$\gamma(c) = (0, c) + U$

(i) Is $L_{\to} = \rfloor$?

$\beta \circ f(a) = (f(a), 0) + U$

$\gamma \circ g(a) = (0, g(a)) + U$

$(f(a), 0) - (0, g(a)) = (f(a), -g(a)) \in U$

$\therefore \beta \circ f(a) = \gamma \circ g(a)$.

(ii) Let $\beta' : B \longrightarrow D$, $\gamma' : C \longrightarrow D$ and $\beta' \circ f = \gamma' \circ g$

Define $\theta : L \longrightarrow D$

$$\theta((b, c) + U) := \beta'(b) + \gamma'(c)$$

Well-defined?

If $(b, c) = (fa, -ga)$

$\Rightarrow \beta'(fa) + \gamma'(-ga) = (\beta' f - \gamma' g)(a) = 0$

Check hom.

Is $\beta' = \theta \circ \beta$? $\theta\beta(b) = \theta((b, 0) + U) = \beta'(b) + 0$ ✓.

& Uniqueness - exercise.

Follows Rotman.

[Other versions in the literature. Rotman easiest]

$(I, \leq)$ quasi-ordered: $\leq$ transitive and reflexive

View as category: $\tilde{I}$

Objects I $\quad \mathrm{Hom}_{\tilde{I}}(x,y) = \begin{cases} \{ i^x_y \} & \text{if } x \leq y \\ \phi & \text{else.} \end{cases}$

Direct system :=

a functor $F : \tilde{I} \to \mathcal{C}$

i.e. $\forall i \in I$, $F_i \in \mathcal{C}$

if $i \leq j$, have $\varphi^i_j : F_i \to F_j$

$\varphi^i_i = id_{F_i}$

$i \leq j \leq k$:

$$\begin{tikzcd} F_i \arrow[r, "\varphi^i_j"] \arrow[rr, bend right, "\varphi^i_k"'] & F_j \arrow[r, "\varphi^j_k"] & F_k \end{tikzcd}$$

$\varphi^i_k = \varphi^j_k \circ \varphi^i_j$

Ex (1) I with $i \le j \iff i = j$

direct system is just a family of [modules ....]

(2) $I = \{1, 2, 3\}$ $1 < 2$, $1 < 3$

A direct system is a diagram

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & F_2 \\ F_3 & \end{tikzcd}$$

$\mathcal{F} = (F_i, \varphi^i_j)$

$$\begin{tikzcd}
\varinjlim F_i \arrow[rr, dashed, "\exists!\,\beta"] & & X \\
& F_i \arrow[ul, "\alpha_i"'] \arrow[ur, "f_i"] \arrow[d, "\varphi^i_j"] & \\
& F_j \arrow[uul, "\alpha_j"] \arrow[uur, "f_j"'] &
\end{tikzcd}$$

$(i \le j)$

such that $\forall i \quad \beta \alpha_i = f_i$

Thm 3.4: $\exists$ (sketch) for modules:

$$\sum_j F_j = \{ (a_j) : \text{almost all } a_i = 0 \}$$

$\cup$

U = submodule generated by $(0, \dots 0\ a_i, \dots -\varphi^i_j(a_i), 0 \dots)$
$i \le j$

Show $\sum_j F_j / U \simeq \varinjlim F_i$

analogous to pushout.

Rk: $\varinjlim F_i$ very large

eg F_i = 1-dim $\in$ K-Mod
field

$(I, \leq)$ trivial quasiorder

then $\varinjlim F_i = \sum F_i$

∞ dim, all sequences with $\leq$ finite # nonzero terms.

R-Mod has $\varinjlim$ by 3.4
but R-mod does not.