Analysis I Sheet 7 MT05

Absolute convergence, power series

1 Give either a proof, or a counterexample to, each of the following assertions:

(i) If $n^2 a_n \to 0$ as $n \to \infty$ then $\sum a_n$ converges.
(ii) If $n a_n \to 0$ as $n \to \infty$ then $\sum a_n$ converges.
(iii) If $\sum a_n$ converges then $\sum a_n^2$ converges.
(iv) If $\sum a_n$ converges absolutely then $\sum a_n^2$ converges.
(v) If $\sum a_n$ converges absolutely then $|a_n| < 1/n$ for all sufficiently large n.

2 Find the radius of convergence R of each of the following power series.

(i) $\sum n^{2005} x^n$, (ii) $\sum x^{\binom{n}{2}}$, (ii) $\sum \dfrac{x^n}{2^n n^4}$, (iii) $\sum 2^n x^{n!}$, (iv) $\sum \dfrac{(nx)^n}{n!}$.

Optional *extension:* If you can, determine whether or not the series converges for $x = R$ and for $x = -R$.

3 (a) Write down the sum of the power series $\sum_{n=0}^{\infty} x^n$ valid for $|x| < 1$. By multiplying the series by itself 'term-by-term' find the sum of $\sum_{n=1}^{\infty}(n+1)x^n$

(b) Prove that $\sum_{k=0}^{k=n} \binom{k}{r} = \binom{n+1}{r+1}$.

[*The convention is that* $\binom{k}{r} = 0$ *when* $k < r$. *An easy way to prove this identity is to count the number of ways we choose* r + 1 *from* n + 1 *by first choosing the largest one,* $k+1$ *say, then choosing* r *smaller ones.*]

(c) Prove that for all $|x| < 1$, $(1-x)^{-k} = \sum_{n=0}^{\infty} \binom{n+k}{n} x^n$.

(d) Evaluate (for each n, k) the finite sum $\sum_{r=0}^{n} (-1)^r \binom{k}{r}\binom{n-r+k}{n-r}$

4 We have seen that $\cos x$ and $\sin x$ have power series expansions valid for all x. The function $\sec x + \tan x$ has a power series expansion $\sum_{n=0}^{\infty} a_n \dfrac{x^n}{n!}$, valid for $|x| < \pi/2$. Find the coefficients a_n for $n \leqslant 7$.

[*You may wish to take power series expansions in the identity* $(\sec x + \tan x)\cos x = 1 + \sin x$ *and use the theorem on multipication of series.*]

5 Let ω be the complex number $\mathrm{e}^{2\pi\mathrm{i}/3}$. Verify that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

Find a power series expansion for $\mathrm{e}^z + \mathrm{e}^{\omega z} + \mathrm{e}^{\omega^2 z}$.

Evaluate

$$\sum_{n=0}^{\infty} \frac{8^n}{(3n)!}, \qquad \sum_{n=0}^{\infty} \frac{27^n}{(3n+1)!}.$$

Are your answers real numbers?