# Linear Algebra 14: Diagonalisability of self-adjoint linear transformations

Thursday 1 December 2005

Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Diagonalisability Version 1
- Diagonalisability Version 2
- Self-adjoint projection operators
- Diagonalisability Version 3
- An example
- The future

**Throughout the lecture** $V$ is a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$ and $T : V \to V$ is self-adjoint linear.

## Diagonalisability, Version 1

Theorem. *There is an orthonormal basis of $V$ consisting of eigenvectors of $T$.*

Proof.

# Diagonalisability, Version 2

Theorem. (1) *If* $A \in \mathsf{M}_{n\times n}(\mathbb{R})$ *and* $A^{\mathsf{tr}} = A$ *then there exists* $U \in \mathsf{O}(n)$ *such that* $U^{-1}AU = D$ *where* $D$ *is a diagonal matrix.* (*Recall:* $U \in \mathsf{O}(n)$ *means that* $U^{-1} = U^{\mathsf{tr}}$ *.*)

(2) *If* $A \in \mathsf{M}_{n\times n}(\mathbb{C})$ *and* $\bar{A}^{\mathsf{tr}} = A$ *then there exists* $U \in \mathsf{U}(n)$ *such that* $U^{-1}AU = D$ *where* $D$ *is a diagonal matrix with real coefficients.* (*Recall:* $U \in \mathsf{U}(n)$ *means that* $U^{-1} = \bar{U}^{\mathsf{tr}}$ *.*)

## Self-adjoint projection operators

Lemma. *Let* $P : V \to V$ *be a projection operator* (*idempotent*). *Then* $P$ *is self-adjoint if and only if* $\operatorname{Im} P = (\operatorname{Ker} P)^{\perp}$.

Proof.

## Diagonalisability, Version 3

The Spectral Theorem. *If the distinct eigenvalues of $T$ are $\lambda_1, \ldots, \lambda_k$, then there are uniquely determined self-adjoint projection operators $P_i : V \to V$ for $1 \leqslant i \leqslant k$ such that $P_i P_j = 0$ whenever $i \neq j$ and*

$$\sum_i P_i = I \quad \textit{and} \quad T = \sum_i \lambda_i P_i \,.$$

Note that $P_1, \ldots, P_k$ is a partition of the identity (see Lect. 2).

# Schools 1999, Paper a1, Question 4

Let $V$ be a finite-dimensional complex inner product space. Let $A$ be a linear transformation of $V$; define the *adjoint* $A^*$ of $A$ and show that it is unique. Show also that $\mathrm{Ker}\,(A) = \mathrm{Ker}\,(A^*A)$. If $A$, $B$ are linear transformations of $V$ show that $(AB)^* = B^*A^*$.

What does it mean to say that $S$ is *self-adjoint*? Suppose $S$ is self-adjoint. Show that

$$(tr(S))^2 \leqslant r(S)tr(S^2)\,,$$

where $r$ denotes the rank, and $tr$ denotes the trace of a linear transformation of $V$. [You may use without proof that a self-adjoint linear transformation has only real eigenvalues and that there exists an orthonormal basis of eigenvectors of $V$.]

Deduce that for an arbitrary linear transformation $A$

$$(tr(A^*A))^2 \leqslant r(A)tr((A^*A)^2)\,.$$

## The future

The syllabus and synopsis have both been covered. Christmas has been in the shops for weeks.

**So tomorrow's lecture is cancelled.**

Instead, students who wish to discuss any aspect of my two algebra courses this term (or, indeed, anything of interest) are cordially invited to meet me at the usual time 3pm–4pm in my room in Queen's College (ask the lodge porter for directions).

To others—to ALL—I wish a productive vacation and a very happy holiday season.

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