Oxford University Department of Mathematics

Part A Further Linear Algebra, MT 2005: Exercise Sheet 4

Adjoints of linear transformations (on an inner-product space); eigenvalues, eigenvectors and diagonalisation of self-adjoint linear transformations. [See for example KAYE AND WILSON, *Linear Algebra*, Chapter 13. Alternative sources are: HALMOS, *Finite-dimensional vector spaces*, (second edition), §§70–79; HERSTEIN, *Topics in Algebra* (second edition), Ch. 6, §10; COHN, *Algebra*, Vol. 1 (recently re-published as *Classic algebra*), Ch. 8; and many other books.]

Note: the following five problems are offered to give focus to tutorials. The wise and intelligent student will be trying many other exercises, however, from books, past examination papers, and other such sources.

1. Prove that a real symmetric $n \times n$ matrix A has n linearly independent real eigenvectors. Deduce that there is an orthogonal matrix P such that $P^{\mathrm{tr}}AP$ is diagonal. Show further that if all the eigenvalues of A are positive then there exists an invertible $n \times n$ matrix T such that $T^{\mathrm{tr}}AT = I_n$.

2. [An old FHS question] (i) Let A, B be real symmetric $n \times n$ matrices. Suppose that all the eigenvalues of A are positive and let T be an invertible $n \times n$ matrix such that $T^{\mathrm{tr}}AT = I_n$. Show that $T^{\mathrm{tr}}BT$ is symmetric and deduce that there exists an invertible $n \times n$ matrix S such that $S^{\mathrm{tr}}AS = I_n$ and $S^{\mathrm{tr}}BS$ is diagonal. Show also that the diagonal entries of $S^{\mathrm{tr}}BS$ are the roots of the equation $\det(xA - B) = 0$.

(ii) Show that if $Q(x, y, z) = 3x^2 + 5y^2 + 3z^2 + 2xy - 2xz + 2yz$ and $R(x, y, z) = x^2 + y^2 + z^2 + 10xy + 2xz - 6yz$ then there exist linear forms l, m, n in x, y, z such that $Q(x, y, z) = l^2 + m^2 + n^2$ and $R(x, y, z) = l^2 + \sqrt{2}\,m^2 - \sqrt{2}\,n^2$.

3. [Part of FHS 1990, A1, 4.] Let V be the vector space of all $n \times n$ real matrices with the usual addition and scalar multiplication. For A, B in V, let $\langle A, B\rangle = \mathrm{Trace}(AB^t)$, where B^t denotes the transpose of B.

(i) Show that this defines an inner product on V.

(ii) Let P be an invertible $n \times n$ matrix and let $\theta : V \to V$ be the linear transformation given by $\theta(A) = P^{-1}AP$. Find the adjoint θ^* of θ.

(iii) Prove that θ is self-adjoint if and only if P is either symmetric or skew-symmetric (P is *skew-symmetric* if $P^t = -P$).

4. [Part of an old FHS question.] Consider the vector space of real-valued polynomials of degree $\leqslant 1$ in a real variable t, equipped with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t$. Let D be the operation of differentiation with respect to t. Find the adjoint D^*.

5. [FHS 1988, A1, 3.] Let V be a finite-dimensional real inner product space, $T : V \to V$ a linear transformation and T^* its adjoint. Prove the following:

(i) $(\mathrm{im}\, T)^{\perp} = \ker T^*$

(ii) $\ker T^*T = \ker T$

(iii) $\dim\ker(TT^*) = \dim\ker(T^*T)$

(iv) if v is an eigenvector of T^*T with eigenvalue $\lambda \neq 0$, then Tv is an eigenvector of TT^* with eigenvalue λ.

Deduce that there exists an orthogonal linear transformation P such that $P^{-1}TT^*P = T^*T$. [You may assume that any self-adjoint linear transformation has an orthonormal basis of eigenvectors.]

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