Analysis I Sheet 1 MT05

Real numbers, arithmetic, ordering, (real) absolute value

1 Let $a, b, c \in \mathbb{R}$. Prove that

(i) If $ab = ac$ and $a \neq 0$, then $b = c$;
(ii) If $ab = 0$, then $a = 0$ or $b = 0$;
(iii) $-(a + b) = (-a) + (-b)$;
(iv) $a \geqslant b$ if and only if $a + c \geqslant b + c$;
(v) If $a < b$ and $c < 0$, then $ac > bc$.

[*You may use the axioms for the real numbers as stated in lectures or in the book of Bartle and Sherbert, and, as always, you may use anything which has been* proved *in lectures.*]

2 Let $a, b \in \mathbb{R}$. Prove that

(i) $||a| - |b|| \leqslant |a - b|$;
(ii) $|ab| = |a|\,|b|$;
(iii) $\max(a, b) = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$.

Give a geometrical interpretation of (iii).

[*You may use any properties of the absolute value discussed in lectures, but you may not use square roots because their existence has yet to be proved.*]

3 Find all $x \in \mathbb{R}$ such that $4 < |x + 2| + |x - 1| < 5$.

4 For each of the following conditions, find and sketch the set of all pairs $(x, y) \in \mathbb{R}^2$ satisfying the condition:

(i) $|x| + |y| = 1$;
(ii) $|x| - |y| \geqslant 2$;
(iii) $1 < |xy| \leqslant 2$.

5 By the end of term we will be able to prove that there is a real number e with the property that

$$0 < n!\left(\mathrm{e} - \sum_{r=0}^{n} \frac{1}{r!}\right) < \frac{1}{n}$$

for all integers $n \geqslant 1$. Assuming only this property of e, prove that e is irrational.

[*Hint: Assume on the contrary that* $\mathrm{e} = p/q$; *choose a suitable value for* n *and deduce that there would be an integer* k *such that* $0 < k < 1$.]