Linear Algebra 10: Inner product spaces, II

Monday 21 November 2005

Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Gram–Schmidt Process (from last time)
- A worked example: FHS 1999, Paper a1, Qn 3
- Complex inner product spaces
- Uniqueness of finite-dimensional inner product space

FHS 1999, Paper a1, Question 3

Let V be a finite-dimensional vector space over $\mathbb{R}$. Define what is meant by saying that V *has an inner product* $\langle\ ,\ \rangle$ *over* $\mathbb{R}$.

Let $\{b_1, b_2, \ldots, b_n\}$ be a basis for V. Prove that there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of V such that for each $k = 1, 2, \ldots, n$ the set $\{e_1, e_2, \ldots, e_k\}$ is a basis for the subspace spanned by $\{b_1, b_2, \ldots, b_k\}$. Deduce that there are $t_{ij} \in \mathbb{R}$ with $t_{ii} \neq 0$ such that

$$b_1 = t_{11}e_1\,, \quad b_2 = t_{12}e_1 + t_{22}e_2\,, \quad \ldots\,, \quad b_k = t_{1k}e_1 + \cdots + t_{kk}e_k\,, \quad \ldots$$

(for $1 \leqslant k \leqslant n$). Show that $\langle b_i, b_j\rangle = \sum_{k=1}^{n}\langle b_i, e_k\rangle\langle b_j, e_k\rangle$ (for $1 \leqslant i, j \leqslant n$).

Now let G be the $n \times n$ matrix with (i,j)th entry equal to $\langle b_i, b_j\rangle$. Show that $\operatorname{Det} G = \prod_{i=1}^{n} t_{ii}^2$ and deduce that G is non-singular. Show also that $\operatorname{Det} G \leqslant \prod_{i=1}^{n}\langle b_i, b_i\rangle$.

Complex inner product spaces

Definition: Let V be a vector space over $\mathbb{C}$. An inner product on V is a function $B : V \times V \to \mathbb{C}$ such that for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{C}$

C(1) $B(\alpha u + \beta v, w) = \alpha B(u, w) + \beta B(v, w)$

C(2) $B(u, v) = \overline{B(v, u)}$

C(3) if $u \neq 0$ then $B(u, u) > 0$ [B is positive definite]

Note: A complex inner product is sesquilinear: $B(u, \alpha v + \beta w) = \bar{\alpha} B(u, v) + \bar{\beta} B(u, w)$.

Note. Often we find $\langle u, v \rangle$ used for inner products. Often they are called Hermitian forms.

Note. As in the real case we define $||u|| := \langle u, u \rangle^{\frac{1}{2}}$.

Examples of complex inner product spaces

Example 1: $V = \mathbb{C}^n$ and $\langle u, v \rangle = u^{\mathsf{tr}}\overline{v}$.

Example 2: $V = \{f : [0,1] \to \mathbb{C} \mid f$ is continuous$\}$ and

$$\langle u, v \rangle = \int_0^1 f(t)\overline{g(t)}\,\mathrm{d}t\,.$$

Orthogonality in complex inner product spaces

Let V be a complex inner product space.

Orthogonality is defined just as in real inner product spaces.

A finite dimensional complex inner product space has an orthonormal basis.

If $U \leqslant V$ then $V = U \oplus U^{\perp}$.

The Gram–Schmidt orthogonalisation process manufactures an orthonormal set from a linearly independent set in the same way as in the real case.

Uniqueness of inner product spaces

Observation. Let V be a real inner product space of dimension n. Then there is an isomorphism (bijective linear transformation) $\varphi : V \to \mathbb{R}^n$ such that if $\varphi\, u = (x_1, \ldots, x_n)$ and $\varphi\, v = (y_1, \ldots, y_n)$ then $\langle u, v \rangle = \sum x_i y_i$.

We say that V is isometric with $\mathbb{R}^n$.

Observation. Let V be a complex inner product space of dimension n. Then there is an isomorphism $\varphi : V \to \mathbb{C}^n$ such that if $\varphi\, u = (x_1, \ldots, x_n)$ and $\varphi\, v = (y_1, \ldots, y_n)$ then $\langle u, v \rangle = \sum x_i \overline{y_i}$.

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